4. *On the M_2-waves (Sezawa-waves).*

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1. Introduction.

Many years ago we found[1] that there are generally two kinds of different wave types of the dispersive Rayleigh-waves transmitted along the surface of a stratified body. But, our previous studies being restricted mainly to the fundamental nature of these waves, no detailed condition of the range of possible existence of these waves has yet been determined, owing to the extremely complicated forms of the mathematical solutions. In the present investigations, we shall extend our studies to the condition of the range of possible existence of dispersive Rayleigh-waves. Next, as we are now in a position to confirm the existence of M_2-waves in actual seismic waves, we shall deal with the general natures of the M_2-waves to a greater extent.

2. The dispersion curves of the M_2-waves.

The equation of dispersion curves of the Rayleigh-waves transmitted through a stratified body originally obtained by us is

$$\frac{4r's'}{f^2}\left(2-\frac{k'^2}{f^2}\right)\eta-\frac{r's'}{f^2}\left\{4\vartheta+\left(2-\frac{k'^2}{f^2}\right)^2\zeta\right\}\cosh r'H\cosh s'H$$

$$+\frac{r'}{f}\varphi\left\{\frac{s}{f}\left(2-\frac{k'^2}{f^2}\right)^2-4\frac{rs'^2}{f^3}\right\}\cosh r'H\sinh s'H$$

$$+\frac{s'}{f}\varphi\left\{\frac{r}{f}\left(2-\frac{k'^2}{f^2}\right)^2-\frac{4sr'^2}{f^3}\right\}\sinh r'H\cosh s'H$$

$$+\left\{\left(2-\frac{k'^2}{f^2}\right)^2\vartheta+\frac{4r'^2s'^2}{f^4}\zeta\right\}\sinh r'H\sinh s'H=0, \quad \ldots\ldots(1)$$

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1) K. SEZAWA and K. KANAI, "Discontinuity in the Dispersion Curves of Rayleigh Waves", *Bull. Earthq. Res. Inst.*, **13** (1935), 238.

K. SEZAWA and K. KANAI, "The M_2-Seismic Waves", *Bull. Earthq. Res. Inst.*, **13** (1935), 471.

where

$$\left.\begin{aligned}
&\varphi = \frac{\mu' k^2 k'^2}{\mu f^4},\ \zeta = \frac{4rs}{f^2}\left(\frac{\mu'}{\mu}-1\right)^2-\alpha^2,\ \eta = \frac{2rs}{f^2}\left(\frac{\mu'}{\mu}-1\right)\beta-\alpha\gamma,\\
&\vartheta = \frac{rs}{f^2}\beta^2-\gamma^2,\ \alpha = \frac{2\mu'}{\mu}-\left(2-\frac{k^2}{f^2}\right),\ \beta = \frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-2,\\
&\gamma = \frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-\left(2-\frac{k^2}{f^2}\right),\\
&r^2 = f^2-h^2,\ s^2 = f^2-k^2,\ r'^2 = f^2-h'^2,\ s'^2 = f^2-k'^2,\\
&h^2 = \rho p^2/(\lambda+2\mu),\ k^2 = \rho p^2/\mu,\ h'^2 = \rho' p^2/(\lambda'+2\mu'),\ k'^2 = \rho' p^2/\mu',
\end{aligned}\right\}\quad (2)$$

in which H is the thickness of the stratum, $2\pi/f$ the wave length, ρ', λ', μ'; ρ, λ, μ the densities and elastic constants of the stratum and the subjacent medium respectively.

In our previous studies, the dispersion curves corresponding to the M_2-waves of very small wave length and the vicinity of critical condition were left untouched, owing to the extremely complex numerical calculations.

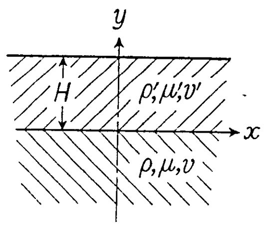

Fig. 1.

