2. *Seismic Focus without Rayleigh-Waves.*

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1. Introduction.

It is clearly verified, practically and theoretically that when some force acts upon or within a semi-infinite elastic body, certain waves with special velocity are propagated along the plane surface. Today, these waves are called Rayleigh-waves and nobody doubts their existence. However the mechanism of the generation of these waves are not yet fully understood. Although great efforts are being made[1] to make clear this problem, we do not know how the energy of Rayleigh-waves is supplied and how the waves with velocity different from either P or S can grow and be maintained. And it seems to us that if a certain origin without Rayleigh-waves should exist, and further, if it should lack some property which is inherent in an origin which radiates Rayleigh-waves, then this property must be closely connected with the generation of them. However, can such a seismic origin exist? Can it have or lack such a special property? We will treat these problems in the following articles.

2. Solutions satisfying boundary conditions.

At first we will assume a seismic origin of a dilatational type. We take a scalar potential

$$\Phi = A_n^m \cdot (hR)^{-\frac{1}{2}} H_{n+\frac{1}{2}}^{(2)}(hR) \bar{P}_n^m(\cos\theta) \exp\{\pm im\varphi\} \exp\{ipt\} \quad \ldots\ldots (2.1)$$

in which R, θ, φ are spherical coordinates

and $\quad h = ap = p/V_P = p/\{(\lambda+2\mu)/\rho\}^{\frac{1}{2}}$, ($\rho$, λ, μ are density and Lamé's constants)

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1) e.g. H. JEFFREYS, "The Formation of Love Waves", *Beitr. z. Geophys.*, **30** (1931), 336, K. SEZAWA and K. KANAI, "The Action of Soil Loyers and of the Ocean as Dynamic Dampers to Seismic Surface Waves, and Notes on a Few Previous Papers," *Bull. Earthq. Res. Inst.*, **18** (1940), 150~168. T. HIRONO, "Mathematical Theory of Shallow Earthquake," *Geophys. Mag.*, **21** (1949), 1.

$$\bar{P}_n^m(\zeta) = (1-\zeta^2)^{m/2}\left(\frac{d}{d\zeta}\right)^m P_n(\zeta)$$

$H_\nu^{(\ )}(\zeta)$ means Hankel's function.

If this focus is situated at $z=-d$ (xy-plane coincides with the free surface and z-axis is taken vertically upwards), then the above expression takes the following form[2] (omitting time factor exp{ipt})

$$\Phi = A_n^m \mu_n^m h^{-2}\exp\{\pm im\varphi\}\int_0^\infty J_m(rs)\, P_n^m\{(h^2-s^2)^{\frac{1}{2}}/h\}\exp\{-i(h^2-s^2)^{\frac{1}{2}}(d+z)\}$$
$$\cdot\{h/(h^2-s^2)^{\frac{1}{2}}\}sds$$
$$r^2 = x^2+y^2$$
$$\mu_n^m = \pi^{-\frac{1}{2}}2^{\frac{1}{2}}i^{n-2m} \qquad \text{...............(2.2)}$$
$$P_n^m(\zeta) = (\zeta^2-1)^{m/2}\left(\frac{d}{d\zeta}\right)^m P_n(\zeta)$$

and $J_m(\ )$ means Bessel function.

As already known by the past study of the generation of Rayleigh-waves, the displacement vector ϑ in the above case is obtained by the next formula[3]

$$\vartheta = \mathrm{grad}(\Phi+\Phi^1) + \mathrm{rot}\ \mathrm{rot}\,(0,0,\Pi^1) \qquad \text{...........(2.3)}$$

or

$$\vartheta_x = \frac{\partial}{\partial x}(\Phi+\Phi^1) + \frac{\partial^2}{\partial x\partial z}\Pi^1$$
$$\vartheta_z = \frac{\partial}{\partial z}(\Phi+\Phi^1) - \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Pi^1 \qquad \text{.............(2.4)}$$

