5. *Electromagnetic Induction within the Earth and Its Relation to the Electrical State of the Earth's Interior. Part I (1)*

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Introduction.

As shown by Gauss[1], it is possible to separate the earth's magnetic field into parts of external and internal origins. Applying this method to the field of the daily variation, Schuster[2] was the first who found that the major part originates above the earth, but there is also a part of internal origin. According to his investigation, the latter may be the secondary field due to the electric currents induced in the earth by the variation of the external one. This hypothesis seems to be able to explain the relation between the two parts when the electrical conductivity (σ) and the magnetic permeability (μ) are distributed suitably in the earth. After Schuster's investigation, we made a good number of studies concerning this problem. Their brief outlines will be mentioned as follows.

In the above mentioned paper, Schuster, in co-operation with Lamb, showed that the conducting sphere must be smaller than the earth, that is, the outer layer of the earth must be non-conductive.

Using the observed data of 21 stations distributed widely over the earth, Chapman[3] made a more complete analysis of the daily variation field. If the thickness of the non-conducting shell is about 250 *km* and the electrical conductivity of the inner core amounts to 3.6×10^{-13} *emu*, the relation between the amplitude and phase of main harmonics becomes reasonable, while the magnetic permeability is assumed to be unity.

Chapman and Whitehead[4] studied electromagnetic induction by periodic magnetic variation in a conducting sphere whose permeability has any value. They showed that we can determine the ratio σ/μ from the observed data,

1) K. F. GAUSS, "*Allgemeine Theorie der Erdmagnetismus*", p. 13.
2) A. SCHUSTER, *Phil. Trans. Roy. Soc. London* A, 180 (1889), 467.
3) S. CHAPMAN, *Phil. Trans. Roy. Soc. London* A, 218 (1919), 1.
4) S. CHAPMAN and T. T. WHITEHEAD, *Trans. Cambr. Phil. Soc.*, 22 (1922), 463.

but not σ and μ separately. They also investigated the possible influence of the induced field produced by the electric current which is flowing in sea. As the electrical conductivity of sea-water amounts to the order of $10^{-10} \sim 10^{-11}$ in electromagnetic unit, it is of importance to estimate to what extent the varying magnetic field may be affected by the presence of sea. According to their estimate, the field must be considerably affected by the presence of a comparatively shallow sea which spreads all over the earth with uniform depth. In practice, as the distribution of land and sea is very irregular, the induced electric current in an ocean must presumably be far less intense, on the average, than it would be in the world-wide sea on account of the complicated ocean boundaries.

As pointed out by Chapman[5], the average field of magnetic storms is mainly of external origin but also has a part produced within the earth. As well as in the case of the daily variation, the latter is naturally interpreted as the induced by the variation of external origin. Chapman and Whitehead[1] made an attempt to determine the electrical conductivity from storm-time induction. Constructing the mathematical theory of electromagnetic induction by aperiodic field, they got $\sigma = 1 \sim 4 \times 10^{-13}$ *emu* in the conducting core. Their calculation was, however, somewhat rough. According to the later work, the conductivity must be higher.

The mathematics of electromagnetic induction in a sphere was fully investigated by Price[6, 7]. In co-operation with him, Chapman[8] studied the electrical state of the earth's interior consistent with the storm-time variation. According to their study, it was required $\sigma = 4.4 \times 10^{-13}$ *emu* for a core of almost exactly the same size in daily variation. It is remarkable that the electrical conductivity which is in accord with the daily variation and the storm-time variation differ from each other. The slower the change, the deeper do the induced electric currents penetrate into the earth. As the storm-time variation is slower than the daily one, the conductivity, which is consistent with the former is interpreted to correspond to the deeper part of the earth's interior than the one obtained from the latter. Then the earth's interior can not be uniformly conducting.