We shall now examine more closely the cases in which the ratios of the rigidity of the subjacent medium to that of the stratum are 2, 5, 20, ∞, the densities of both media being the same and Poisson's condition of elastic

Table I. $\mu/\mu'=5,\ \rho/\rho'=\lambda/\mu=\lambda'/\mu'=1.$

$p/f.\sqrt{\rho'/\mu'}$		1.0	1.2247	1.414	1.4491	1.5497
L/H	M_2	0	0.9079	—	1.358	1.782
	M	—	2.904	3.85	4.050	4.812

Table II. $\mu/\mu'=20,\ \rho/\rho'=\lambda/\mu=\lambda'/\mu'=1.$

$p/f.\sqrt{\rho'/\mu'}$		1.0	1.0954	1.500	1.6432
L/H	M_2	0	0.5981	1.387	1.929
	M	—	1.818	3.484	3.9077

Table III. $\mu/\mu' = \infty,\ \rho/\rho' = \lambda/\mu = \lambda'/\mu' = 1.$

$p/f \cdot \sqrt{\rho'/\mu'}$		1.0	1.0954	1.4142	1.6432	1.8974
L/H	M_2	0	0.609	1.20	1.861	—
	M	—	2.098	3.120	3.730	4.37

constants being satisfied, and also the case in which the rigidity ratio is $\mu/\mu' = 1.65$ and the density ratio $\rho/\rho' = 1.14$. The results are given in Tables I, II, III (other values taken from previous papers[2]) and plotted in Figs. 2~6.

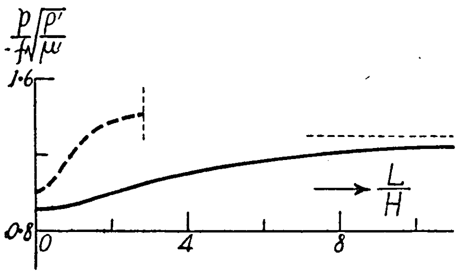

Fig. 2. The dispersion curves of a case in which $\lambda/\mu = \lambda'/\mu' = 1$, $\rho/\rho' = 1$, $\mu/\mu' = 2$.

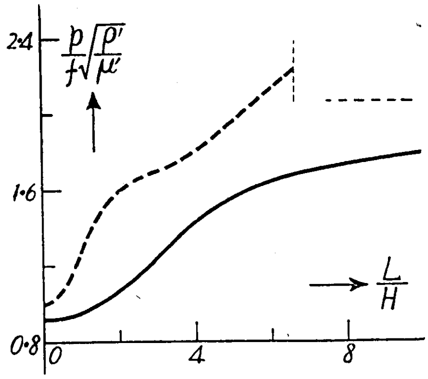

Fig. 3. The dispersion curves of a case in which $\lambda/\mu = \lambda'/\mu' = 1$, $\rho/\rho' = 1$, $\mu/\mu' = 5$.

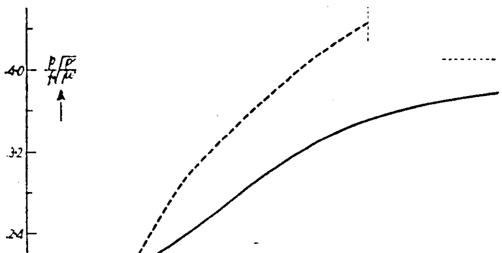

Fig. 4. The dispersion curves of a case in which $\lambda/\mu = \lambda'/\mu' = 1$, $\rho/\rho' = 1$, $\mu/\mu' = 20$.

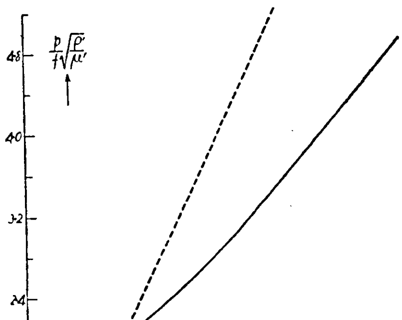

Fig. 5. The dispersion curves of a case in which $\lambda/\mu = \lambda'/\mu' = 1$, $\rho/\rho' = 1$, $\mu/\mu' = \infty$.

2) *Bull. Earthq. Res. Inst.*, **3** (1927), 1; **13** (1935), 238; 239; 472.

It will be seen from these figures that M_2-waves exist within a narrow range, that is, a range of relatively small wave length, while M-waves exist in throughout whole range of wave length. The velocity of transmission of the M_2-waves at critical condition reaches to the velocity of transverse waves of subjacent medium and those at $L/H\to 0$ tends to the velocity of the Stoneley-waves in case the Stoneley-wave can exist and to the transverse waves of stratum in case it cannot.

