2. *Rayleigh Waves Propagated along the Plane Surface of Horizontally Isotropic and Vertically Aeolotropic Elastic Body.*

By Yasuo SATÔ,

Earthquake Research Institute.

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1. Introduction.

Since the publication of the celebrated paper on surface waves by Lord Rayleigh in 1885, considerable development has been made by a number of investigations in the field of the surface waves. But among these studies we find only a few papers concerning the theory of Rayleigh waves in aeolotropic media, none of which gives definite clue as to the conditions of existence and nature of these waves, although Dr. Nagaoka's[1] famous experiment on the elastic constants of rocks proved, in as early as 1900, strong anisotropy in most of the rocks he examined, and recent observations show that the materials in the earth's crust is not always isotropic. Therefore the present discus ion is projected. Among various kinds of aeolotropy, we will consider only, in this paper, the case of horizontally isotropic medium, in view of the fact that this is the most important case in actual seismology.[2] This case has already been treated by some writers among whom Messrs. H. Homma[3] and R. Stoneley[4] are the most remarkable. The present paper being the continuation of the study of the former author, most of the equations in the first half of this paper have been obtained by Mr. Homma, to whom the writer expresses his hearty thanks for the kind permission given him to reproduce them.

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1) H. NAGAOKA, "Elastic Constants of Rocks and the Velocity of Seismic Waves," *Publ. Earthq. Invest. Commit.*, 4 (1900), 47.

2) N. NASU and S. OMOTE, *Zisin.* 13 (1941), 91.

3) S. HOMMA, "On Rayleigh Waves in Horizontally Isotropic Elastic Body" (In Japanese), *Kensinzihô*, 12 (1942), 97.

4) R. STONELEY, "The Seismological Implications of Aeolotropy in Continental Structure," *M. N. R. A. S. Geo. Sup.*, 5 (1943), 343.

2. Equation of motion and boundary conditions.

If we confine ourselves to the problem of Rayleigh waves which are propagated along x-axis and do not vary in the direction of y-axis, then the equations of motion become as follows.* (z-axis is taken vertically downwards and free surface coincides with $z=0$.)

$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} = M_1 \dfrac{\partial^2 u}{\partial x^2} + N \dfrac{\partial^2 u}{\partial z^2} + (M_3 - N) \dfrac{\partial^2 w}{\partial x \partial z} \\ \rho \dfrac{\partial^2 w}{\partial t^2} = (M_3 - N) \dfrac{\partial^2 u}{\partial x \partial z} + N \dfrac{\partial^2 w}{\partial x^2} + M_3 \dfrac{\partial^2 w}{\partial z^2} \end{cases} \qquad \text{(2.1)}$$

where ρ is the density, u and w are the components of displacement in x and z direction respectively, and

$$\begin{cases} X_x = M_1 (x_x + y_y) - 2N_2 y_y + (M_3 - 2N_1) z_z, & Y_z = N_1 y_z \\ Y_y = M_1 (x_x + y_y) - 2N_2 x_x + (M_3 - 2N_1) z_z, & Z_x = N_1 x_z \\ Z_z = (M_3 - 2N_1)(x_x + y_y) + M_2 z_z, & \\ N = N_1 & \end{cases} \qquad \text{(2.2)}$$

If we assume as

$$\begin{cases} u = A_k \exp(-\sigma m Z + imx - imVt) \\ w = iB_k \exp(\quad \text{"} \quad) \end{cases} \qquad \text{(2.3)}$$

then we have

$$\begin{cases} (M_1 - V^2\rho - \sigma^2 N) A_k = \sigma (M_3 - N) B_k \\ \sigma (M_3 - N) A_k = (\sigma_2 M_2 + V^2 \rho - N) B_k \end{cases} \qquad \text{(2.4)}$$

Eliminating A and B, we at once have the ordinary velocity equation,

$$M_2 N \sigma^4 + \{M_3^2 - 2M_3 N - M_1 M_2 + (M_2 + N) V^2 \rho\} \sigma^2 + (M_1 - V^2\rho)(N - V^2\rho) = 0 . \qquad \text{(2.5)}$$

Solving the above equation with respect to σ^2, and denoting the two roots by σ_1^2 and σ_2^2, we have

$$\begin{cases} u = A_1 \exp(-m\sigma_1 z) + A_2 \exp(-m\sigma_2 z) \\ w = i B_1 \exp(\quad \text{"} \quad) + iB_2 \exp(\quad \text{"} \quad) \end{cases} \qquad \text{(2.6)}$$

in which the common factor $\exp(imx\text{-}imvt)$ is omitted for brevity.

