3. *Mathematical Study of the Propagation of Waves upon Stratified Medium.* (2)

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(Read Dec. 16, 1947.—Received Dec. 20, 1950.)

In the previous paper,[1] we have shown the way of getting asymptotic expansions of the integral of the form $\int_a^b \frac{\cos}{\sin} \xi x \, F(\xi)\, d\xi$, and the main results obtained are as follows;

When $F(\xi)$ is differentiable at the closed domain $[a, b]$

$$\begin{aligned}\int_a^b F(\xi) \frac{\cos}{\sin} \xi x \, d\xi = & \, x^{-1}[\pm F(b) \frac{\sin}{\cos} bx \mp F(a) \frac{\sin}{\cos} ax] \\ & + x^{-2}[F'(b) \frac{\cos}{\sin} bx - F'(a) \frac{\cos}{\sin} ax] \\ & + x^{-3}[\mp F''(b) \frac{\sin}{\cos} bx \pm F''(a) \frac{\sin}{\cos} ax] \qquad ..(2.1),\ (2.2) \\ & + x^{-4}[-F'''(b) \frac{\cos}{\sin} bx + F'''(a) \frac{\cos}{\sin} ax] + \dots\end{aligned}$$

When $F(\xi)$ contains $\sqrt{\xi^2 - a^2}$

$$\begin{aligned}\int_a^b F(\xi) \frac{\cos}{\sin} \xi x \, d\xi = & \, x^{-\frac{1}{2}} F_{-1}(a) \sqrt{\pi} \cdot \frac{\cos}{\sin} (ax + \pi/4) \\ & + x^{-1}[\pm F(b) \frac{\sin}{\cos} bx \mp F_0(a) \frac{\sin}{\cos} ax] \qquad ..(3.5),\ (3.6) \\ & \mp x^{-\frac{3}{2}} F_1(a) \frac{1}{2} \sqrt{\pi} \cdot \frac{\sin}{\cos} (ax + \pi/4) + x^{-2}[\cdots] + \dots\end{aligned}$$

in which $F_{-1}(a)$, $F_1(a)$... are defined by the following expression

$$F(\xi) = F_{-1}(a) \ (\xi - a)^{-\frac{1}{2}} + F_0(a) + F_1(a) \cdot (\xi - a)^{\frac{1}{2}} + F_2(a) \cdot (\xi - a) + \dots..$$

When $F(\xi)$ contains $\sqrt{\xi^2 - b^2}$.

$$\begin{aligned}\int_a^b F(\xi) \frac{\cos}{\sin} \xi x \, d\xi = & \mp i x^{-\frac{1}{2}} F_{-1}(b) \sqrt{\pi} \cdot \frac{\sin}{\cos} (bx + \pi/4) \\ & + x^{-1}[\pm F_0(b) \frac{\sin}{\cos} bx \mp F(a) \frac{\sin}{\cos} ax] \qquad ..(3.7),\ (3.8) \\ & - i x^{-\frac{3}{2}} F_1(b) \frac{1}{2} \sqrt{\pi} \cdot \frac{\cos}{\sin} (bx + \pi/4) + x^{-2}[\cdots] + \dots\end{aligned}$$

in which $F_{-1}(b)$, $F_1(b)$... are defined by a similar expression with the above case.

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1) Y. SATÔ, *Bull. Earthq. Res. Inst.*, **26** (1948), 1.

When $F(\xi)$ contains $\sqrt{\xi^2-c^2}$, where $a<c<b$.

$$\int_a^b F(\xi)\frac{\cos}{\sin}\xi x d\xi = x^{-\frac{1}{2}}F_{-1}(c)\sqrt{\pi}\cdot\exp\{-i(cx\pm\pi/4)\}$$
$$+\,x^{-1}[F(b)\frac{\sin}{\cos}bx - F(a)\frac{\sin}{\cos}ax] \quad \ldots\ldots(4.1)$$
$$\mp x^{-\frac{3}{2}}F_1(c)\frac{1}{2}\sqrt{\pi}\cdot\exp\{-i(cx\mp\pi/4)\}+x^{-2}[\cdots]+\cdots$$

and for convenience of practical use, we have described a small lemma;

When $F(\xi)$ is expressed as

$$F(\xi) = A_{-1}(\xi)\cdot(\xi^2-c^2)^{-\frac{1}{2}} + A_0(\xi) + A_1(\xi)(\xi^2-c^2)^{\frac{1}{2}}+\cdots\cdot \quad \ldots(5.1)$$

where $A_{-1}(\xi)$, $A_0(\xi)\ldots$ are functions that are identically zero, or not zero and differentiable at $\xi=c$, then $F(\xi)$ may be developed in the following scheme

$$F(\xi) = A_{-1}(c)\cdot(2c)^{-\frac{1}{2}}(\xi-c)^{-\frac{1}{2}}+A_0(c)$$
$$+\{-A_{-1}(c)/4c + A_1'(c) + A_1(c)\cdot 2c\}(2c)^{-\frac{1}{2}}(\xi-c)^{\frac{1}{2}}+\cdots \quad (5.2)$$

§ 6. Examples.

In the first report we could not give applied examples of our formulae owing to limited space. Now we will show it in this report.

As the first example we will adopt

$$I(x) = I_c(x) + iI_s(x) = \frac{2}{\pi}\int_1^\infty \frac{1}{\sqrt{(\xi^2-1)}}\exp(i\xi x)d\xi^{\,2)} \quad \ldots(6.1)$$

As we can easily see, the branch points of the integrand is ± 1, and the lower and upper limits of the integral are 1 and ∞, therefore this is an example of formula (3.5) and (3.6). Using the same notations $F(\xi)$, $A_{-1}(\xi)$ etc. with the previous paper,

$$a = 1\ (\text{branch point}),\quad b=\infty,$$
$$F(\xi) = \frac{2}{\pi}(\xi^2-1)^{-\frac{1}{2}} \quad \ldots(6.2)$$
$$A_{-1}(\xi) = \frac{2}{\pi},\quad A_n(\xi)=0\quad (n\geqq 0)$$

Therefore from (5.2)

$$F(\xi) = \frac{2}{\pi}\cdot 2^{-\frac{1}{2}}(\xi-1)^{-\frac{1}{2}} + \left\{-\frac{2}{\pi}\cdot\frac{1}{4}\right\}2^{-\frac{1}{2}}(\xi-1)^{\frac{1}{2}} + \cdots \qquad \ldots.(6.3)$$

Placing the above expression into (3.6), we have

$$I_s(x) \equiv J_0(x) = x^{-\frac{1}{2}}\cdot\frac{2^{\frac{1}{2}}}{\pi}\sqrt{\pi}\cdot\sin\left(x+\frac{\pi}{4}\right) - x^{-1}[\ 0\ ]$$

$$+x^{-\frac{3}{2}}\frac{2^{-\frac{3}{2}}}{\pi}\cdot\frac{1}{2}\sqrt{\pi}\cdot\cos\left(x+\frac{\pi}{4}\right)+\cdots$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left[\cos\left(x-\frac{\pi}{4}\right)+\frac{1}{8x}\sin\left(x-\frac{\pi}{4}\right)+\cdots\right] \qquad \ldots.(6.4)$$

Similarly from (3.5)

$$I_c(x) \equiv Y_0(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\cdot\left[\cos\left(x+\frac{\pi}{4}\right)+\frac{1}{8x}\sin\left(x+\frac{\pi}{4}\right)+\cdots\right] \ldots(6.5)$$

This is the well-known asymptotic expansion of Bessel functions $J_0(x)$[3] and $Y_0(x)$[4].

