5. *On the Group Velocity of Dispersive Surface Waves.*

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1. Introduction.

The problem concerning group velocity of seismic waves already was paid attentions to by Nagaoka,[1] Gutenberg[2] about a quarter of a century ago. Several years ago Stoneley[3] showed that, as to be expected from the approximate theory of waves, the Love-waves and the dispersive Rayleigh-waves of minimum group velocity tend to predominate in seismograms.

From our recent investigations[4] by means of analysing the seismic records of the two earthquakes of shallow origin, we found that the apparent transmission velocity of the peculiar phases agrees closely with the minimum group velocity of the M_2-waves.

It seems worth while to investigate further the problem concerning the group velocity of dispersive surface waves because we are now in a position to ascertain the existence of the M_2-waves in actual seismic waves. In the present paper we shall deal with the problem of group velocity of Love-waves, dispersive Rayleigh-waves and M_2-waves mathematically.

2. Group velocity of the Love-waves.

In the case of a one-stratified body, the velocity[5] of Love-waves is expressed by

$$\tan\left(\frac{2\pi}{\lambda}\sqrt{c^2-1}\right)=\frac{\mu}{\mu'}\sqrt{\frac{1-\frac{\rho\mu'}{\rho'\mu}c^2}{c^2-1}}, \qquad (1)$$

1) H. NAGAOKA, "Group Velocity in Distant Earthquakes", *Proc. Phys.-Math. Soc., Japan*, **3** (1906), 44.

2) B. GUTENBERG, "Über Gruppengeschivindigkeit bei Erdbebenwellen", *Phys. Z. S.*, **27** (1926), 114.

3) R. STONELEY, "Surface-waves associated with 20° Discontinuity", *M.N.R.A.S., Geophys. Suppl.*, **4** (1937), 39.

4) K. KANAI, "On the Existence of the M_2-waves in Actual Seismic Disturbances", *Bull. Earthq. Res. Inst.*, **26** (1948), 57.

5) A. E. H. LOVE, "Some Problem of Geodynamics", (Cambridge, 1911), 162.

where $\lambda = L/H$, $c^2 = \rho' p^2/\mu' f^2$, $2\pi/f = L =$ wave length, $2\pi/p = T =$ period, $H =$ thickness of the stratum, ρ, ρ'; μ, μ'; v, v' densities, rigidities and velocities of transverse waves of the subjacent medium and the stratum respectively.

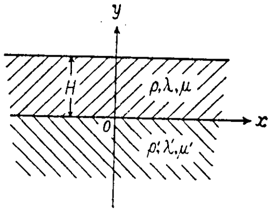

Fig. 1.

The equation of group velocity of waves is written in the form

$$U = c - \lambda \frac{\partial c}{\partial \lambda}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where U is the ratio of the group velocity of waves to the velocity of transverse waves of the stratum (v'). From (1) and (2), we get

$$U = c - \frac{2(c^2-1)}{c\Phi}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where

$$\Phi = \frac{4\pi}{\lambda}\sqrt{c^2-1}\left(1-\frac{\rho\mu'}{\rho'\mu}c^2\right)+\left(1-\frac{\rho\mu'}{\rho'\mu}\right)\sin\frac{4\pi}{\lambda}\sqrt{c^2-1}. \qquad \ldots(4)$$

By means of equations (1) and (3), we determined the dispersion curves of Love-waves and calculated the group velocity of those with the following ratios of densities and rigidities.

(i) $\dfrac{\rho}{\rho'} = 1$; $\dfrac{\mu}{\mu'}\left\{=\left(\dfrac{v}{v'}\right)^2\right\} = \dfrac{5}{4},\ 2,\ \dfrac{10}{3},\ 6,\ 10,\ 25,$

(ii) $\dfrac{\rho}{\rho'} = 1.1$; $\dfrac{\mu}{\mu'} = 2.2,\ 4.4,\ 11,\ 27.5\left\{\text{i.e.}\left(\dfrac{v}{v'}\right)^2 = 2,\ 4,\ 10,\ 25\right\},$

(iii) $\dfrac{\rho}{\rho'} = 2$; $\dfrac{\mu}{\mu'} = 4,\ 10,\ 25,\ 50\left\{\text{i.e.}\left(\dfrac{v}{v'}\right)^2 = 2,\ 5,\ 12.5,\ 25\right\}.$

The results of calculations of the fourteen cases given above are plotted in Figs. 2~15. In these figures, phase velocities and group velocities are indicated respectively by full lines and broken ones.