and stress components are

$$\begin{cases} \widehat{xz} = \mu\left\{2\dfrac{\partial^2}{\partial x\partial z}(\Phi+\Phi^1) + \left(\dfrac{\partial^3}{\partial x\partial z^2}-\dfrac{\partial^3}{\partial x^3}-\dfrac{\partial^3}{\partial x\partial y^2}\right)\Pi^1\right\} \\ \widehat{zz} = \lambda\nabla^2(\Phi+\Phi^1) + 2\mu\dfrac{\partial^2}{\partial z^2}(\Phi+\Phi^1) - 2\mu\left\{\dfrac{\partial^3}{\partial x^2\partial z}+\dfrac{\partial^3}{\partial x^2\partial z}\right\}\Pi^1 \end{cases} \text{..(2.5)}$$

where Φ^1 is interpreted as reflected P-waves, and Π^1 as reflected S waves. The expressions of Φ^1 ane Π^1 take the following forms;

2) Y. SATÔ, "Transformation of Wave-functions related to the Transformation of Coordinates Systems." *Bull. Earthq. Res. Inst.*, **28** (1949), 1.

3) e.g. S. SYÔNO, "On the Propagation of Tremors over the Plane Surface of a Semi-Infinite Perfect Elastic Solid." (1) *Geophys. Mag.*, **12** (1938), 67.

$$\begin{cases} \Phi^1 = A_n^m \mu_n^m h^{-2} \exp\{\pm im\varphi\} \int_0^\infty S\cdot J_m(rs)\, P_n^m\{(h^2-s^2)^{\frac{1}{2}}/h\} \\ \qquad\qquad \cdot \exp\{-i(h^2-s^2)^{\frac{1}{2}}(d-z)\}\cdot\{h/(h^2-s^2)^{\frac{1}{2}}\} sds \\ \Pi^1 = A_n^m \mu_n^m h^{-2} \exp\{\pm im\varphi\} \int_0^\infty T\cdot J_m(rs)\, P_n^m\{(h^2-s^2)^{\frac{1}{2}}/h\} \\ \qquad\qquad \cdot \exp\{-i(h^2-s^2)^{\frac{1}{2}}d + i(k^2-s^2)^{\frac{1}{2}}z\}\cdot\{h/(h^2-s^2)^{\frac{1}{2}}\} sds \end{cases} \quad \ldots\ldots(2.6)$$

The explicit expression of S and T must be determined by the boundary conditions which are known as

$$\widehat{xz} = \widehat{yz} = \widehat{zz} = 0 \quad \text{at} \quad z = 0 \quad \ldots\ldots(2.7)$$

Therefore

$$\begin{cases} \lambda\nabla^2(\Phi+\Phi^1) + 2\mu\dfrac{\partial^2}{\partial z^2}(\Phi+\Phi^1) - 2\mu\left(\dfrac{\partial^3}{\partial x^2\partial z} + \dfrac{\partial^3}{\partial y^2\partial z}\right)\Pi^1 = 0 \\ 2\dfrac{\partial}{\partial z}(\Phi+\Phi^1) + \left(\dfrac{\partial^2}{\partial z^2} - \dfrac{\partial^2}{\partial x^2} - \dfrac{\partial^2}{\partial y^2}\right)\Pi^1 = 0 \end{cases} \quad \ldots\ldots(2.8)$$

Since Φ and Φ^1 satisfy the differential equation of the form $\nabla^2\Phi = -h^2\Phi$ and Π^1 satisfies $\nabla^2\Pi^1 = -k^2\Pi^1$ ($k^2 = b^2p^2 = p^2/V_s^2 = p^2/(\mu/\rho)$), (2.8) is somewhat simplified by the aid of this relation and the following two expressions must hold at the boundary surface.