Now, we will show another example.

According to H. Lamb[5] horizontal displacement u_0 of a point situated on the surface of a semi-infinite elastic solid when a concentrated force is applied vertically to the origin is expressed by

$$u_0 = -\frac{iQ}{2\pi\mu}\int_{-\infty}^{\infty}\frac{\xi(2\xi^2-k^2-2\alpha\beta)}{F(\xi)}\exp(i\xi x)d\xi \qquad \ldots..(6.6)$$

in which

$$F(\xi) = (2\xi^2-k^2)^2 - 4\xi^2\alpha\beta$$

$\alpha = \sqrt{(\xi^2-h^2)}$, $\beta = \sqrt{(\xi^2-k^2)}$ and Q is a constant.

Since $F(\xi)$, α and β are even functions

$$u_0 = \frac{Q}{\pi\mu}\int_0^{\infty}G(\xi)\sin\xi x d\xi \qquad \ldots..(6.7)$$

where $G(\xi) = \xi(2\xi^2-k^2-2\alpha\beta)/F(\xi)$

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2) WATSON, *Theory of Bessel Fun tions*, p. 180, expression (14) and (15).

3) *Ibid.*, p. 194.

4) *Ibid.*, p. 199.

5) H. LAMB, *Phil. Trans.* A **203** (1904), 1. Expression (52).

$G(\xi)$ has more than one branch point, that is $\xi^2 = h^2$ and $\xi^2 = k^2$, and we must somewhat modify the previously obtained formula (4.1). However, the procedure is easy and we have not to state it precisely in this paper.

From (6.7)

$$\begin{aligned} G_{-1}(h) &= 0 \\ G_0(h) &= G(h) = h/(2h^2 - k^2) \\ G_1(h) &= [(\xi - h)^{-\frac{1}{2}}\{G(\xi) - G_0(h)\}]_{\xi=h} \\ &= i2^{\frac{3}{2}} h^{\frac{3}{2}} k^2 (k^2 - h^2)^{\frac{1}{2}} (2h^2 - k^2)^{-3} \end{aligned} \qquad \ldots\ldots(6.8)$$

and

$$\begin{aligned} G_{-1}(k) &= 0 \\ G_0(k) &= G(k) = 1/k^2 \\ G_1(k) &= [(\xi - k)^{-\frac{1}{2}}\{G(\xi) - G_0(k)\}]_{\xi=k} \\ &= 2^{\frac{3}{2}}(k^2 - h^2)^{\frac{1}{2}}/k^{\frac{5}{2}} \end{aligned} \qquad \ldots\ldots(6.9)$$

The asymptotic expansion of u_0 is, using the latter formula of (4.1) and adding the effect at two branch points, h and k, as follows;

$$\begin{aligned} u_0 = & \frac{Q}{\pi\mu} x^{-\frac{3}{2}} \cdot i \frac{2^{\frac{3}{2}} h^{\frac{3}{2}} k^2 \sqrt{k^2 - h^2}}{(2h^2 - k^2)^3} \cdot \frac{1}{2}\sqrt{\pi} \cdot \exp\{-i(hx + \pi/4)\} \\ & + \frac{Q}{\pi\mu} x^{-\frac{3}{2}} \cdot 2^{\frac{3}{2}} \frac{\sqrt{k^2 - h^2}}{k^{5/2}} \cdot \frac{1}{2}\sqrt{\pi} \cdot \exp\{-i(kx + \pi/4)\} + \cdots\cdots \end{aligned} \qquad (6.10)$$

This is the very expression which H. Lamb had obtained in his famous paper.[6] Of course the vertical component v_0 may be also obtained in a similar manner.

PART II. GENERATION AND PROPAGATION OF WAVES UPON STRATIFIED MEDIUM.

§ 7. Fundamental equations.

Equations of the motion of elastic body in two dimensional space referring to cartesian coordinates are

6) H. LAMB, *Phil. Trans.* A **203** (1904), 1. Expression (90).

$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} = (\lambda + \mu)\dfrac{\partial \varDelta}{\partial x} + \mu \nabla^2 u \\ \rho \dfrac{\partial^2 w}{\partial t^2} = (\lambda + \mu)\dfrac{\partial \varDelta}{\partial z} + \mu \nabla^2 w \end{cases} \qquad \ldots.(7.1)$$

where ρ.... density ; λ, μ.... Lamé's constants
u, w.... displacement component in x and z direction

$$\varDelta = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

Using displacement potentials $\varPhi$ and $\varPsi$,

$$\begin{cases} u = \partial \varPhi/\partial x + \partial \varPsi/\partial z \\ w = \partial \varPhi/\partial z - \partial \varPsi/\partial z \end{cases} \qquad \ldots.(7.2)$$

Introducing (7.2) into (7.1) we have

$$\begin{cases} \partial^2 \varPhi/\partial t^2 = V_p^2 \nabla^2 \varPhi \\ \partial^2 \varPsi/\partial t^2 = V_s^2 \nabla^2 \varPsi \end{cases} \quad \text{where} \quad \begin{matrix} V_p^2 = (\lambda + 2\mu)/\rho \\ V_s^2 = \mu/\rho \end{matrix} \qquad \ldots.(7.3)$$

When we may assume that $\varPhi$ and $\varPsi$ are proportional to $\exp(ipt)$ we get from the above equations

$$\begin{cases} (\nabla^2 + h^2)\, \varPhi = 0 \\ (\nabla^2 + k^2)\, \varPsi = 0 \end{cases} \quad \text{where} \quad \begin{matrix} h \equiv ap \equiv p/V_P \\ k \equiv bp \equiv p/V_S \end{matrix} \qquad \ldots.(7.4)$$

And the solutions of this equation are written in the following form

$$\begin{cases} \varPhi = \exp(\pm \alpha z + ifx + ipt) \\ \varPsi = \exp(\pm \beta z + ifx + ipt) \end{cases} \quad \text{where} \quad \begin{matrix} \alpha = \sqrt{(f^2 - h^2)} \\ \beta = \sqrt{(f^2 - k^2)} \end{matrix} \qquad \ldots.(7.5)$$

Displacement components u and w are expressed by the linear combination of the terms obtained by the introduction of (7.5) into (7.2).

On the other hand stress components are expressed in the following form

$$\begin{cases} \widehat{xx}/\mu = (\lambda/\mu)\varDelta + 2\partial u/\partial x = -k^2\varPhi - 2\partial^2\varPhi/\partial z^2 + 2\partial^2\varPsi/\partial x\partial z \\ \widehat{xz}/\mu = \partial w/\partial x + \partial u/\partial z = 2\partial^2\varPhi/\partial z\partial x - k^2\varPsi - 2\partial^2\varPsi/\partial x^2 \\ \widehat{zz}/\mu = (\lambda/\mu)\varDelta + 2\partial w/\partial z = -k^2\varPhi - 2\partial^2\varPhi/\partial x^2 - 2\partial^2\varPsi/\partial x\partial z \end{cases} \qquad \ldots.(7.6)$$

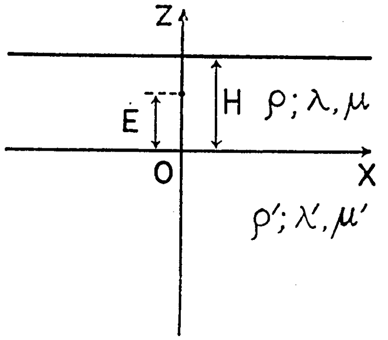

Fig. 1.