Next, we determined graphically the conditions of the minimum group velocities by means of Figs. 2~15. The relations of the minimum group velocity and phase velocity as well as the period of waves corresponding with the minimum group velocity to the velocity ratio of the two mediums, under the condition in which the density ratio of the two mediums is kept

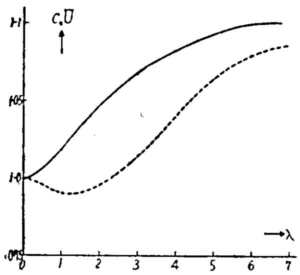

Fig. 2. $\rho/\rho'=1$, $\mu/\mu'=1.25$.

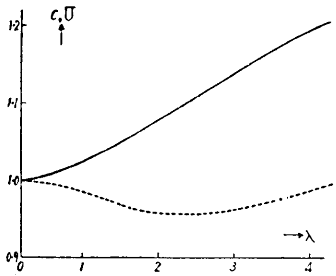

Fig. 3. $\rho/\rho'=1$, $\mu/\mu'=2$.

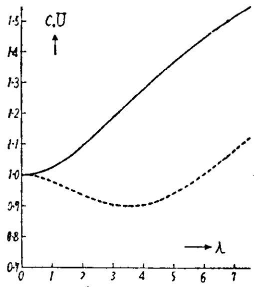

Fig. 4. $\rho/\rho'=1$, $\mu/\mu'=10/3$.

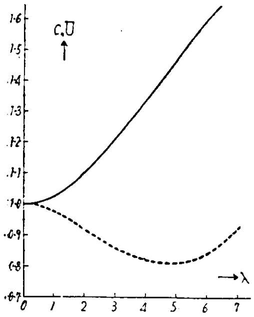

Fig. 5. $\rho/\rho'=1$, $\mu/\mu'=6$.

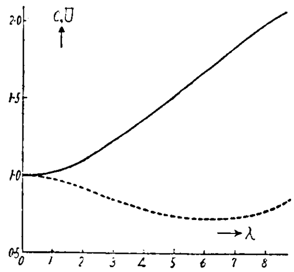

Fig. 6. $\rho/\rho'=1$, $\mu/\mu'=10$.

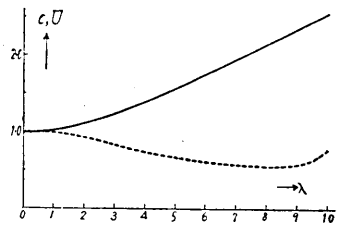

Fig. 7. $\rho/\rho'=1$, $\mu/\mu'=25$

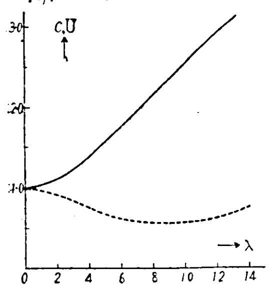

Fig. 8. $\rho/\rho'=1.1$, $\mu/\mu'=2.2$.

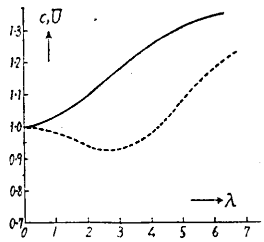

Fig. 9. $\rho/\rho'=1.1$, $\mu/\mu'=4.4$.

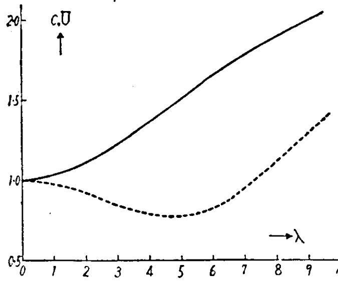

Fig. 10. $\rho/\rho'=1.1$, $\mu/\mu'=11$.