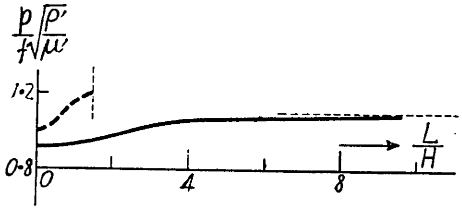

Fig. 6. The dispersion curves of a case in which $\lambda/\mu=\lambda'/\mu'=1$, $\rho/\rho'=1.14$, $\mu/\mu'=1.65$.

3. The determination of the range of possible existence of dispersive Rayleigh-waves.

Since the decrease in amplitudes in the subjacent medium as to the distance from the boundary surface is given by e^{-ry}, e^{-sy}, in case either of f^2-h^2, f^2-k^2 is negative, the condition of the surface waves fails. Hence, the critical condition of the possible existence of Rayleigh-waves is determined by the condition that either of f^2-h^2 and f^2-k^2 changes its sign from positive to negative. From the nature of things $h^2<k^2$, the condition under consideration is given by $f^2=k^2$. Then, the velocity equation (1) reduces itself to

$$\frac{4r's'}{f^2}\left(2-\frac{k'^2}{f^2}\right)\eta-\frac{r's'}{f^2}\left\{4\vartheta+\left(2-\frac{k'^2}{f^2}\right)^2\zeta\right\}\cosh r'H\cosh s'H$$
$$-\frac{4rr's'^2}{f^4}\varphi\cosh r'H\sinh s'H+\frac{r's'}{f^2}\varphi\left(2-\frac{k'^2}{f^2}\right)^2\sinh r'H\cosh s'H$$
$$+\left\{\frac{4r'^2s'^2}{f^4}\zeta+\left(2-\frac{k'^2}{f^2}\right)^2\vartheta\right\}\sinh r'H\sinh s'H=0, \quad \dots\dots\dots(3)$$

where

$$\left.\begin{aligned}
&\varphi=\frac{\mu'k'^2}{\mu f^2},\ \zeta=-\alpha^2,\ \eta=-\alpha\gamma,\ \vartheta=-\gamma^2,\\
&\alpha=\frac{2\mu'}{\mu}-1,\ \beta=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-2,\ \gamma=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-1,\\
&\frac{k'^2}{f^2}=\left(\frac{v}{v'}\right)^2,\ \frac{r^2}{f^2}=1-\frac{1}{\lambda/\mu+2},\ \frac{r'^2}{f^2}=1-\frac{1}{\lambda'/\mu'+2}\left(\frac{v}{v'}\right)^2,\\
&\frac{s'^2}{f^2}=1-\left(\frac{v}{v'}\right)^2.
\end{aligned}\right\}\ (4)$$

In special case $(v/v')^2=1$, equations (3) and (4) become

$$\frac{4r'}{f}\eta-\frac{r'}{f}(4\vartheta+\zeta)\cosh r'H+\frac{r}{f}\varphi\sinh r'H+(fH)\vartheta\sinh r'H=0, \quad (5)$$

where

$$\left.\begin{aligned}&\varphi=\frac{\mu'}{\mu},\ \zeta=-\left(\frac{2\mu'}{\mu}-1\right)^2,\ \eta=-\left(\frac{2\mu'}{\mu}-1\right)\left(\frac{\mu'}{\mu}-1\right),\\ &\vartheta=-\left(\frac{\mu'}{\mu}-1\right)^2,\ \frac{r^2}{f^2}=1-\frac{1}{\lambda/\mu+2},\ \frac{r'^2}{f^2}=1-\frac{1}{\lambda'/\mu'+2}.\end{aligned}\right\}\ldots.(6)$$

Using equations (3)~(6), the special cases for Poisson's ratio ($\sigma=1/4$ and $\sigma'=1/4$, that is to say, $\lambda/\mu=\lambda'/\mu'=1$) are on calculation. The results of calculation for the three cases, $(v/v')^2=1/2$, 2, 1 are shown in Figs. 7, 8, 9. In the double hatched area of these figures M- and M_2-waves can exist, while M-waves can exist in the hatched area only and in the blank area no Rayleigh type waves can exist. From Figs. 8 and 9, it will be seen that M_2-waves can exist only within a narrow range, that is, a range of relatively small wave length, in $v/v'\geq 1$, while it is possible for M-waves to exist within a range from zero to infinity of wave length under the same condition.