In this expression both (A_1, B_1) and (A_2, B_2) must be so chosen as to satisfy the equation (2. 4).

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* We use the expressions derived by Mr. S. Homma.

Next we must consider the boundary conditions. Since at $z=0$, normal tractions must vanish, we substitute (2.6) into the expressions of Z_z and Z_x in (2.2), and equating the results to zero, we have,

$$\begin{cases}(M_3-2N)(A_1+A_2)-M_2(\sigma_1 B_1+\sigma_2 B_2)=0\\ \sigma_1 A_1+\sigma_2 A_2+B_1+B_2=0\end{cases} \quad \text{.............................}(2.7)$$

We may eliminate A_1, A_2, B_1, B_2, from (2.4) and (2.8). The result

$$\begin{vmatrix}\sigma_1 & \sigma_2 & 1 & 1\\ M_3-2N & M_3-2N & -\sigma_1 M_2 & -\sigma_2 M_2\\ M_1-r^2\rho-\sigma_1^2 N & 0 & -\sigma_1(M_3-N) & 0\\ 0 & M_1-V^2\rho-\sigma_2^2 N & 0 & -\sigma_2(M_3-N)\end{vmatrix}=0 \quad \text{...........}(2.8)$$

is the usual characteristic equation.

3. Examination of the characteristic equation.

For the sake of convenience we use the following notations

$$\begin{cases}M_1\equiv\alpha N, \quad M_2\equiv\beta N, \quad M_3\equiv\gamma N\\ \tau\equiv V^2\rho/N\equiv(V/V_s)^2, \quad \text{(where } V_s \text{ is the velocity of S wave)}\end{cases} \quad \text{...}(3.1)$$

Then from (2.5)

$$\begin{cases}\sigma_1^2+\sigma_2^2=[-\gamma^2+2\gamma+\alpha\beta-(\beta+1)\tau]/\beta\\ \sigma_1^2\sigma_2^2=(\alpha-\tau)(1-\tau)/\beta\end{cases} \quad \text{....................................}(3.2)$$

Now we will develop the determinant (2.8) and simplify it by means of the above relation. Since $\sigma_1=\sigma_2$ cannot satisfy both of the equation of motion and boundary conditions, we divide the expression above obtained by $(\sigma_1-\sigma_2)$; then we get an important equation

$$\tau(\tau-\alpha)=[\beta\tau-K]\sigma_1\sigma_2 \quad \text{..}(3.3)$$

where

$$K\equiv\alpha\beta-(\gamma-2)^2 \quad \text{..}(3.4)$$

From (3.3) and the second equation of (3.2) we can eliminate $\sigma_1\sigma_2$ and obtain finally the characteristic equation which determines the velocity of Rayleigh-type waves. That is

$$f(\tau)\equiv\beta(\beta-1)\tau^3-\beta(2K+\beta-\alpha)\tau^2+K(K+2\beta)\tau-K^2=0 \quad \text{...........}(3.5)$$

On getting this equation, however, we took the square of both members of (3.3), therefore, there is some possibility that some extraneous roots might be involved in this equation. At first we assumed σ_1 and σ_2 to be both positive. While $\tau=V^2/V_s^2$ must, of course, be positive. Thus, among the roots of $f(\tau)=0$ we must adopt only those which satisfy the following conditions.

1) τ must be positive (since $\tau=V^2/V_s^2$),
2) $\tau-\alpha$ and $\beta\tau-K$ must have same sign (since both sides of (3.3) must have same sign).

4. Possible range of values of elastic constants.*

Following Love, we write elastic constants as c_{ij} $(i, j=1, 2, 3, 4, 5, 6)$, then the strain energy function W of horizontally isotropic body is written as follows:

$$2W=c_{11}(x_x+y_y)^2+c_{33}z_z^2-4c_{66}x_xy_y+2c_{13}(x_x+y_y)z_z$$
$$+c_{44}(y_z+z_x)^2+c_{66}x_y^2 \quad \ldots\ldots(4.1)$$

Since W must be positive definite form, we see at once $c_{11}>0$ by putting every strain component except x_x to zero.