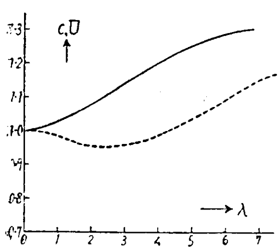

Fig. 11. $\rho/\rho'=1.1$, $\mu/\mu'=27.5$.

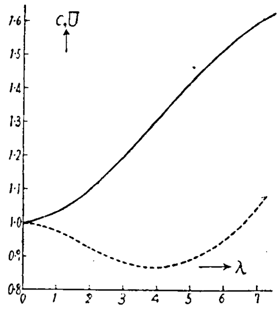

Fig. 12. $\rho/\rho'=2$, $\mu/\mu'=4$.

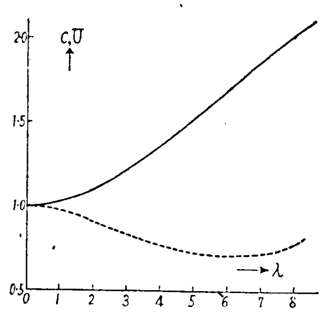

Fig. 13. $\rho/\rho'=2$, $\mu/\mu'=10$.

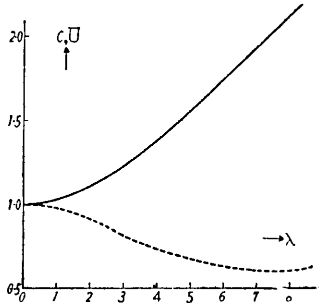

Fig. 14. $\rho/\rho'=2$, $\mu/\mu'=25$.

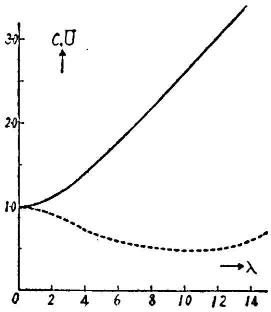

Fig. 15. $\rho/\rho'=2$, $\mu/\mu'=50$.

constant in three cases, namely, $\rho/\rho'=1$, 1.1, 2, are shown in Tables I, II, III and plotted in Figs. 16, 17, 18.

Table I. Minimum group velocity of Love-waves. $\rho/\rho'=1$.

$(v/v')^2$	5/4	2	10/3	6	10	25
U	0.990	0.957	0.901	0.814	0.727	0.566
Tv'/H	1.20	2.18	2.84	3.31	3.61	3.79
c	1.025	1.10	1.23	1.435	1.69	2.24

Table II. Minimum group velocity of Love-waves. $\rho/\rho'=1.1$.

$(v/v')^2$	2	4	10	25
U	0.953	0.868	0.715	0.552
Tv'/H	2.22	2.95	3.57	3.82
c	1.11	1.30	1.70	2.35

Table III. Minimum group velocity of Love-waves. $\rho/\rho'=2$.

$(v/v')^2$	2	5	5/4	25
U	0.927	0.776	0.695	0.487
Tv'/H	2.33	3.30	3.69	3.83
c	1.15	1.50	2.04	2.60

It will be seen from Fig. 16 that the value of minimum group velocity of Love-waves decreases with the increase in the velocity ratio of the

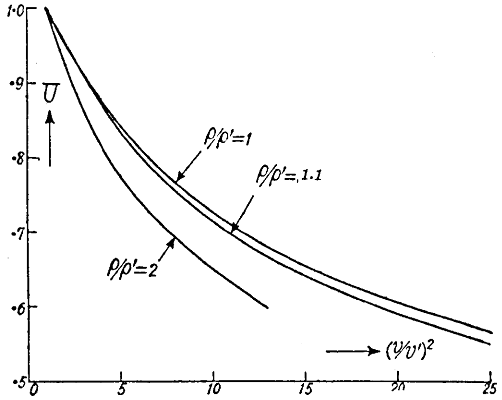

Fig. 16. Love-waves. U is in the ratio of minimum group velocity to v'.