$$\begin{cases} \displaystyle\int_0^\infty\Big[\Big\{-(\gamma^2-2)h^2 + 2\frac{\partial^2}{\partial z^2}\Big\}\cdot\exp\{-i(h^2-s^2)^{\frac{1}{2}}z\} \\ \qquad +\Big\{-(\gamma^2-2)h^2 + 2\dfrac{\partial^2}{\partial z^2}\Big\}\cdot S\exp\{i(h^2-s^2)^{\frac{1}{2}}z\} \\ \qquad +\Big\{2k^2\dfrac{\partial}{\partial z} + 2\dfrac{\partial^3}{\partial z^3}\Big\}\cdot T\exp\Big\{i(k^2-s^2)^{\frac{1}{2}}z\Big\}\Big] \\ \qquad \cdot J_m(rs)\, P_n^m\{(h^2-s^2)^{\frac{1}{2}}/h\}\cdot\exp\{-i(h^2-s^2)^{\frac{1}{2}}d\}\cdot\{h/(h^2-s^2)^{\frac{1}{2}}\}sds \\ \displaystyle\int_0^\infty\Big[2\frac{\partial}{\partial z}\exp\{-i(h^2-s^2)^{\frac{1}{2}}z\} + 2\frac{\partial}{\partial z}S\exp\{i(h^2-s^2)^{\frac{1}{2}}z\} + \Big\{2\frac{\partial^2}{\partial z^2}+k^2\Big\}T\Big] \\ \qquad \cdot J_m(rs)\, P_n^m\{(h^2-s^2)^{\frac{1}{2}}/h\}\cdot\exp\{-i(h^2-s^2)^{\frac{1}{2}}d\}\cdot\{h/(h^2-s^2)^{\frac{1}{2}}\}sds \end{cases} \quad \ldots\ldots(2.9)$$

where $\gamma^2 = (\lambda+2\mu)/\mu$

Since these two integral expressions must hold without regard to r, we may equate the two parentheses [ ] to zero; therefore we have

$$\begin{cases} \{-\gamma^2 h^2 + 2s^2\}S + i2(k^2 - s^2)^{\frac{1}{2}} s^2 T = -\{-\gamma^2 h^2 + 2s^2\} \\ i2(h^2 - s^2)^{\frac{1}{2}} S + (2s^2 - k^2)T = i2(h^2 - s^2) \end{cases} \quad \ldots\ldots\ldots\ldots\ldots (2.10)$$

Solving these equations

$$S = H/D, \qquad T = J/D \quad ,$$

where

$$\begin{aligned} D &= (2s^2 - k^2)^2 + 4s^2 \tilde{\alpha}\tilde{\beta} \quad , \qquad \tilde{\alpha} = (h^2 - s^2)^{\frac{1}{2}} \\ H &= -(2s^2 - k^2)^2 + 4s^2\tilde{\alpha}\tilde{\beta} \quad , \qquad \tilde{\beta} = (k^2 - s^2)^{\frac{1}{2}} \ldots\ldots (2.11) \\ J &= i4(2s^2 - k^2)\tilde{\alpha} \end{aligned}$$

We intoroduce (2.11) into (2.2) and (2.6), and get

$$\begin{cases} \Phi = A_n^m \mu_n^m h^{-2} \cdot \exp\{\pm im\varphi\} \int_0^\infty J_m(rs) P_n^m\{\tilde{\alpha}/h\} \exp\{-i\tilde{\alpha}(d+z)\}(h/\tilde{\alpha})sds \\ \Phi^1 = A_n^m \mu_n^m h^{-2} \cdot \exp\{\pm im\varphi\} \int_0^\infty \frac{H}{D} J_m(rs) P_n^m\{\tilde{\alpha}/h\} \exp\{-i\tilde{\alpha}(d-z)\}(h/\tilde{\alpha})sds \\ \Pi^1 = A_n^m \mu_n^m h^{-2} \cdot \exp\{\pm im\varphi\} \int_0^\infty \frac{J}{D} J_m(rs) P_n^m\{\tilde{\alpha}/h\} \exp\{-i\tilde{\alpha}d + i\tilde{\beta}z\}(h/\tilde{\alpha})sds \end{cases}$$

$$\ldots\ldots\ldots\ldots (2.12)$$

3. Displacement components of Rayleigh-waves.

The vertical distribution of the amplitude of Rayleigh-waves is uniquely determined by Poisson's ratio of the elastic medium; so that, if the surface amplitude of both components of displacement vanish, not even a single train of Rayleigh-wave can exist in this medium.