Similarly we obtain the following relations,

$$c_{11}>0,\ c_{33}>0,\ c_{44}>0,\ c_{66}>0 \quad \ldots\ldots(4.2)$$

Next, if we take $y_z=z_x=x_y=0$, W becomes

$$2W=c_{11}(x_x+y_y)^2+c_{33}z_z^2-4c_{66}x_xy_y+2c_{13}(x_x+y_y)z_z$$

So that, using the notations $\xi=x_x/z_z$, $\eta=y_y/z_z$, we get

$$F\equiv 2W/z_z^2=c_{11}(\xi+\eta)^2+c_{33}-4c_{66}\xi\eta+2c_{13}(\xi+\eta).$$

Here, F must also be positive. The condition is equivalent to

$$[F_{\xi\xi}F_{\eta\eta}-F_{\xi\eta}^2]_{\xi=\xi_0,\eta=\eta_0}>0 \text{ and } [F_{\xi\xi}]_{\xi=\xi_0,\eta=\eta_0}>0$$
$$\text{or} \quad (c_{11}-c_{66})\,c_{66}>0 \quad \text{and} \quad c_{11}>0 \quad \ldots\ldots(4.3)$$

where ξ_0, η_0 satisfies

$$\begin{cases} F_\xi=c_{11}\xi+(c_{11}-2c_{66})\eta+c_{13}=0 \\ F_\eta=c_{11}\eta+(c_{11}-2c_{66})\xi+c_{13}=0 \end{cases} \quad \ldots\ldots(4.4)$$

From this we have $\xi_0=\eta_0=-c_{13}/2(c_{11}-c_{66})$ ......(4.5)

Substituting these values into $F>0$ and simplifying, we have

$$(c_{11}-c_{66})c_{33}>c_{13}^2 \quad \ldots\ldots(4.6)$$

Now, we will collect here the conditions obtained above.

* This article is due to Mr. Homma's instruction.

$$\begin{cases} c_{11}>0, \quad c_{33}>0, \quad c_{44}>0, \quad c_{66}>0 \\ (c_{11}-c_{66})c_{33}>c_{13}^2 \end{cases} \qquad \text{......(4.7)}$$

Our previous notations M_1, M_2, etc. bear the following relations with the notations in this article.

$$c_{11}=M_1\equiv\alpha N\ , \quad c_{33}=M_2\equiv\beta N\ , \quad c_{44}=N\ ,$$
$$c_{13}=M_3-2N\equiv(\gamma-2)N \qquad \text{......(4.8)}$$

Substitution of which, and a new notation $\delta\equiv C_{66}/N$ into (4.7), gives the following relations.

$$\alpha>0\ , \quad \beta>0\ , \quad \alpha>\delta>0\ , \quad (\alpha-\delta)\ \beta>(\gamma-2)^2 \qquad \text{......(4.9)}$$

In two-dimensional problems in which c_{66} is not involved, we may put $c_{66}=0$, therefore the relations (4.9) is reduced to

$$\alpha>0\ , \quad \beta>0\ , \quad K\equiv\alpha\beta-(\gamma-2)^2>0 \qquad \text{......(4.10)}$$

5. Existence of Rayleigh waves.

Now we will examine the existence of solution of (3.5) under the condition (4. 10). We denote the roots of $f(\tau)=0$ lying between 0 and 1 by τ_1, τ_2 and τ_3 (where $\tau_1<\tau_2<\tau_3$; or at times τ_2 and τ_3 may not exist). The following results are obtained without difficulty:—

i) $f(\tau) \quad =(\beta\tau-K)^2(\tau-1)-\beta\tau^2(\tau-\alpha)$

So that

$$\begin{cases} f(0) &=-K^2<0 \\ f'(0) &=K(K+2\beta)>0 \\ f(K/\beta) &=(K^2/\beta^2)(\gamma-2)^2>0 \\ f(1) &=\beta(\alpha-1) \\ f(\alpha) &=(\gamma-2)^2(\alpha-1) \end{cases} \qquad \text{......(5.1)}$$

ii) $\qquad \tau_n<\alpha \qquad$ ......(5.2)

Proof; By virtue of the second equation of (3.2), it is evident that $\tau_n>\max\ (\alpha,\ 1)$ or $\tau_n<\min\ (\alpha,\ 1)$, but the former case is impossible. For, if it holds, from the first equation of (3.2) we know

$$\sigma_1^2+\sigma_2^2=-[(\gamma-1)^2+(\tau-1)+\beta(\tau-\alpha)]/\beta<0$$

which is contradictory to the fact that σ_1 and σ_2 must be real. Therefore

$$\tau_n<\min\ (\alpha,\ 1).$$

Consequently $\qquad \tau_n<\alpha. \qquad$ Q. E. D.

iii) Necessary and sufficient condition that τ_n should be a true root is $\beta\tau_n < k$.