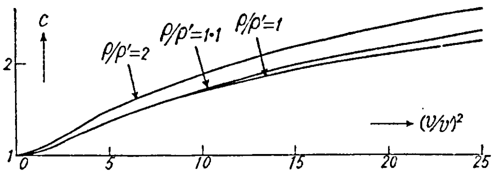

Fig. 17. Love-waves. c is in the ratio of phase velocity correspond to the case in minimum group velocity to v'.

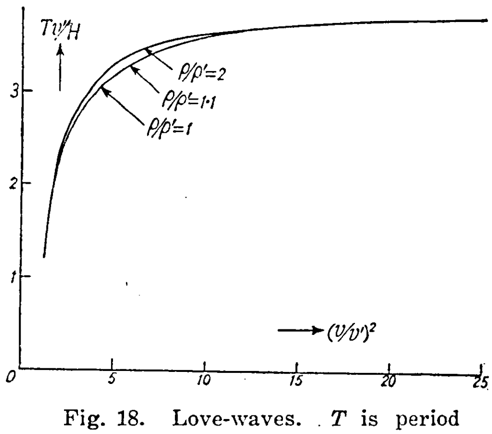

Fig. 18. Love-waves. T is period correspond to case in minimum group velocity.

subjacent medium to the stratum, under the condition that the density ratio of the two mediums, (ρ/ρ'), as kept constant, and increases with the decrease in the density ratio (ρ/ρ') in case the velocity ratio (v/v') is kept constant.

Fig. 17 shows that the period of waves corresponding to the condition in the minimum group velocity scarcely varies as the density ratio of the two mediums, (ρ/ρ'), and the period in question increase with the increase in the velocity ratio of the two mediums, (v/v'), and it takes an asymptotic value beyond a certain ratio of v/v', that is when $(v/v')^2 > 10$.

3. Group velocity of the dispersive Rayleigh-waves.

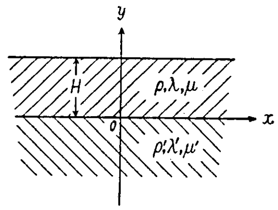

Fig. 19.

It was already found that the velocity equation of dispersive Rayleigh-waves transmitted through a stratified body is expressed by [6]

$$\frac{4r's'}{f^2}\left(2-\frac{k'^2}{f^2}\right)\eta-\frac{r's'}{f^2}\left\{4\vartheta+\left(2-\frac{k'^2}{f^2}\right)^2\zeta\right\}\cosh r'H\cosh s'H$$

$$+\frac{r'}{f}\varphi\left\{\frac{s}{f}\left(2-\frac{k'^2}{f^2}\right)^2-\frac{4rs'^2}{f^3}\right\}\cosh r'H\sinh s'H$$

$$+\frac{s'}{f}\varphi\left\{\frac{r}{f}\left(2-\frac{k'^2}{f'^2}\right)^2-\frac{4sr'^2}{f^3}\right\}\sinh r'H\cosh s'H$$

$$+\left\{\left(2-\frac{k'^2}{f^2}\right)^2\vartheta+\frac{4r'^2s'^2}{f^4}\zeta\right\}\sinh r'H\sinh s'H=0\,, \qquad \ldots\ldots(5)$$

where

$$\left.\begin{aligned}
&\varphi=\frac{\mu'k^2k'^2}{\mu f^4},\ \zeta=\frac{4rs}{f^2}\left(\frac{\mu'}{\mu}-1\right)^2-\alpha^2,\ \eta=\frac{2rs}{f^2}\left(\frac{\mu'}{\mu}-1\right)\beta-\alpha\gamma,\\
&\vartheta=\frac{rs}{f^2}\beta^2-\gamma^2,\ \alpha=\frac{2\mu'}{\mu}-\left(2-\frac{k^2}{f^2}\right),\ \beta=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-2,\\
&\gamma=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-\left(2-\frac{k^2}{f^2}\right),\\
&r^2=f^2-h^2,\ s^2=f^2-k^2,\ r'^2=f^2-h'^2,\ s'^2=f^2-k'^2,\\
&h^2=\rho p^2/(\lambda+2\mu),\ k^2=\rho p^2/\mu,\ h'^2=\rho' p^2/(\lambda'+2\mu'),\ k'^2=\rho' p^2/\mu',
\end{aligned}\right\}(6)$$