§ 8. Simple source in the above layer and boundary conditions.

We wish to determine the component displacements u and w due to a point source of compressional waves at $x=0$, $z=E$ (cf. Fig. 1).

Following H. Nakano's[7] procedure we assume simple harmonic origin

$$\begin{cases} \Phi_0 = QH_0^{(2)}(hr)\exp(ipt) = Q\dfrac{i}{\pi}\exp\{\mp\alpha(z-E)+ifx+ipt\}df, \quad z \gtrless E \\ \Psi_0 = 0 \end{cases} \qquad \ldots.(8.1)$$

where subscription refers to the said origin and the integral sign $\int_{-\infty}^{\infty}$ is abridged for simplicity. The corresponding stress components are (omitting the time factor $\exp(ipt)$)

$$\begin{cases} \widehat{zz_0}/\mu = Q\dfrac{i}{\pi}(2f^2-k^2)\exp\{\mp\alpha(z-E)+ifx\}df \\ \widehat{xz_0}/\mu = Q(\pm)\dfrac{1}{\pi}2f\exp\{\mp\alpha(z-E)+ifx\}df \end{cases} \qquad \ldots.(8.2)$$

Displacements due to this origin are

$$\begin{cases} u_0 = Q(-)\dfrac{1}{\pi}\dfrac{f}{\alpha}\exp\{\mp\alpha(z-E)+ifx\}df \\ w_0 = Q(\mp)i\dfrac{1}{\pi}\exp\{\mp\alpha(z-E)+ifx\}df \end{cases} \qquad \ldots.(8.3)$$

Therefore,

horizontal displacement at $z=0$ is $u_0 = Q(-)\dfrac{1}{\pi}\dfrac{f}{\alpha}\exp\{-\alpha E+ifx\}df$

vertical displacement at $z=0$ is $w_0 = Qi\dfrac{1}{\pi}\exp\{-\alpha E+ifx\}df$

tangential stress at $z=0$ is $\widehat{xz_0} = Q(-)\mu\dfrac{1}{\pi}2f\exp\{-\alpha E+ifx\}df$

7) H. NAKANO, *Jap. Journ. Astro. Geophys.*, **2** (1925), 1.

$$\text{normal stress at } z=0 \text{ is } \widehat{zz_0} = Q\mu\cdot i\frac{1}{\pi}\frac{2f^2-k^2}{\alpha}\exp\{-\alpha E+ifx\}\,df$$

$$\text{tangential stress at } z=H \text{ is } \widehat{xz_0} = Q\mu\cdot\frac{1}{\pi}2f\exp\{-\alpha(H-E)+ifx\}\,df$$

$$\text{normal stress at } z=H \text{ is } \widehat{zz_0} = Q\mu\cdot i\frac{1}{\pi}\frac{2f^2-k^2}{\alpha}\exp\{-\alpha(H-E)+ifx\}\,df \qquad \ldots\ldots(8.4)$$

Conditions required at the boundary surfaces are the continuity of displacement and stress components at $z=0$, and the vanishing of stress at the free suface $z=H$. Therefore we must combine displacements such as derived from potentials in (7.5) and satisfy the boundary conditions. For this purpose we assume

$$\begin{cases} \Phi' = \mathbf{A}\exp\{\alpha' z+ifx\} \\ \Psi' = \mathbf{B}\exp\{\beta' z+ifx\} \end{cases} \qquad \ldots\ldots(8.5)$$

where $\alpha' = \sqrt{(f^2-h'^2)},\quad h' = a'p = p/V_{p'} = p/\{(\lambda'+2\mu')/\rho'\}^{\frac{1}{2}}$

$\beta' = \sqrt{(f^2-k'^2)},\quad k' = b'p = p/V_{s'} = p/\{\mu'/\rho'\}^{\frac{1}{2}}$

in the semi-infinite part; and in the upper stratum

$$\begin{cases} \Phi = \{\mathbf{C}\cosh\alpha z+\mathbf{D}\sinh\alpha z\}\exp(ifx) \\ \Psi = \{\mathbf{E}\cosh\beta z+\mathbf{F}\sinh\beta z\}\exp(ifx) \end{cases} \qquad \ldots\ldots(8.6)$$

Six equations which must hold at boundary surfaces are

$$\begin{cases} \mathbf{A}if+\mathbf{B}\beta'+\mathbf{C}(-if)+\mathbf{F}(-\beta) = Q(-)\frac{1}{\pi}\frac{f}{\alpha}\exp(-\alpha E)\,df \\ \mathbf{A}\alpha'+\mathbf{B}(-if)+\mathbf{D}(-\alpha)+\mathbf{E}(if) = Qi\frac{1}{\pi}\exp(-\alpha E)\,df \\ \mathbf{A}i2\chi f\alpha'+\mathbf{B}\chi(2f^2-k^2)+\mathbf{D}(-i2f\alpha)+\mathbf{E}[-(2f^2-k^2)] = Q(-)\frac{1}{\pi}2f\exp(-\alpha E)\,df \\ \mathbf{A}\chi(2f^2-k^2)+\mathbf{B}(-i2f\beta')+\mathbf{C}[-(2f^2-k^2)]+\mathbf{F}(i2f\beta) = Qi\frac{1}{\pi}\frac{2f^2-k^2}{\alpha}\exp(-\alpha E)\,df \\ \mathbf{C}[-i2f\alpha\mathfrak{S}]+\mathbf{D}[-i2f\alpha\mathfrak{C}]+\mathbf{E}[-(2f^2-k^2)\mathfrak{C}']+\mathbf{F}[(2f^2-k^2)\mathfrak{S}'] = Q\frac{1}{\pi}2f\exp(-\alpha E')\,df \\ \mathbf{C}[-(2f^2-k^2)\mathfrak{C}]+\mathbf{D}[-(2f^2-k^2)\mathfrak{S}]+\mathbf{E}[i2f\beta\mathfrak{S}']+\mathbf{F}[i2f\beta\mathfrak{C}'] = Qi\frac{1}{\pi}\frac{2f^2-k^2}{\alpha}\exp(-\alpha E')\,df \end{cases} \qquad \ldots(8.7)$$

in which

$$\chi = \mu' / \mu, \qquad E' = H \dot{-} E$$

$$\mathfrak{C} = \cosh \alpha H, \quad \mathfrak{S} = \sinh \alpha H$$

$$\mathfrak{C}' = \cosh \beta H, \quad \mathfrak{S}' = \sinh \beta H.$$

§ 9. Formal solution.