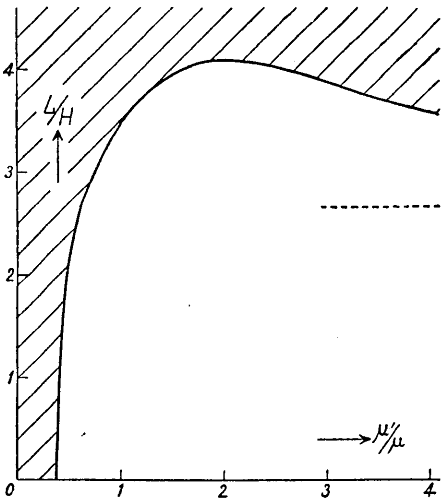

Fig. 7. The range of possible existence of M_2- and M-waves for the case $(v/v')^2=1/2$. In the hatched area M-waves only can exist; in the blank area no surface waves can exist.

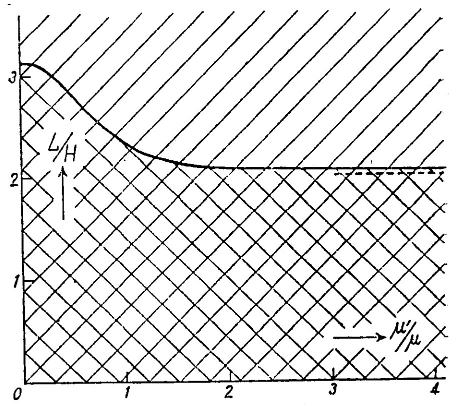

Fig. 8. The range of possible existence of M_2- and M-waves for the case $(v/v')^2=2$. In the double hatched area M_2- and M-waves can exist; in the hatched area M-waves only can exist.

On the other hand, Fig. 7 shows that, under the condition of $v/v' < 1$, no M_2-waves can exist in any wave length, while M-waves exist in relatively large wave length unless the condition of possible existence of the Stoneley-waves be satisfied, that is to say, $\mu'/\mu > 0.39$, and exist for any wave length within the range $\mu'/\mu \leq 0.39$ where Stoneley-waves can exist, under the condition of $v/v' < 1$.[3]

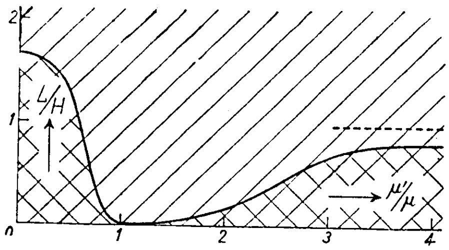

Fig. 9. The range of possible existence of M_2- and M-waves for the case $(v/v')^2=1$. In the double hatched area M_2- and M-waves can exist; in the hatched area M-waves only can exist.

4. Dispersion curves of higher order of M- and M_2-waves.

In order to make the nature of M_2-waves clear, we determined the dispersion curves of M- and M_2-waves of higher order in the bodies whose densities and elastic constants are specified by $\lambda/\mu=\lambda'/\mu'=1$, $\rho/\rho'=1$, $\mu/\mu'=8$. The results are shown in Table IV and plotted in Fig. 10. The types of orbital motions of the free surface corresponding to points A, B, C, D of the four dispersion curves are shown in Figs. 11~14 together with the

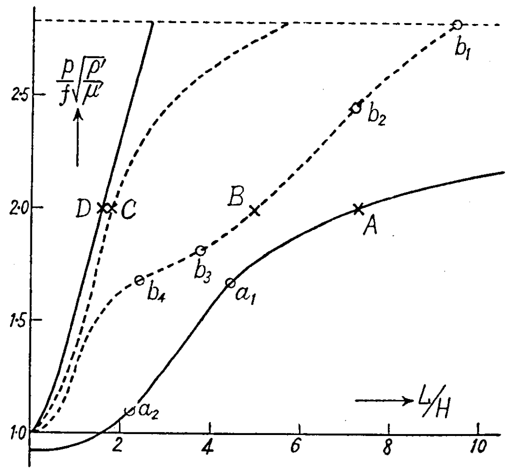

Fig. 10. The dispersion curves of a case in which $\lambda/\mu=\lambda'/\mu'=1$, $\rho/\rho'=1$, $\mu/\mu'=8$. The curves in broken line and full line represent M_2- and M-waves respectively.