6) K. SEZAWA and K. KANAI, "Discontinuity in the Dispersion Curves of Rayleigh Waves", *Bull. Earthq. Res. Inst.*, **13** (1935), 238.

in which H is the thickness of the stratum, $2\pi/f$ the wave length, and ρ, λ, μ; ρ', λ', μ' the densities and elastic constants of the subjacent medium and the stratum respectively.

Dispersion curves of cases $\mu/\mu'=3$, 5, 8, 20 ($\lambda=\mu$, $\lambda'=\mu'$, $\rho/\rho'=1$) and $\mu/\mu'=1.65$ ($\lambda=\mu$, $\lambda'=\mu'$, $\rho/\rho'=1.14$) calculated by means of the above equation are shown by full lines in Figs. 20~24. The third and the fifth cases correspond approximately to the superficial condition of the Kwanto district in Japan and the stratification of Eurasian Continent respectively.

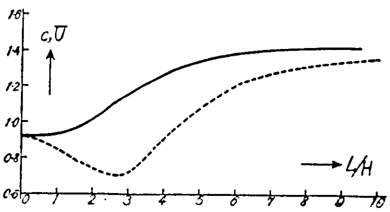

Fig. 20. $\rho/\rho'=1$, $\mu/\mu'=3$.

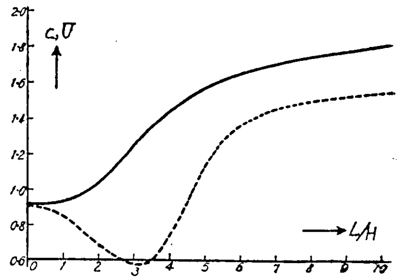

Fig. 21. $\rho/\rho'=1$, $\mu/\mu'=5$.

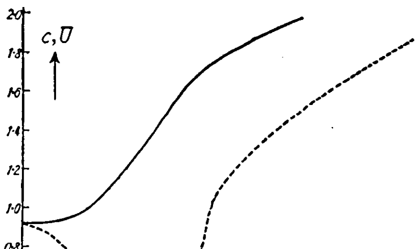

Fig. 22. $\rho/\rho'=1$, $\mu/\mu'=8$.

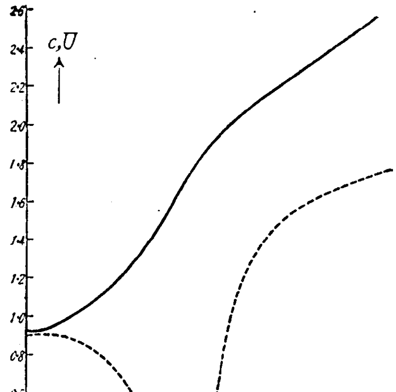

Fig. 23. $\rho/\rho'=1$, $\mu/\mu'=20$.

Using the above dispersion curves, we next determined the group velocity graphically, the results being shown by broken lines in Figs. 20~24. The relations among the minimum group velocity, the wave length corresponding

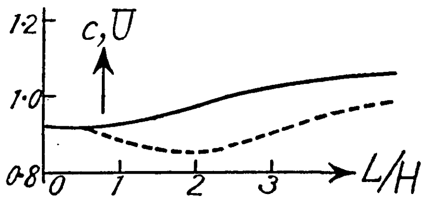

Fig. 24. $\rho/\rho'=1.14$, $\mu/\mu'=1.65$.

to the minimum group velocity and the rigidity ratio of the two mediums are shown in Table IV.

Table IV. Minimum group velocity of Rayleigh-waves, $\lambda/\mu=\lambda'/\mu'=1$.