Introducing (2.12) into (2.4) we have at once

$$\vartheta_x = A_n^m \mu_n^m h^{-2} \frac{\partial}{\partial x} \exp\{\pm im\varphi\} \int_0^\infty J_m(rs) P_n^m\{\tilde{\alpha}/h\} \exp\{-i\tilde{\alpha}d\}$$

$$\cdot \left\{ \exp(-i\tilde{\alpha}z) + \frac{H}{D}\exp(i\tilde{\alpha}z) + \frac{J}{D} i\tilde{\beta}\exp(i\tilde{\beta}z) \right\} (h/\tilde{\alpha})sds \quad ..(3.1)$$

At the free surface $z = 0$ the above expression can be written as

$$\vartheta_x = A_n^m \mu_n^m h^{-2} \cdot 4hk^2 \cdot \frac{\partial}{\partial x} \exp\{\pm im\varphi\} \int_0^\infty J_m(rs) P_n^m\{\tilde{\alpha}/h\} \exp\{-i\tilde{\alpha}d\} \frac{1}{D} \tilde{\beta} sds$$

$$\ldots\ldots (3.2)$$

and ϑ_y is obtained from (3.2) by substituting y for x. Similarly

$$\vartheta_z = A_n^m \mu_n^m h^{-2} \cdot i2hk^2 \cdot \exp\{\pm im\varphi\} \int_0^\infty J_m(rs) P_n^m\{\tilde{\alpha}/h\} \exp\{-i\tilde{\alpha}d\} \frac{1}{D}(2s^2 - k^2) s ds \qquad \ldots\ldots\ldots(3.3)$$

Since

$$J_m(rs) = \frac{1}{2}\{H_m^{(1)}(rs) + H_m^{(2)}(rs)\}, \quad H_m^{(2)}(-rs) = (-)^{m+1} H_m^{(1)}(rs) \qquad \ldots(3.4)$$

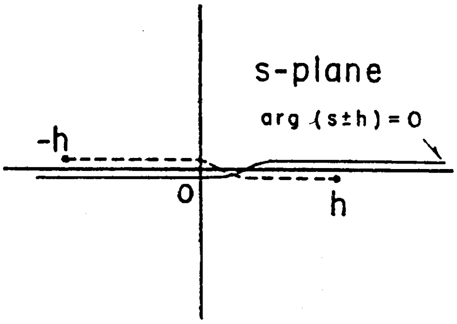

Fig. 1

and the sign of $\tilde{\alpha} = \sqrt{(h^2 - s^2)}$ should be taken as is shown in Fig. 1, the above two integrations are transformed into the next form

$$\vartheta_x = A_n^m \mu_n^m \cdot 2h^{-1} k^2 \frac{\partial}{\partial x} \exp\{\pm im\varphi\} \int_{-\infty}^{\infty} H_m^{(1)}(rs)\, P_n^m\{\tilde{\alpha}/h\} \exp\{-i\tilde{\alpha}d\} \frac{1}{D} \tilde{\beta} s ds \qquad \ldots\ldots\ldots(3.5)$$

and

$$\vartheta_z = A_n^m \mu_n^m \cdot ih^{-1} k^2 \exp\{\pm im\varphi\} \int_{-\infty}^{\infty} H_m^{(1)}(rs) P_n^m\{\tilde{\alpha}/h\} \exp\{-i\tilde{\alpha}d\} \frac{1}{D}(2s^2 - k^2) s ds$$

From these integral representations, displacement components of Rayleigh-waves are, multiplying $2\pi i$ to the residue of the integrand,

$$\left\{ \begin{aligned} \vartheta_x[R] &= 2\pi i A_n^m \mu_n^m \cdot 2h^{-1} k^2 \Big[\frac{\partial}{\partial x} \exp\{\pm im\varphi\} \cdot H_m^{(1)}(rs) \\ &\qquad \cdot P_n^m\{\tilde{\alpha}/h\} \exp\{-i\tilde{\alpha}d\} \frac{1}{D} \tilde{\beta} s \Big]_{s=K} \\ \vartheta_z[R] &= 2\pi i \cdot A_n^m \mu_n^m \cdot ih^{-1} k^2 \Big[\exp\{\pm im\varphi\} \cdot H_m^{(1)}(rs) \\ &\qquad \cdot P_n^m\{\tilde{\alpha}/h\} \exp\{-i\tilde{\alpha}d\} \cdot \frac{1}{D}(2s^2 - k^2) s \Big]_{s=K} \end{aligned} \right. \qquad \ldots\ldots\ldots(3.6)$$

where K is the root of $D = 0$.