From the equations (8.7) we can determine **A, B....F** formally; that is....

$$\mathbf{A} = Q\frac{df}{\pi}A/M, \qquad \mathbf{C} = Q\frac{df}{\pi}C/M$$

$$\mathbf{B} = Q\frac{df}{\pi}B/M, \qquad \mathbf{D} = Q\frac{df}{\pi}D/M$$

$$\mathbf{E} = Q\frac{df}{\pi}E/M$$

$$\mathbf{F} = Q\frac{df}{\pi}F/M$$

where

$$M = \begin{vmatrix} if & \beta' & -if & 0 & 0 & -\beta \\ \alpha' & -if & 0 & -\alpha & if & 0 \\ i2\chi f\alpha' & \chi(2f^2-k^2) & 0 & -i2f\alpha & -(2f^2-k^2) & 0 \\ \chi(2f^2-k'^2) & -i2\chi f\beta & -(2f^2-k^2) & 0 & 0 & i2f\beta \\ 0 & 0 & -i2f\alpha\mathfrak{S} & -i2f\alpha\mathfrak{C} & -(2f^2-k^2)\mathfrak{C}' & -(2f^2-k^2)\mathfrak{S}' \\ 0 & 0 & -(f^2-k^2)\mathfrak{C} & -(2f^2-k^2)\mathfrak{S} & i2f\beta\mathfrak{S}' & i2f\beta\mathfrak{C}' \end{vmatrix}$$

and $A, B, C \dots F$ is obtained by replacing the 1st, 2nd, 3rd...6th column of M by $-\dfrac{f}{\alpha}\exp(-\alpha E)$, $i\exp(-\alpha E)$, $-2f\exp(-\alpha E)$, $i\dfrac{2f^2-k^2}{\alpha}\exp(-\alpha E)$, $2f\exp(-\alpha E')$ and $i\dfrac{2f^2-k^2}{\alpha}\exp(-\alpha E')$ respectively.

Expanding the determinants M and C, D, E, F, we have (A and B are not necessary for the calculation of surface displacements)

$$M = k^4 L_1 N_5 - k^2 \alpha L_2 N_6 \qquad C = -i\frac{1}{\alpha}k^4\mathfrak{C}L_1N_5 + ik^2\mathfrak{C}L_2N_6 \qquad D = k^4L_1P_4 + ik^4\frac{1}{\alpha}\mathfrak{C}L_1N_3 + ik^2\mathfrak{C}L_2N_8$$

$$
\begin{array}{llll}
-k^2\alpha L_3N_3 & -k^2\alpha L_3P_4 - ik^2\mathfrak{C}L_3N_5 & & +ik^2\mathfrak{C}L_3N_3 \\
+k^2\beta L_4N_4 & \qquad -ik^2\dfrac{\beta}{\alpha}\mathfrak{C}L_4N_4 & +k^2\beta L_4P_3 & +ik^2\dfrac{\beta}{\alpha}\mathfrak{C}L_4N_2 \\
-k^2\beta L_5N_1 & -k^2\beta L_5P_2 & +k^2\beta L_5P_1 & \\
+\alpha\beta L_6N_2, & +\alpha\beta L_6P_3 + i\beta\mathfrak{C}L_6N_4, & & -i\beta\mathfrak{C}L_6N_2,
\end{array}
$$

$$
\begin{array}{ll}
E = -k^2\alpha L_2P_4 - ik^2\mathfrak{C}L_2\{N_3+N_5\} & F = -k^4L_1P_2 - ik^4\dfrac{1}{\alpha}\mathfrak{C}L_1N_1 \\
\quad -k^2\beta L_4P_2 - ik^2\dfrac{\beta}{\alpha}\mathfrak{C}L_4N_1 & \quad +k^2\alpha L_2P_3 + ik^2\mathfrak{C}L_2\{N_2+N_4\} \\
\quad -\alpha\beta L_6P_1 + i\beta\mathfrak{C}L_6N_1, & \quad +k^2\alpha L_3P_1 - ik^2\mathfrak{C}L_3N_1
\end{array}
\qquad \ldots\ldots(9.3)
$$

where

$$L_1 = \begin{vmatrix} ifU_1 & \beta'U \\ \alpha'U & -ifU_1 \end{vmatrix} = f^2U_1^2 - \alpha'\beta'U \qquad U = 1+2\chi'/\mathfrak{v}^2$$

$$L_2 = \begin{vmatrix} ifU_1 & \beta'U \\ i2\chi'f\alpha' & f'' \end{vmatrix} = if(U_1f'' - 2\chi'\alpha'\beta'U), \qquad U_1 = U-\chi\sigma^2, \quad \chi = \mu'/\mu$$

$$L_3 = \begin{vmatrix} ifU_1 & \beta'U \\ f'' & -i2\chi'f\beta' \end{vmatrix} = \chi\sigma^2k^2\beta' \qquad \chi' = \chi - 1$$

$$L_4 = \begin{vmatrix} \alpha'U & -ifU_1 \\ i2\chi'f\alpha' & f'' \end{vmatrix} = -\chi\sigma^2k^2\alpha', \qquad f' = 2f^2 - k^2, \quad f'' = 2\chi'f^2 - \chi k'^2$$

$$L_5 = \begin{vmatrix} \alpha'U & -ifU_1 \\ f'' & -i2\chi'f\beta' \end{vmatrix} = L_2, \qquad \gamma^2 = (\lambda + 2\mu)/\mu$$

$$L_6 = \begin{vmatrix} i2\chi'f\alpha' & f'' \\ f'' & -i2\chi'f\beta' \end{vmatrix} = 4\chi'^2f^2\alpha'\beta' - f''^2, \qquad \sigma^2 = \left(\frac{\mu}{\rho}\right)\Big/\left(\frac{\mu'}{\rho'}\right)$$

$$N_1 = \begin{vmatrix} -i2f\alpha\mathfrak{S} & -i2f\alpha\mathfrak{C} \\ -(2f^2-k^2)\mathfrak{C} & -(2f^2-k^2)\mathfrak{S} \end{vmatrix} = -i2ff'\alpha \qquad \mathfrak{v} = \left(\frac{p}{f}\right)\Big/ V_s$$

$$N_2 = \begin{vmatrix} -i2f\alpha\mathfrak{S} & -(2f^2-k^2)\mathfrak{C}' \\ -(2f^2-k^2)\mathfrak{C} & i2f\beta\mathfrak{S}' \end{vmatrix} = 4f^2\alpha\beta\mathfrak{S}\mathfrak{S}' - f'^2\mathfrak{C}\mathfrak{C}'$$

$$N_3 = \begin{vmatrix} -i2f\alpha\mathfrak{S} & -(2f^2-k^2)\mathfrak{S}' \\ -(2f^2-k^2)\mathfrak{C} & i2f\beta\mathfrak{C}' \end{vmatrix} = 4f^2\alpha\beta\mathfrak{S}\mathfrak{C}' - f'^2\mathfrak{C}\mathfrak{S}'$$

$$N_4 = \begin{vmatrix} -i2f\alpha\mathfrak{C} & -(2f^2-k^2)\mathfrak{C}' \\ -(2f^2-k^2)\mathfrak{S} & i2f\beta\mathfrak{S}' \end{vmatrix} = 4f^2\alpha\beta\mathfrak{C}\mathfrak{S}' - f^{2\prime}\mathfrak{S}\mathfrak{C}'$$

$$N_5 = \begin{vmatrix} -i2f\alpha\mathfrak{C} & -(2f^2-k^2)\mathfrak{S}' \\ -(2f^2-k^2)\mathfrak{S} & i2f\beta\mathfrak{C}' \end{vmatrix} = 4f^2\alpha\beta\mathfrak{S}\mathfrak{S}' - f'^2\mathfrak{S}\mathfrak{S}'$$

$$N_6 = \begin{vmatrix} -(2f^2-k^2)\mathfrak{C} & -(2f^2-k^2)\mathfrak{S}' \\ i2f\beta\mathfrak{S}' & i2f\beta\mathfrak{C}' \end{vmatrix} = -i2f\beta f', \qquad \begin{matrix} \mathfrak{E} = \exp(-\alpha E) \\ \mathfrak{E}' = \exp(-\alpha E') \end{matrix}$$