3) The results obtained approximately by J. G. Scholte somewhat differ from the precise results of the present paper.

J. G. SCHOLTE, "The Range of Existence of Rayleigh and Stoneley Waves", *M. N. R. A. S., Geophys. Suppl.*, **5** (1947), 120–126.

Added note:

As we mentioned in the previous paper,[4] our numerical calculations concerned with Stoneley waves were carried out by means of the equations which were led by introducing $f^2=k^2$ or $f^2=k'^2$ into (4). The form of the equations used but not written in the paper is reduced to that[5] of (1) and (2) of Scholte's paper.

4) K. SEZAWA and K. KANAI, "The Range of Possible Existence of Stoneley waves, and Some Related Problems", *Bull. Earthq. Res. Inst.*, **17** (1939), 3.

5) *loc. cit.*, 3), 120.

Table IV. $\mu/\mu'=8$, $\rho/\rho'=\lambda/\mu=\lambda'/\mu'=1$.

$p/f.\sqrt{\rho'/\mu'}$		1.0	1.1	1.32	1.44	1.56	1.68	1.8166	2.000	2.4495
L/H	M_2	0	0.62832	1.0665	1.2913	1.617	2.436	3.763	4.942	7.154
	M	—	2.2258	3.0998	3.5508	—	4.434	5.427	7.270	24.05

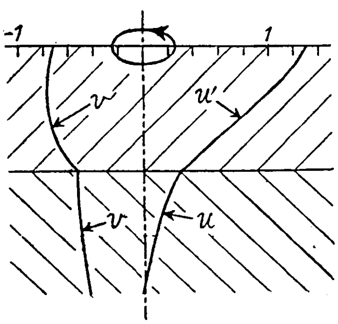

Fig. 11. The distributions of displacements and orbital motion of case corresponding to point *A* in Fig. 10.

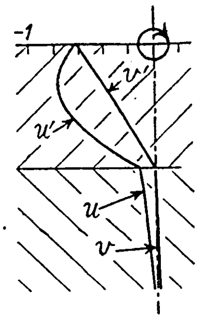

Fig. 12. The distributions of displacements and orbital motion of case corresponding to point *B* in Fig. 10.

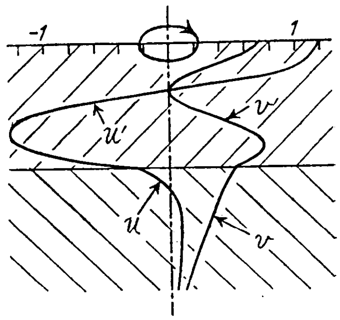

Fig. 13. The distributions of displacements and orbital motion of case corresponding to point *C* in Fig. 10.

distribution of displacements at different depths.

It will be seen that there are two kinds of orbital motion, one in the same sense as that of the gravitational waves and the other in the same sense as that of the usual Rayleigh-waves transmitted on the surface of a semi-infinite body. Then, it will be reasonable to consider that the dispersion curves *C* and *D* correspond to the first order of dispersion curves of M_2-waves and M-waves respectively and there will be an infinite number of dispersion curves which correspond to higher order of the M_2-waves and the M-waves.

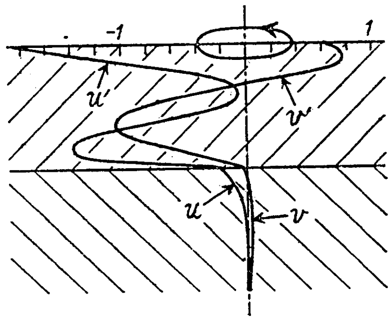

Fig. 14. The distributions of displacements and orbital motion of case corresponding to point *D* in Fig. 10.