4. Superposition of two seismic origins.

Now we will consider another seismic origin which is specified by a scalar potential Φ' situated at the same point with the former one.

$$\Phi' = A_{n'}^{m}\mu_{n'}^{m}\exp\{\pm im\varphi\}\int_0^\infty J_m(rs)P_{n'}^{m}\{a/h\}\cdot\exp\{-i\tilde{\alpha}(z+d)\}(h/\tilde{\alpha})sds \quad \ldots(4.1)$$

Displacement vector ϑ' due to this origin is readily gained from (3.5) and the phase of Rayleigh-waves is

$$\left\{\begin{aligned} \vartheta_x'[R] &= 2\pi i\cdot A_{n'}^{m}\mu_{n'}^{m}\cdot 2h^{-1}k^2\Big[\frac{\partial}{\partial x}\exp\{\pm im\varphi\}\cdot H_m^{(1)}(rs) \\ &\qquad\cdot P_{n'}^{m}\{\tilde{\alpha}/h\}\exp\{-i\tilde{\alpha}d\}\frac{1}{D}\tilde{\beta}s\Big]_{s=K} \quad \ldots\ldots(4.2) \\ \vartheta_z'[R] &= 2\pi i\cdot A_{n'}^{m}\mu_{n}^{m}\cdot ih^{-1}k^2\Big[\exp\{\pm im\varphi\}\cdot H_m^{(1)}(rs) \\ &\qquad\cdot P_{n'}^{m}\{\tilde{\alpha}/h\}\exp\{-i\tilde{\alpha}d\}\frac{1}{D}(2s^2-k^2)s\Big]_{s=K} \end{aligned}\right.$$

Comparing the two pairs of expressions (3.6) and (4.2), we easily notice that $\vartheta_x[R]$, $\vartheta_z[R]$ and $\vartheta_x'[R]$, $\vartheta_z'[R]$ are quite similar in form; the only difference between them are the factors $A_n^m\mu_n^m P_n^m\{\tilde{\alpha}/h\}_{s=K}$ and $A_{n'}^m\mu_{n'}^m P_{n'}^m\{\tilde{\alpha}/h\}_{s=K}$. Therefore, if the two origins are superposed, and besides, if the relation

$$\Big[A_n^m\mu_n^m P_n^m\{\tilde{\alpha}/h\} + A_{n'}^m\mu_{n'}^m P_{n'}^m\{\tilde{\alpha}/h\}\Big]_{s=K} = 0 \quad \ldots\ldots(4.3)$$

holds, then the Rayleigh-waves generated from these origins, which are situated at the same point $x=y=0$ and $z=-d$, will cancel each other and no Rayleigh-waves will be observed.

As is clear from the expressions (3.5) or (4.1), above relation holds without regard to epicentral distance r, azimuthal angle φ and focal depth d. Moreover, the condition (4.3) contains no function concerning time, hence Rayleigh-waves can never be generated from the seismic origin of the above type, even if the time factor may take an arbitrary functional form.

5. Numerieal calculations.

In this article we will perform the numerical calculations assuming $\lambda=\mu$.

When $\lambda = \mu$

$$K^2/h^2 = 3(3+\sqrt{3})/4 = 3.54904$$
$$[\bar{\alpha}/h]_{s=K} = -i\,1.59656 \qquad \text{.......................(5.1)}$$

therefore we have from (4.3)

$$\frac{A^m_{n'}}{A^m_n} = -\frac{\mu^m_n}{\mu^m_{n'}}\left[\frac{P^m_n\{\bar{\alpha}/h\}}{P^m_{n'}\{\bar{\alpha}/h\}}\right]_{s=K} = -i^{n-n'}\frac{P^{(m)}_n\{-i1.59656\}}{P^{(m)}_{n'}\{-i1.59656\}} \qquad \text{.....(5.2)}$$

where $P^{(m)}_n(\zeta) = (d/d\zeta)^m P_n(\zeta)$

Results of calculations are as follows;

$A^0_1/A^0_0 = -0.62637,$

$A^0_2/A^0_0 = -0.23129,$ $A^0_2/A^0_1 = -0.39627,$

$A^0_3/A^0_0 = -0.07956,$ $A^0_3/A^0_1 = -0.12702,$ $A^0_3/A^0_2 = -0.34398$

$A^1_2/A^1_1 = -0.20878,$

$A^1_3/A^1_1 = -0.04850,$ $A^1_3/A^1_2 = -0.23257$

$A^2_3/A^2_2 = -0.12527$

..................(5.3)