$$P_1 = \begin{vmatrix} 2f\mathfrak{E}' & -i2f\alpha\mathfrak{S} \\ i\frac{1}{\alpha}f'\mathfrak{E}' & -f'\mathfrak{C} \end{vmatrix} = -2ff'\mathfrak{E}^{-1}, \qquad P_3 = \begin{vmatrix} 2f\mathfrak{E}' & -f'\mathfrak{C}' \\ i\frac{f'}{\alpha}\mathfrak{E}' & i2f\beta\mathfrak{S}' \end{vmatrix}$$

$$P_2 = \begin{vmatrix} 2f\mathfrak{E}' & -i2f\alpha\mathfrak{C} \\ i\frac{1}{\alpha}f'\mathfrak{E}' & -f'\mathfrak{S} \end{vmatrix} = P_1, \qquad P_4 = \begin{vmatrix} 2f\mathfrak{E}' & -f'\mathfrak{S}' \\ i\frac{f'}{\alpha}\mathfrak{E}' & i2f\beta\mathfrak{C}' \end{vmatrix}.$$

§ 10. Horizontal and vertical displacements.

With these preparations we can calculate the expression of u and w, by introducing (8.6)—**C**, **D**, **E** and **F** contained in this expression are obtained in § 9—into (7.2), we have

$$\text{and} \begin{cases} u = [if\{\mathbf{C}\cosh\alpha z + \mathbf{D}\sinh\alpha z\} + \beta\{\mathbf{E}\sinh\beta z + \mathbf{F}\cosh\beta z\}]\exp(ifx) \\ w = [\alpha\{\mathbf{C}\sinh\alpha z + \mathbf{D}\cosh\alpha z\} - if\{\mathbf{E}\cosh\beta z + \mathbf{F}\sinh\beta z\}]\exp(ifx) \end{cases} \quad \ldots\ldots(10.1)$$

of which the values at the free surface are

$$\begin{cases} u = [if\{\mathbf{C}\mathfrak{C} + \mathbf{D}\mathfrak{S}\} + \beta\{\mathbf{E}\mathfrak{S} + \mathbf{F}\mathfrak{C}\}]\exp(ifx) \\ w = [\alpha\{\mathbf{C}\mathfrak{S} + \mathbf{D}\mathfrak{C}\} - if\{\mathbf{E}\mathfrak{C} + \mathbf{F}\mathfrak{S}\}]\exp(ifx) \end{cases} \quad \ldots\ldots(10.2)$$

or, using (9·1)

$$\begin{cases} u = Q\dfrac{1}{\pi}\dfrac{1}{M}[i\{C\mathfrak{C} + D\mathfrak{S}\} + \beta\{E\mathfrak{S} + F\mathfrak{C}\}]\exp(ifx)df \\ w = Q\dfrac{1}{\pi}\dfrac{1}{M}[\alpha\{C\mathfrak{S} + D\mathfrak{C}\} - if\{E\mathfrak{C} + F\mathfrak{S}\}]\exp(ifx)df \end{cases} \quad \ldots\ldots(10.3)$$

in which $\int_{-\infty}^{\infty}$ is omitted for brevity.

Since the integrand of u is, excluding $\exp(ifx)$, an odd function of f, and that of w is even, (10.3) may be transformed as

$$\begin{cases} u = Q\dfrac{1}{\pi}\cdot i2\displaystyle\int_0^\infty \dfrac{1}{M}[if\{C\mathfrak{C}+D\mathfrak{S}\}+\beta\{E\mathfrak{S}+F\mathfrak{C}\}]\sin(fx)df \\ w = Q\dfrac{1}{\pi}\cdot 2\displaystyle\int_0^\infty \dfrac{1}{M}[\alpha\{C\mathfrak{S}+D\mathfrak{C}\}-if\{E\mathfrak{C}+F\mathfrak{S}\}]\cos(fx)df \end{cases} \quad \ldots(10.4)$$

This is exactly the same form of integration that we learned in the first part, therefore, in order to obtain the asymptotic expansion of the integral we have only to calculate the coefficients of the expressions at branch points and upper and lower limits of the integral. We will consider them in order.

For convenience of description we will simply write

$$\begin{cases} u = Q\dfrac{1}{\pi}i2\displaystyle\int_0^\infty U(f)\sin(fx)df \\ w = Q\dfrac{1}{\pi}\;2\displaystyle\int_0^\infty W(f)\cos(fx)df \end{cases} \quad \ldots\ldots(10.5)$$

At first, we will take up the effect of the lower limit of the integral. Terms concerning this point are, using the formulae in § 2 ((2.1) and (2.2)), written down as

$$\begin{aligned} u\,; &\quad x^{-1}U(0)\cos(0x) - x^{-2}U'(0)\sin(0x) + \cdots\cdots \\ w\,; &\; - x^{-1}W(0)\sin(0x) - x^{-2}W'(0)\cos(0x) + \cdots\cdot \end{aligned} \quad \ldots.(10.6)$$

Since $U(f)$ is an odd and $W(f)$ an even function of f, the above terms all vanish, and no effect of lower limits remains.

Similarly the effect of upper limit vanishes in our case. Therefore, it is sufficient for us to consider the effects near the branch points, which are $f=h, k, h'$ and k'. From the experience of our past study, these branch points correspond with the various phases of seismic waves; that is to say, $f=k$ and h correspond with S and P waves in the upper layer, and $f=k'$ and h' with S and P waves in the substratum medium, which in this paper, is denoted by S, P and S', P' respectively.

Thus our present task is reduced to the problem of how to expand the two functions $U(f)$ and $W(f)$ to the form of (5.1) at the points $f=h, k, h,$

and k'. We express the expansion of any function $F(f)$ at the point $f = k$ as

$$ {}_sF(f) = {}_sF_{-1}\cdot\beta^{-1} + {}_sF_0 + {}_sF_1\cdot\beta + {}_sF_2\cdot\beta^2 + \cdots\cdot $$

Similarly

$$ \begin{aligned} {}_PF(f) &= {}_PF_{-1}\cdot\alpha^{-1} + {}_PF_0 + {}_PF_1\cdot\alpha + {}_PF_2\cdot\alpha^2 + \cdots\cdot \\ {}_{S'}F(f) &= {}_{S'}F_{-1}\cdot\beta'^{-1} + {}_{S'}F_0 + {}_{S'}F_1\cdot\beta' + {}_{S'}F_2\cdot\beta'^2 + \cdots\cdot \\ {}_{P'}F(f) &= {}_{P'}F_{-1}\cdot\alpha'^{-1} + {}_{P'}F_0 + {}_{P'}F_1\cdot\alpha' + {}_{P'}F_2\cdot\alpha'^2 + \cdots\cdot \end{aligned} \tag{10.7} $$

in which

$$ \beta = \sqrt{(f^2 - k^2)}, \qquad \alpha = \sqrt{(f^2 - h^2)} $$
$$ \beta' = \sqrt{(f^2 - k'^2)}, \qquad \alpha' = \sqrt{(f^2 - h'^2)} $$

and the suffixes S, P, S' and P' mean the expansion with respect to β, α, β' and α' respectively. (This notation will be often used in future.)