Proof; Substitute (5.2) into (3.3), then we immediately find the result.

With the above preparation we can conclude easily; that, when $\alpha < 1$, there exists one and only one root that satisfies the required condition (i. e. one and only one sort of Rayleigh-wave exists).

Proof; $K/\beta = \alpha - (\gamma-2)^2/\beta < \alpha$.

$\therefore \quad 0 < K/\beta < \alpha < 1.$

From (ii) and Fig. 1 we can find

$$\beta\tau_1 < K \qquad \text{but} \qquad \beta\tau_2 > K,$$

therefore τ_1 is the only root that satisfies the required conditions.

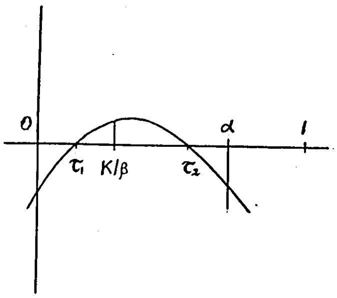

Fig. 1.

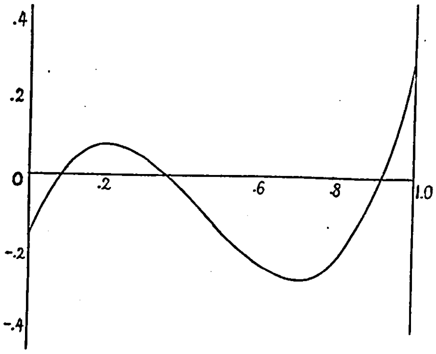

Fig. 2.

iv) Next, the case when $\alpha > 1$ will be examined. Since $f(1) > 0$ in this case, we can show a case where three roots exist between 0 and 1. As for example;

$$\alpha = 1.1 \ , \quad \beta = 3 \ , \quad \gamma = 3.7 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.3)$$

for which

$$K = 0.41 \ , \quad K/\beta = 0.1367$$

$$\tau_1 = 0.0850 \ , \quad \tau_2 = 0.3605 \quad \tau_3 = 0.9146 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.4)$$

so that, as will be seen from Fig. 2, three roots exist. However, neither τ_2 nor τ_3 satisfies $\beta\tau_n < K$,(for $\beta\tau_2 = 1.0314 > 0.41 = K$, $\beta\tau_3 = 2.7437 > 0.41$). Therefore from the criterion iii) τ_1 is the only root.

The above case is a single example, but starting from this example, we can derive a general theorem about the existence of Rayleigh waves.

Roots of $f(\tau)=0$ are the continuous function of parameter α, β and γ. Therefore

$$\beta\tau_2-K=(\text{function of } \alpha,\ \beta,\ \gamma)\equiv\phi(\alpha,\beta,\gamma) \quad\cdots\cdots(5.5)$$

is also a continuous function of α, β and γ.

and from (5.4)

$$\phi(1.1,\ \ 3,\ \ 3.7)=0.6714>0 \quad\cdots\cdots(5.6)$$

In order to arrive at a case where τ_2 is a true root, we must start from the above example, and varying α, β and γ continuously find a combination of three parameters such that*

$$\phi(\alpha,\ \ \beta,\ \ \gamma)<0, \quad\cdots\cdots(5.7)$$

but on the way from (5.6) to (5.7) we must pass a point where

$$\phi(\alpha,\ \ \beta,\ \ \gamma)=0. \quad\cdots\cdots(5.8)$$

However, this is impossible. For, from the first equation of (5.1)

$$\begin{aligned} f(\tau_2)=0&=\phi(\alpha,\ \ \beta,\ \ \gamma)-\beta\tau_2^2(\tau_2-\alpha) \\ &= \qquad\qquad\qquad -\beta\tau_2^2(\tau_2-\alpha) \end{aligned} \quad\cdots\cdots(5.9)$$

this is self-contradictory.

Therefore we can conclude finally that ;

"One and only one sort of Rayleigh waves exists, in horizontally isotropic but vertically aeolotropic semi-infinite elastic medium."

The writer wishes to express his heartiest thanks to Mr. S. Homma. Without his instruction and encouragement, this paper could have never been completed.

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2. 水平等方彈性体表面を傳はるレーリー波

地震研究所　佐藤泰夫

半無限の水平等方彈性体表面をつたはるレーリー波は常に唯一種類存在する.

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*If there exists another domain isolated from the one involving the point $\alpha=1.1$, $\beta=3$, $\gamma=3.7$, then the proof is not complete, but the author carried out a comprehensive numerical calculation only to find no such domain.