ρ/ρ'	1.14	1			
μ/μ'	1.65	3	5	8	20
U	0.84	0.71	0.61	0.45	0.24
L/H	2.0	2.5	3.0	3.4	3.8

4. Group velocity of the M_2-waves.

By means of equation (5), we calculated the dispersion curves of M_2-waves and determined the group velocity of them graphically. We selected six cases of stratification, namely $\mu/\mu'=2, 3, 5, 8, 20$ under the conditions of $\rho=\rho'$, $\lambda=\mu$, $\lambda'=\mu'$ and $\mu/\mu'=1.65$, $\rho/\rho'=1.14$, $\lambda/\mu=\lambda'/\mu'=1$. The first, the sixth and the fourth cases correspond approximately to the stratifications of the Pacific Ocean, the Eurasian Continent and the superficial condition of the Kwanto district in Japan respectively. The phase velocities and the

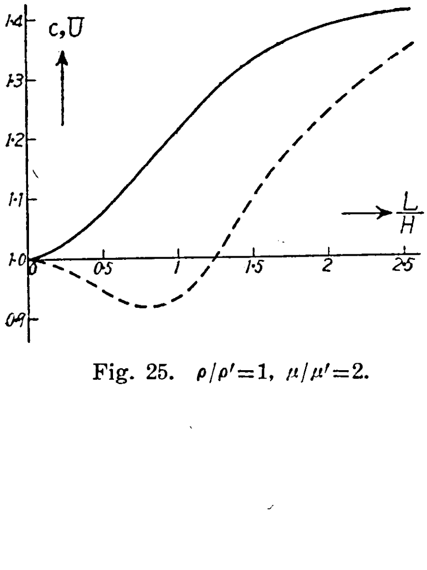

Fig. 25. $\rho/\rho'=1$, $\mu/\mu'=2$.

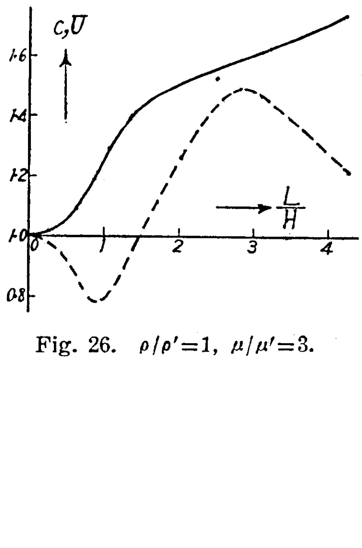

Fig. 26. $\rho/\rho'=1$, $\mu/\mu'=3$.