However the coefficients ${}_sF_0$, ${}_sF_2$,..., ${}_PF_0$,... are not always involved in the final expression (cf. (5.2)), therefore we need not always calculate these quantities for our present purpose. It is sufficient for us to obtain only ${}_sU_{-1}$, ${}_sU_1$, ${}_PU_{-1}$ etc. for getting the asymptotic values of displacement components.

Applying the similar way of description with (10.7) to the functions M, C, D, E and F, we have

$$ \begin{cases} {}_PU(f) = if\cdot\dfrac{{}_PC_0}{{}_PM_1}\cdot\alpha^{-1} + \cdots\cdot \\ {}_PW(f) = \{{}_PC_0\cdot H + {}_PD_1 - if\cdot{}_PE_2 - if\cdot{}_PF_2\}\dfrac{1}{{}_PM_1}\cdot\alpha + \cdots\cdot \end{cases} $$

$$ \begin{cases} {}_sU(f) = \{if\cdot{}_sC_2\mathfrak{C} + if\cdot{}_sD_2\mathfrak{S} + {}_sF_1\}\dfrac{1}{{}_sM_1}\beta + \cdots\cdot \\ {}_sW(f) = \{\alpha\cdot{}_sC_2\mathfrak{S} + \alpha\cdot{}_sD_2\mathfrak{C} - if\cdot{}_sE_2 - if\cdot{}_sF_2\}\dfrac{1}{{}_sM_1}\beta + \cdots\cdot \end{cases} $$

$$ \begin{cases} {}_{P'}U(f) = \Big[\{if\cdot{}_{P'}C_1\cdot\mathfrak{C} + if\cdot{}_{P'}D_1\cdot\mathfrak{S} + \beta\cdot{}_{P'}E_1\mathfrak{S}' + \beta\cdot{}_{P'}F_1\mathfrak{C}'\} \\ \qquad - \dfrac{{}_{P'}M_1}{{}_{P'}M_0}\{if\cdot{}_{P'}C_0\cdot\mathfrak{C} + if\cdot{}_{P'}D_0\cdot\mathfrak{S} + \beta\cdot{}_{P'}E_0\cdot\mathfrak{S}' + \beta\cdot{}_{P'}F_0\cdot\mathfrak{C}'\}\Big]\dfrac{1}{{}_{P'}M_0}\cdot\alpha' + \cdots \\ {}_{P'}W(f) = \Big[\{\alpha\cdot{}_{P'}C_1\cdot\mathfrak{S} + \alpha\cdot{}_{P'}D_1\cdot\mathfrak{C} - if\cdot{}_{P'}E_1\cdot\mathfrak{C}' - if\cdot{}_{P'}F_1\mathfrak{S}'\} \\ \qquad - \dfrac{{}_{P'}M_1}{{}_{P'}M_0}\{\alpha\cdot{}_{P'}C_0\cdot\mathfrak{S} + \alpha\cdot{}_{P'}D_0\cdot\mathfrak{C} - if\cdot{}_{P'}E_0'\mathfrak{C} - if\cdot{}_{P'}F_0\mathfrak{S}'\}\Big]\dfrac{1}{{}_{P'}M_0}\cdot\alpha' + \cdots \end{cases} $$

$$\begin{cases} {}_{s'}U(f) = \Big[\{if\cdot{}_{s'}C_1\cdot\mathfrak{C}+if\cdot{}_{s'}D_1\cdot\mathfrak{S}+\beta\cdot{}_{s'}E_1\mathfrak{S}'+\beta\cdot{}_{s'}F_1\cdot\mathfrak{C}'\}. \\ \qquad -\dfrac{{}_{s'}M_1}{{}_{s'}M_0}\{if\cdot{}_{s'}C_0\cdot\mathfrak{C}+if\cdot{}_{s'}D_0\cdot\mathfrak{S}+\beta\cdot{}_{s'}E_0\cdot\mathfrak{S}'+\beta\cdot{}_{s'}F_0\cdot\mathfrak{C}'\}\Big]\dfrac{1}{{}_{s'}M_0}\cdot\beta'+\cdots \\ {}_{s'}W(f) = \Big[\{\alpha\cdot{}_{s'}C_1\cdot\mathfrak{S}+\alpha\cdot{}_{s'}D_1\cdot\mathfrak{C}-if\cdot{}_{s'}E_1\cdot\mathfrak{C}'-if\cdot{}_{s'}F_1\cdot\mathfrak{S}'\} \\ \qquad -\dfrac{{}_{s'}M_1}{{}_{s'}M_0}\{\alpha\cdot{}_{s'}C_0\cdot\mathfrak{S}+\alpha\cdot{}_{s'}D_0\mathfrak{C}-if\cdot{}_{s'}E_0\cdot\mathfrak{C}'-if\cdot{}_{s'}F_0\mathfrak{S}'\}\Big]\dfrac{1}{{}_{s'}M_0}\cdot\beta'+\cdots \end{cases} \quad \ldots\ldots(10.8)$$

§ 10.1 Preparation for the above calculations.

We have thus obtained the expansions of $U(f)$ and $W(f)$. We have to proceed and get the coefficients ${}_{r}M_0$, ${}_{r}M_1$, ${}_{s}C_2$.... which are involved in the expression (10.8). For this purpose, we will prepare some formulae which will help us in the following calculations.

$M(f)$, $C(f)$, $D(f)$.... contain N_k $(k = 1, 2, \ldots 6)$ and $P_k\,(k = 1, 2, 3, 4)$, and this N_k and P_k contain α and β, therefore we will at first express these quantities as power series with respect to α and β.

Power series with respect to α;

$$\begin{cases} N_1 = -i2ff'\alpha \\ N_2 = \{-f'^2\mathfrak{C}' + (4f^2\beta H\mathfrak{S}' - \dfrac{1}{2}f'^2H^2\mathfrak{C}')\alpha^2 + 0(\alpha^4)\} \\ N_3 = \{-f'^2\mathfrak{S}' + (4f^2\beta H\mathfrak{C}' - \dfrac{1}{2}f'^2H^2\mathfrak{S}')\alpha^2 + 0(\alpha^4)\} \\ N_4 = \{(4f^2\beta\mathfrak{S}' - f'^2H\mathfrak{C}')\alpha + 0(\alpha^3)\} \\ N_5 = \{(4f^2\beta\mathfrak{C}' - f'^2H\mathfrak{S}')\alpha + 0(\alpha^3)\} \\ N_6 = -i2ff'\beta \end{cases} \quad \ldots\ldots(10.9)$$

$$\begin{cases} P_1 = -2ff'(1 + \alpha H + 0(\alpha^2)) \\ P_2 = P_1 \\ P_3 = \left(i\dfrac{1}{\alpha}f'^2\mathfrak{C}' + i4f^2\beta\mathfrak{S}'\right) \\ P_4 = \left(i\dfrac{1}{\alpha}f'^2\mathfrak{S}' + i4f^2\beta\mathfrak{C}'\right) \end{cases}$$

Power series with respect to β;

$$\begin{cases} N_1 = -i2ff'a \\ N_2 = \{-f'^2\mathfrak{C} + (4f^2aH\mathfrak{S} - \frac{1}{2}f'^2H^2\mathfrak{C})\beta^2 + 0(\beta^4)\} \\ N_3 = \{(4f^2a\mathfrak{S} - f'^2H\mathfrak{C})\beta + 0(\beta^3)\} \\ N_4 = \{-f'^2\mathfrak{S} + (4f^2aH\mathfrak{C} - \frac{1}{2}f'^2H^2\mathfrak{S})\beta^2 + 0(\beta^4)\} \\ N_5 = \{(4f^2a\mathfrak{C} - f'^2H\mathfrak{S})\beta + 0(\beta^3)\} \\ N_6 = -i2ff'\beta \end{cases}$$

$$\begin{cases} P_1 = -2ff'\mathfrak{C}^{-1} \\ P_2 = P_1 \\ P_3 = \left\{i\frac{1}{a}f'^2 + (i4f^2H + i\frac{1}{a}f'^2\frac{1}{2}H^2)\beta^2 + 0(\beta^4)\right\} \\ P_4 = \left\{\left(i\frac{1}{4}f^2 + i\frac{1}{a}f'^2H\right)\beta + 0(\beta^3)\right\} \end{cases} \quad \ldots\ldots(10.10)$$

Introducing (10.9) and (10.10) into (9.2) and (9.3), we have expansions of M, C, D, E and F as the power series of α, β, α' and β'.