6. Conclusions.

By the calculations so far obtained, we have seen that seismic origin which radiates no Rayleigh-waves can exist. If two origins which are of the type (2.1) and (4.1) and which satisfy one of the relations shown in (5.3), are superposed, we will observe no Rayleigh-waves. Thus we have made clear the first problem in §1, however, we cannot reply in the affirmative to the second problem. Even if we examine the nature of the origin specified by (5.3), we can scarcely discover the property which we had expected at the beginning of this study.

When we look at seismograms we often notice that while one seismogram shows a remarkably large surface wave phase, another seismogram on the contrary shows hardly any evidence of the same phase. Hitherto these phenomena were attributed solely to the depth of foci, and not to the mechanism of them. But, as we have seen in this paper we should also pay attention to the latter cause.

Next, consider a seismic origin of a dilatational type

$$A^{m\prime}_n\cdot(hR)^{-\frac{1}{2}}H^{(2)}_{n+\frac{1}{2}}(hR)\,\bar{P}^m_n(\cos\theta')\exp\{\pm im\varphi'\}\exp\{ipt\}$$
$$+\,A^{m\prime}_{n'}\cdot(hR)^{-\frac{1}{2}}H^{(2)}_{n'+\frac{1}{2}}(hR)\,\bar{P}^m_{n'}(\cos\theta')\exp\{\pm im\varphi'\}\exp\{ipt\}, \quad ..(6.1)$$

where the spherical coordinates R, θ', φ' refer to the same origin with above, but to the different axes x', y', z', and $A_n^{m'}$ and $A_{n'}^{m}$ satisfy the relation (5.2). If the axes x', y', z' coincide with the former axes x, y, z, the focus (6.1) naturally radiate no Rayleigh-waves. Now, if z' axis does not coincide with z, what will happen in this case?

For brevity we will take a simple example; putting $m = 0, n = 0, n' = 1$ into (6.1) we have

$$\Big[A_0^{0'}\cdot(hR)^{-\frac{1}{2}}H_{\frac{1}{2}}^{(2)}(hR) + A_1^{0'}\cdot(hR)^{-\frac{1}{2}}H_{\frac{3}{2}}^{(2)}(hR)\,P_1(\cos\theta')\Big]\exp\{ipt\}\qquad\text{(6.2)}$$

Referring to (x, y, z)-system, (6.2) can be transformed into the following form

$$\Big[A_0^{0'}\cdot(hR)^{-\frac{1}{2}}H_{\frac{1}{2}}^{(2)}(hR) + A_1^{0'}\cdot(hR)^{-\frac{1}{2}}H_{\frac{3}{2}}^{(2)}(hR)\Big\{\mathfrak{A}_1^0 P_1(\cos\theta) + (\mathfrak{A}_1^1\cos\varphi + \mathfrak{B}_1^1\sin\varphi)\,\bar{P}_1^1(\cos\theta)\Big\}\Big]\exp\{ipt\}\qquad\text{(6.3)}$$

A glance at the last line will show that this focus generates Rayleigh-waves. Hence the said property is not inherent in the mechanism of the focus itself, but consists in the relation between the free surface and the focus.

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2. レーリー波を發生しない震源

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半無限体内にある震源で，レーリー波を發生しないやうなものが存在し得ることを証明した．簡單な例をあげれば，(自由表面に垂直に軸をとり，これを基準にして θ をはかれば)

$$\varPhi = (hR)^{-\frac{1}{2}}\Big[H_{\frac{1}{2}}^{(2)}(hR) - 0.23129\,H_{\frac{5}{2}}^{(2)}(hR)\,P_2(\cos\theta)\Big]f(t)$$

で表はされる震源などはその一つである．ここに $f(t)$ は t の勝手な函数である．

しかしこの震源が，何か明かにわかる特殊の性質をもつてゐるか，といへば，答は否定的である．又たとへ上と同じ力の加はり方をしても，z 軸が自由表面に垂直でない場合にはレーリー波は必ず發生する．