5. Distribution of displacements at different depths.

We also studied closely the distribution of displacements at different depths. Horizontal and vertical components of displacement within the stratum and in the subjacent medium is reduced to

$$\left.\begin{aligned} u'\frac{\Phi f}{A} &= \frac{s'}{f}\Big(C\cosh r'y + D\sinh r'y + \frac{s'}{f}E\sinh s'y + F\cosh s'y\Big), \\ v'\frac{\Phi f}{A} &= i\Big(\frac{r's'}{f^2}C\sinh r'y + \frac{r's'}{f^2}D\cosh r'y + \frac{s'}{f}E\cosh s'y + F\sinh s'y\Big), \end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned} u\frac{\Phi f}{A} &= \frac{\mu' s' k'^2}{\mu f^3}\Big(\frac{s}{f}Be^{sy} - \Phi e^{ry}\Big), \\ v\frac{\Phi f}{A} &= \frac{i\mu' s' k'^2}{\mu f^3}(Be^{sy} - \Phi e^{ry}), \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

where

$$\left.\begin{aligned} C &= -\frac{2sr'}{f^2}\varphi\cosh s'H + \frac{2r's'}{f^2}\zeta\sinh s'H - \Big(2-\frac{k'^2}{f^2}\Big)\eta\sinh r'H, \\ D &= -2\vartheta\cosh s'H - \frac{2rs'}{f^2}\varphi\sinh s'H + \Big(2-\frac{k'^2}{f^2}\Big)\eta\cosh r'H, \\ E &= \frac{2r'}{f}\eta\cosh s'H - \frac{r'}{f}\Big(2-\frac{k'^2}{f^2}\Big)\zeta\cosh r'H + \frac{r}{f}\Big(2-\frac{k'^2}{f^2}\Big)\varphi\sinh r'H, \\ F &= -\frac{2r's'}{f^2}\eta\sinh s'H + \frac{sr'}{f^2}\Big(2-\frac{k'^2}{f^2}\Big)\varphi\cosh r'H + \Big(2-\frac{k'^2}{f^2}\Big)\vartheta\sinh r'H, \\ B &= -\frac{2r'}{f}\gamma\cosh s'H + \frac{4rr's'}{f^3}\Big(\frac{\mu'}{\mu}-1\Big)\sinh s'H + \frac{r'}{f}\Big(2-\frac{k'^2}{f^2}\Big)\alpha\cosh r'H \\ &\quad -\frac{r}{f}\Big(2-\frac{k'^2}{f^2}\Big)\beta\sinh r'H, \\ \Phi &= \frac{2sr'}{f^2}\beta\cosh s'H - \frac{2r's'}{f^2}\alpha\sinh s'H - \frac{2sr'}{f^2}\Big(2-\frac{k'^2}{f^2}\Big)\Big(\frac{\mu'}{\mu}-1\Big)\cosh r'H \\ &\quad +\Big(2-\frac{k'^2}{f^2}\Big)\gamma\sinh r'H. \end{aligned}\right\} \quad (9)$$

The results of calculations of the cases corresponding to points b_1, b_2, b_3, b_4, a_1, a_2 indicated in Fig. 10, are shown in Figs. 15~20. These figures show that the distribution of displacements of waves corresponding to M_2-waves is somewhat different from that of M-waves. It appears that there is the maximum of horizontal displacements having no horizontal nodal plane in the stratum for M_2-waves excepting relatively small wave length, while it

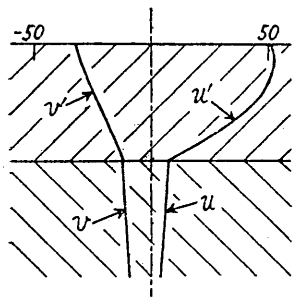

Fig. 15. The distributions of displacements of case corresponding to point b_1 in Fig. 10.

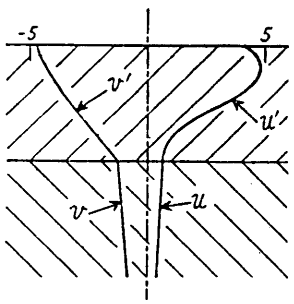

Fig. 16. The distributions of displacements of case corresponding to point b_2 in Fig. 10.

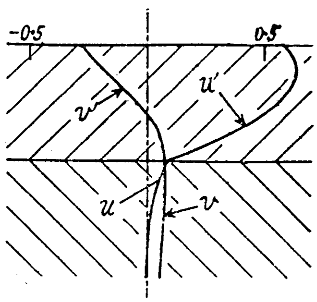

Fig. 17. The distributions of displacements of case corresponding to point b_3 in Fig. 10.