§ 10-2 Expression by the non-dimensional quantities.

In the preceding articles we have obtained the expressions of $_PU(f)$, $_PW(f)$, $_SU(f)$ and so on; however, these are not the most convenient form for our present purpose. Because, for numerical calculations, non-dimensional quantities are always preferable. Therefore, we will rewrite the above expression by means of such quantities.

We will employ the following notations which are all dimensionless.

$$\begin{aligned} &\xi \equiv fH, \\ &\alpha_1 \equiv \alpha/f, \qquad \alpha_2 \equiv \alpha'/f, \\ &\beta_1 \equiv \beta/f, \qquad \beta_2 \equiv \beta'/f. \end{aligned} \quad \ldots\ldots(10.11)$$

(Since α, β, α' and β' are even-functions of f, $\alpha_1, \beta_1, \alpha_2$ and β_2 must be assumed as odd-functions of ξ, although they do not contain ξ.)

Further, by putting the sign $\sim$ to any function F, which is a simultaneous polynomial of the n-th degree of f, we mean

$$\tilde{F}(\xi) = F(f) / f^n. \quad \ldots.(10.12)$$

Now we will come back to the first equation of (10.7). Employing the above notations

$$\begin{aligned} {}_sF(f) &= f^n \cdot {}_s\tilde{F}(\xi) \\ &= f^n \cdot \{{}_s\tilde{F}_{-1} \cdot \beta_1^{-1} + {}_s\tilde{F}_0 + {}_s\tilde{F}_1 \cdot \beta_1 + {}_s\tilde{F}_2 \cdot \beta_1^2 + \cdots\} \\ &= {}_s\tilde{F}_{-1} \cdot \beta^{-1} f^{n+1} + {}_s\tilde{F}_0 \cdot f^n + {}_s\tilde{F}_1 \cdot \beta f^{n-1} + {}_s\tilde{F}_2 \cdot \beta^2 f^{n-2} + \cdots. \end{aligned}$$

On the other hand, the same function is written in the first equation of (10.7)

$${}_sF(f) = {}_sF_{-1} \cdot \beta^{-1} + {}_sF_0 + {}_sF_1 \cdot \beta + {}_sF_2 \cdot \beta^2 + \cdots.$$

Therefore

$$\begin{aligned} {}_sF_{-1} &= {}_s\tilde{F}_{-1} \cdot f^{n+1} \\ \text{or} \qquad {}_sF_k &= {}_s\tilde{F}_k \cdot f^{n-k} \end{aligned} \quad \ldots.(10.13)$$

Similar reasoning of course holds in the expansions of ${}_PF(f)$, ${}_{S'}F(f)$ and ${}_{P'}F(f)$.

Since M is the 10-th degree and C, D, E and F the 9-th degree of f,

$$\begin{aligned} &{}_sM_k = {}_s\tilde{M}_k \cdot f^{10-k} \\ &{}_sC_k = {}_s\tilde{C}_k \cdot f^{9-k}, \quad {}_sD_k = {}_s\tilde{D}_k \cdot f^{9-k} \text{ etc.} \end{aligned} \quad \ldots..(10.14)$$

Introducing the above relations into (10.8) we have

$$\left\{ \begin{aligned} {}_PU(f)\,df &= i \frac{{}_P\tilde{C}_0}{{}_P\tilde{M}_1} \xi \frac{d\xi}{\sqrt{(\xi^2 - \omega^2/\gamma^2)}} \\ {}_PW(f)\,df &= \{\xi \cdot {}_P\tilde{C}_0 + {}_P\tilde{D}_1 - i_P\tilde{E}_2 - i_P\tilde{F}_2\} \frac{1}{{}_P\tilde{M}_1} \frac{1}{\xi} \sqrt{(\xi^2 - \omega^2/\gamma^2)} \cdot d\xi \end{aligned} \right. \quad \ldots..(10.15)$$

Now, we will simply express the above expressions as

$${}_PU(f)df = {}_PU \cdot \frac{1}{\alpha_1} d\xi, \quad {}_PW(f) = {}_PW \cdot \alpha_1 d\xi,$$

and similarly

$$\begin{aligned} {}_sU(f)df &= {}_sU \cdot \beta_1 d\xi, & {}_sW(f) &= {}_sW \cdot \beta_1 d\xi, \\ {}_{P'}U(f)df &= {}_{P'}U \cdot \alpha_2 d\xi, & {}_{P'}W(f) &= {}_{P'}W \cdot \alpha_2 d\xi, \\ {}_{S'}U(f)df &= {}_{S'}U \cdot \beta_2 d\xi, & {}_{S'}W(f) &= {}_{S'}W \cdot \beta_2 d\xi. \end{aligned} \quad \ldots..(10.16)$$

New notations $_PU$, $_SW$ etc. will be easily found by comparing the above three sets of expressions (10.8), (10.15) and (10.16).

As is explained in § 10, four phases of waves P, S and P', S' appear corresponding to the four pairs of expressions in (10.16). We will express them as $u[P]$, $w[P]$ and so on hereafter. Of course another important phase exists; that is the Rayleigh-waves (we express them by R), which results from the poles of the integrands $U(f)$ and $W(f)$. Hence we may write as

$$\begin{cases} u = u[P] + u[S] + u[P'] + u[S'] + u[R] \\ w = w[P] + w[S] + w[P'] + w[S'] + w[R] \end{cases} \qquad \ldots\ldots (10.17)$$

and the component waves $u[P]$, $w[P]$.... are easily obtained by referring to (4.1), (5.1) (10.5) (10.15) and (10.16).