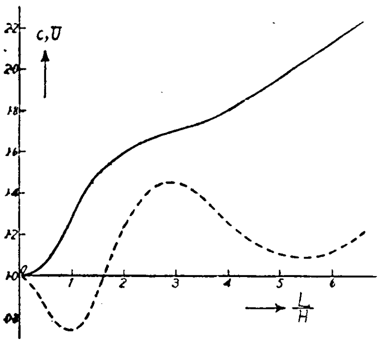

Fig. 27. $\rho/\rho'=1, \mu/\mu'=5.$

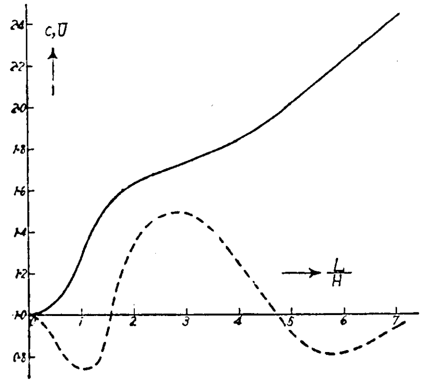

Fig. 28. $\rho/\rho'=1, \mu/\mu'=8.$

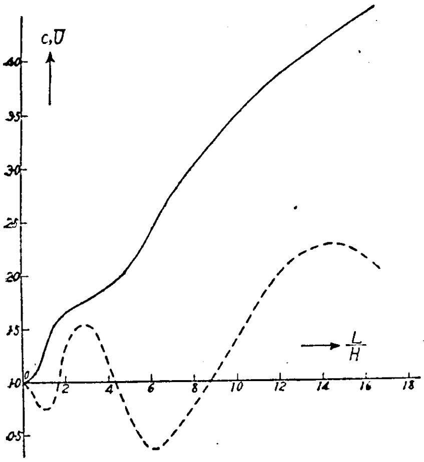

Fig. 29. $\rho/\rho'=1, \mu/\mu'=20.$

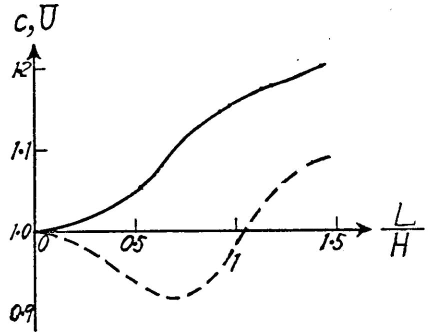

Fig. 30. $\rho/\rho'=1.14, \mu/\mu'=1.65.$

group velocities of the above cases are shown respectively by full lines and broken lines in Figs. 25~30.

It will be seen from Figs. 25~30 that the stationary state of group velocity of M_2-waves is only in the minimum state when the rigidity ratio of the two mediums (μ/μ') is small, such a phenomenon is similiar to the feature of the group velocity of Love-waves as well as Rayleigh-waves, but the number of the state mentioned above increases with increase in the rigidity ratio of two mediums, (μ/μ'). The relations among the first minimum group velocity, wave length corresponding to the minimum group velocity and rigidity ratio of the two mediums are shown in Table V.

Table V. Minimum group velocity of M_2-waves, $\lambda/\mu=\lambda'/\mu'=1$.

ρ/ρ'	1.14	1				
μ/μ'	1.65	2	3	5	8	20
U	0.93	0.92	0.79	0.74	0.74	0.75
L/H	0.66	0.83	0.96	0.98	1.07	1.03

5. Comparison among the feature of group velocity of Love-waves, dispersive Rayleigh-waves and M_2-waves.

The relations among the minimum group velocity, wave length as well as phase velocity corresponding to the condition in the minimum group velocity and rigidity ratio of two mediums, (μ/μ'), under the condition $\rho/\rho'=1$, $\lambda/\mu=\lambda'/\mu'=1$, are shown in Figs. 31~33.

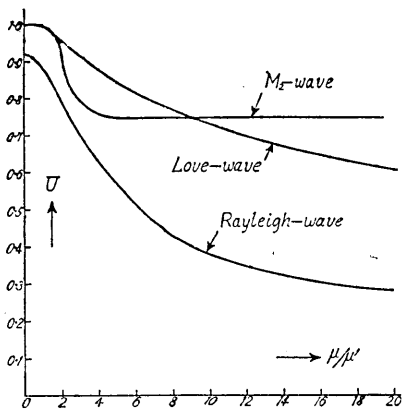

Fig. 31. Relation between the minimum group velocity and ratio of μ/μ'.

From Fig. 31 we found that the value of minimum group velocity of the three kinds of dispersive surface waves tends to decrease with the increase in the rigidity ratio of two mediums, (μ/μ'). The values mentioned above of dispersive Rayleigh-waves are the smallest among the three kinds of waves and decrease rapidly with the increase in the ratio μ/μ', and those of Love-waves decrease somewhat gradually with the increase in the ratio μ/μ', while those of M_2-waves, in case they are beyond a certain ratio of

μ/μ', namely $\mu/\mu'>4$, tend to be an asymptotic line, and take the largest value among the three kinds of waves beyond a certain ratio of μ/μ', namely $\mu/\mu'>8$, in the present case.

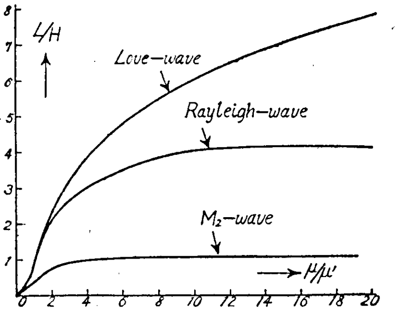

Fig. 32. Relation between wave length corresponding to condition in minimum group velocity and ratio of μ/μ'.