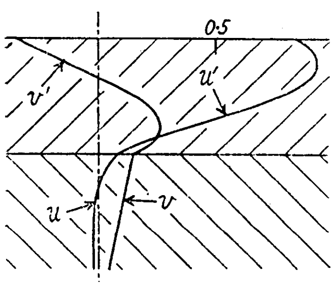

Fig. 18. The distributions of displacements of case corresponding to point b_4 in Fig. 10.

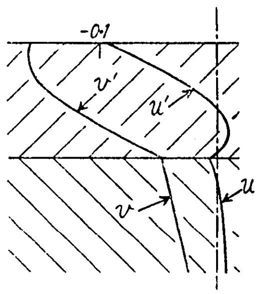

Fig. 19. The distributions of displacements of case corresponding to point a_1 in Fig. 10.

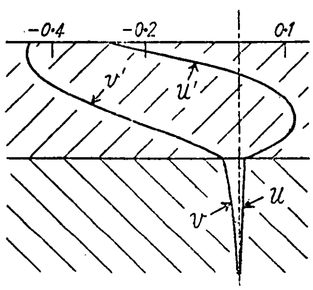

Fig. 20. The distributions of displacements of case corresponding to point a_2 in Fig. 10.

seems no particular feature in the distribution of displacements of M-waves. It would be useful to identify the M_2-waves with the peculiar waves in the seismic records at different depths of the earth.

6. Concluding remarks.

From mathematical investigations it is ascertained that the two kinds of dispersive Rayleigh-waves, which we have previously found, fairly differ in their nature. It has been ascertained fairly that M_2-waves can exist in a narrow range, that is the range of relatively small wave length, in $v/v' \geq 1$, and impossible to exist in any wave length in $v/v' < 1$. While, M-waves exist in any wave length in $v/v' \geq 1$, and in the range $v/v' < 1$ for $\mu'/\mu \leq 0.39$, and are possible to exist for relatively large wave length in the range $\mu'/\mu > 0.39$ under the condition of $v/v' < 1$.

The waves corresponding to M_2-waves including higher order are transmitted with an orbit of the same sense as that of gravitational waves, while those corresponding to M-waves including higher order do so with an orbit of the same sense as that of usual Rayleigh-waves.

It should be born in mind that there is the maximum of horizontal component of displacements having no horizontal nodal plane in the stratum for M_2-waves excepting relatively small wave length, while there are no peculiar feature for M-waves. As a matter of fact, by utilizing this feature, we should be able to confirm to a greater extent the existence of the M_2-waves in actual seismic disturbance.

These results were found from complex numerical calculations, and not immediately from the mathematical expressions.

In concluding this paper I wish to express my thanks to Miss S. Yoshizawa for her kind assistance in preparing this paper.

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4. M_2 波 (妹澤波) について

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一つの表面層がある場合の2種類の表面波，即ち分散性レーレー波及び M_2 波，の性質を数理的にしらべた。$\lambda/\mu=\lambda'/\mu'=1$ の場合，v, v' を夫々下層，表面層の横波の速度とすると，M_2 波は $v/v'\geq 1$ の場合には比較的に短波長の範圍に存在し得るが，$v/v'<1$ の場合には實在し得ない。M 波の方は $v/v'\geq 1$ の場合には，あらゆる波長にわたつて存在し得，$v/v'<1$ の場合には $\mu'/\mu\leq 0.39$ の範圍ではあらゆる波長にわたつて存在し得るが，$\mu'/\mu>0.39$ の範圍では比較的に長波長のところにのみ存在することができる。

M_2 波と M 波について，地表面よりの深さと振幅との關係をしらべたところ，M_2 波の水平成分の分布に限つて，多くの場合表面層中に極大ができることがわかつた。この性質は地表面における波點の運動軌跡が普通のレーレー波と逆である性質と相まつて，實際の地震記象を解析して M_2 波の實在性を確める上に相當に役立つことであろう。

尙，M_2 波は，M 波の中の表面層中に節のできる，いわゆる，M 波の高次の波とは異ることを確めておいた。