$$\begin{cases} u[P] = \tilde{Q}\dfrac{i2}{\pi}\Big[x^{-\frac{1}{2}}\cdot {}_PU\cdot(2\omega/\gamma)^{-\frac{1}{2}}\sqrt{\pi}\cdot\exp\{-i(hx-\pi/4)\}\Big] \\ w[P] = \tilde{Q}\dfrac{2}{\pi}\Big[-x^{-\frac{3}{2}}\cdot {}_PW\cdot(2\omega/\gamma)^{-\frac{1}{2}}\dfrac{1}{2}\sqrt{\pi}\cdot\exp\{-i(hx-\pi/4)\}\Big] \end{cases}$$

$$\begin{cases} u[S] = \tilde{Q}\dfrac{i2}{\pi}\Big[x^{-\frac{3}{2}}\cdot {}_SU\cdot(2\omega)^{\frac{1}{2}}\cdot\dfrac{1}{2}\sqrt{\pi}\cdot\exp\{-i(kx+\pi/4)\}\Big] \\ w[S] = \tilde{Q}\dfrac{2}{\pi}\Big[-x^{-\frac{3}{2}}\cdot {}_SW\cdot(2\omega)^{\frac{1}{2}}\cdot\dfrac{1}{2}\sqrt{\pi}\cdot\exp\{-i(kx-\pi/4)\}\Big] \end{cases}$$

$$\begin{cases} u[P'] = \tilde{Q}\dfrac{i2}{\pi}\Big[x^{-\frac{3}{2}}\cdot {}_{P'}U\cdot(2\omega\sigma/\gamma)^{\frac{1}{2}}\cdot\dfrac{1}{2}\sqrt{\pi}\cdot\exp\{-i(h'x+\pi/4)\}\Big] \\ w[P'] = \tilde{Q}\dfrac{2}{\pi}\Big[-x^{-\frac{3}{2}}\cdot {}_{P'}W\cdot(2\omega\sigma/\gamma)^{\frac{1}{2}}\cdot\dfrac{1}{2}\sqrt{\pi}\cdot\exp\{-i(h'x-\pi/4)\Big] \end{cases}$$

$$\begin{cases} u[S'] = \tilde{Q}\dfrac{i2}{\pi}\Big[x^{-\frac{3}{2}}\cdot {}_{S'}U\cdot(2\omega\sigma)^{\frac{1}{2}}\cdot\dfrac{1}{2}\sqrt{\pi}\cdot\exp\{-i(k'x+\pi/4)\}\Big] \\ w[S'] = \tilde{Q}\dfrac{2}{\pi}\Big[-x^{-\frac{3}{2}}\cdot {}_{S'}W\cdot(2\omega\sigma)^{\frac{1}{2}}\cdot\dfrac{1}{2}\sqrt{\pi}\cdot\exp\{i(k'x-\pi/4)\}\Big] \end{cases}$$

$$\ldots\ldots (10.18)$$

in which $\tilde{Q} = Q/H$

Of these expressions $u[S]$ and $w[S]$ always vanish, which fact is anticipated from the nature of waves and their paths, and may be verified by slight calculations.

$u[R]$ and $w[R]$ can be calculated in a similar manner from (10.3)

$$\begin{cases} u[R] = 2\pi i\cdot\sum_n Q\frac{1}{\pi}\frac{1}{M'(f)}[if\{C\mathfrak{C} + D\mathfrak{S}\} + \beta\{E\mathfrak{S} + F\mathfrak{C}\}]_{f=K_n} \\ w[R] = 2\pi i\cdot\sum_n Q\frac{1}{\pi}\frac{1}{M'(f)}[\alpha\{C\mathfrak{S} + D\mathfrak{C}\} - if\{E\mathfrak{C} + F\mathfrak{S}\}]_{f=K_n} \end{cases} \quad \ldots(10.19)$$

where K_n implies the n-th root of $M(f) = 0$ and the notation $\sum$ implies the summation with respect to every branch of dispersion curve of Rayleigh-waves.

Since $M(f) = f^{10}\widetilde{M}(\xi)$ and $M(K_n) = 0$,

$$\left[\frac{d}{df}M(f)\right]_{f=K_n} = f^9\left\{10\,\widetilde{M}(\xi) + \xi\frac{d}{d\xi}\widetilde{M}(\xi)\right\} = K_n^9\left[\xi\frac{d}{d\xi}\widetilde{M}(\xi)\right]_{\xi=\Xi_n}. \quad \ldots\ldots(10.20)$$

in which $\Xi_n = K_n H$

Therefore (10.19) may be written in the following form:

$$\begin{cases} u[R] = 2\widetilde{Q}\sum\left\{1\Big/\frac{d}{d\xi}\widetilde{M}(\xi)\right\}\cdot\left[-\{\widetilde{C}\mathfrak{C} + \widetilde{D}\mathfrak{S}\} + i\beta_1\{\widetilde{E}\mathfrak{S} + \widetilde{F}\mathfrak{C}\}\right]_{\xi=\Xi_n} \\ w[R] = 2\widetilde{Q}\sum\left\{1\Big/\frac{d}{d\xi}\widetilde{M}(\xi)\right\}\cdot\left[i\alpha_1\{\widetilde{C}\mathfrak{S} + \widetilde{D}\mathfrak{C}\} + \{\widetilde{E}\mathfrak{C} + \widetilde{F}\mathfrak{S}\}\right]_{\xi=\Xi_n} \end{cases} \quad \ldots\ldots(10.21)$$

Now, in our study, we will take only two branches $\xi = \Xi_0$ and Ξ_1; that is to say the ordinary Rayleigh-waves and M_2-waves.[8]

In the next article we will show the numerical calculations of the above expressions corresponding to several values of p, or period, assuming some appropriate values for the physical constants and the structure of media.

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8) K. SEZAWA and K. KANAI, *Proc. Imp. Acad.*, **11** (1935), 13, 96; *Bull. Earthq. Res. Inst.*, **13** (1935), 237, 471. **18** (1940), 1.
K. SEZAWA, *Bull. Earthq. Res. Inst.*, **16** (1938), 1.
K. KANAI, *Bull. Earthq. Res. Inst.*, **26** (1948), 57.

Errata of 1st Paper.

Page	Line			
2	4	$\int_{ax}^{bx} F\left(\frac{z}{x}\right)\frac{1}{x}dx$	Read	$\int_{ax}^{bx} F\left(\frac{z}{x}\right)\cos z\cdot\frac{1}{x}dz$
,,	6	$F''(b)$	,,	$-F''(b)$
,,	,,	$[F'''(b)$,, $-F'''(a)$,, $]$	,,	$[-F'''(b)$,, $+F'''(a)$,, $]$
,,	9	(1.2)	,,	(2.2)
,,	14	$F_{-1}(a)\frac{1}{2}(\xi-a)^{\frac{1}{2}}$	,,	$F_{-1}(a)(\xi-a)^{-\frac{1}{2}}$
3	10	$x^{-\frac{1}{2}}$	,,	$x^{-\frac{3}{2}}$
4	21	(4c)(+	,,	(4c)+
,,	28	$\alpha=\sqrt{\xi^2-k^2}$	,,	$\alpha=\sqrt{\xi^2-h^2}$

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3. 層のある媒質を傳はる表面波の研究 (2)

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第一報において我々は $\int_a^b F(\xi)\frac{\cos}{\sin}\xi x\,d\xi$ の形の定積分を x の漸近級数に展開する一つの方法を示したのであつた。その結果は本文に再錄してある (2.1), (2.2); (3.5), (3.6); (3.7), (3.8); (4.1) である。

この方法は，新しいやり方であるので，一二の例題を扱ふはづであつたが，前にはそのよゆうがなかつたので，本報告においてまずそれを行つた。

以上が第一部であり，第二部として，半無限彈性体の上に一様な表面層がのつてゐる場合，その層内に震源があつた時に，十分遠くはなれた所ではどのやうな波が傳はつてくるか，といふ問題を理論的にといた。途中の計算はやや複雜であるが，上に求めた積分の計算法によつて，比較的簡便に結果の式をもとめる事ができる。

實際の場合に適合するやうな數値を入れての計算は次の報告で發表する豫定である。