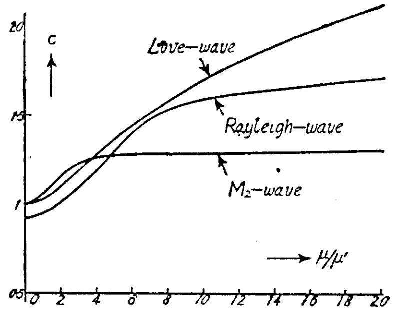

Fig. 33. Relation between phase velocity corresponding to condition in minimum group velocity and ratio of μ/μ'.

Fig. 32 shows that the period of waves of all the three kinds of dispersive surface waves corresponding to the condition in minimum group velocity increases with the increase in the rigidity ratio of the two mediums, (μ/μ'). The periods above mentioned of M_2-waves and dispersive Rayleigh-waves take an asymptotic line when they go beyond a certain ratio of μ/μ', namely $\mu/\mu'>4$ and $\mu/\mu'>8$ respectively, but those of Love-waves are still increasing in such ratio of μ/μ' as 20.

6. Concluding remarks.

From the results of the mathematical investigations concerned with the nature of group velocity of Love-waves, dispersive Rayleigh-waves and M_2-waves, we found that there is a little difference between the values of minimum group velocity of M_2-waves and Love-waves, when the rigidity ratio of subjacent medium to stratum is small, namely μ/μ' is more than several times smaller, while that of Rayleigh-waves is smaller than or at least half of the other two.

Concerning the period of waves corresponding to the minimum group velocity, Love-waves rank first, next dispersive Rayleigh-waves, then come M_2-waves, and the difference among them are considerably large.

On the other hand, when μ/μ' is more than several times smaller, the values of minimum group velocity of the three kinds of dispersive surface waves differ little, and the periods of waves corresponding to the condition in the minimum group velocity of Love-waves and Rayleigh-waves differ little but that of M_2-waves is smaller than that is about one third of the other two.

These results were obtained by complex numerical calculations, and not immediately from the types of the mathematical expressions.

These results of the present investigations will be useful in classifying the phase of seismic waves by means of analysing the records of actual seismic disturbances.

Especially, the features of M_2-waves, that is, the values of minimum group velocity and period of waves corresponding to the minimum group velocity scarecely vary as the rigidity ratio of subjacent medium to stratum beyond a certain ratio of μ/μ', will be utilizable in ascertaining the possible existence of M_2-waves in actual seismic disturbances.

In conclusion I wish to express my thanks to Mrs. S. Izumi who has assisted me in preparing this paper.

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5. 分散性表面波の群速度について

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ラブ波，分散性レーレー波並びに M_2 波の群速度の性質を数理的に研究した。2層の密度が等しい場合に，下層の剛性が表面層の剛性に比べて大きいとき，即ち μ/μ' が數倍以上のとき，には M_2 波とラブ波の極小群速度の値は餘り變らないが，レーレー波のそれは 1/2 或はそれ以上小さい。極小群速度にあたるところの波の周期はラブ波のものが最も大きく，レーレー波，M_2 波の順に小さく，各々の間の差違は相當に大きい。

μ/μ' が數倍以下のときには，3種類の波の極小群速度の値は大して變らない。極小群速度にあたるところの波の周期は，ラブ波とレーレー波とはその差違が小さいが，M_2 波の値は兩者の1/3近くも小さい。

これらの數理的研究結果は地震記象を解析して，表面波の各位相をきめる際に，相當に役立つであろう。又，特に M_2 波の極小群速度の値と，そのときの波の周期の値とが，μ/μ' が或程度大きくなると，μ/μ' に關してほとんど變らなく一定の値をとる性質は，地震記象の上で M_2 波の實在性を確める上に非常に便利な筈である。