K. Terada[9] discussed electromagnetic induction in a sphere consisting of many concentric shells whose electrical conductivity differ each other. Applying his theory to the induction by 12-hourly component of daily variation on

5) S. CHAPMAN, *Proc. Roy. Soc.* A, 95 (1918), 61.
6) A. T. PRICE, *Proc. Lond. Math. Soc.* ser. 2, 31 (1930), 217.
7) A. T. PRICE, *Proc. Lond. Math. Soc.* ser. 2, 33 (1931), 233.
8) S. CHAPMAN and A. T. PRICE, *Phil. Trans. Roy. Soc. London*, A, 229 (1930), 427.
9) K. TERADA, *Geophys. Mag*, 13 (1939), 63 and 16 (1948), 5.

the basis of Chapman's analysis, he got the same conclusion with Lahiri and and Price's investigation as mentioned below.

As mentioned above, the deeper part of the earth may be more conductive than the part near the surface. In order to get more precise knowledge about the conductivity-distribution, Lahiri and Price[10] studied electromagnetic induction in a conducting sphere whose conductivity varies proportionally to r^{-m} where r and m denote respectively the radial distance and at the same time a constant. According to their study, it was informed that the conductivity becomes enormously higher at the depth of 700 *km* beneath the surface.

Summing up the results of these studies, it is likely that the electrical conductivity of the earth's interior increases with the depth. Although, these investigations gave us very interesting conclusion, only the daily and storm-time variation were considered up to now. We have, however, magnetic variations of another type such as abrupt beginning of magnetic storm, solar daily disturbance variation, bay-type disturbance, pulsation and disturbance accompanying solar eruption. If we can collect a good many data from observations, these variations may also be available for our purpose. While a few of them vary with periods comparable to that of daily variation, abrupt beginning and pulsation vary very rapidly. To use these variations whose periods differ very much will be desirable for getting more complete knowledge about the conductivity-distribution in the earth. In this paper, electromagnetic induction due to variations of various types will be discussed as far as the writer can get data.

As is known in researches of seismic waves, there is a distinct discontinuity at the depth of 2900 *km* in the earth. In the core, inside this boundary, it seems very likely that the rigidity is so small that we can scarcely observe the transversal waves which come through this region, while the velocity of the longitudinal waves decreases from about 13 to 8 *km/sec* at the boundary. This is an evidence that the physical state of the material differs distinguishably at this level and consequently it may be natural to consider that the electrical conductivity also changes in- and outside this boundary. From this point of view, the writer attempts to determine the electrical conductivity in the earth's core and mantle separately.

As to the influence of sea, about which only an idealized case was investigated by Chapman, Whitehead, Lahiri and Price,[4), 10)] a complete estimate for the real earth is very difficult, because the distribution of land and sea is very complicated. In this paper, the writer, intending to investigate the possible

10) B. N. LAHIRI and A. T. PRICE, *Phil. Trans. Roy. Soc. London*, A, 237 (1939), 509.

influence of ocean, such as the Pacific and Atlantic Oceans, calculate approximately to what extent the daily variation and the sudden commencement of magnetic storm may be affected by the presence of a sea bounded by two meridians $\pi/2$ apart in longitude.

In short, this paper is an extension of studies on electromagnetic induction in the earth which was hitherto developed by many magneticians. It is hoped that the knowledge about the electrical state of the earth's interior obtained in this study may contribute to get the more comprehensive knowledge of the internal constitution of the earth and physical properties of material in such high temperature and pressure as is in the earth.

CHAPTER I. ASSUMPTION ON THE MAGNETIC PERMEABILITY IN THE EARTH.

If μ would be larger compared to unity in the earth, the magnetically induced field must be taken into consideration for our problem together with electromagnetically induced field. Though we have not hitherto any reliable method to infer the distribution of μ in the earth, a few attempts have been tried with respect to this problem.

As pointed out by Chapman and Whitehead[4], the magnetically induced part cannot be separated from the electromagnetically induced part in the case of periodic variation. On the other hand, Chapman and Price[8] showed that it may be possible to make such a separation in the case of storm-time variation. According to them, it is found that the magnetically induced field will oppose the horizontal components and reinforce the vertical components of the external inducing field at the end of magnetic storms because the electromagnetically induced currents may have fairly well died away at that time. Then, if μ differs considerably from unity, the ratio of the external field to the whole field may become appreciably larger in the vertical component than in the horizontal component. They tried to find whether such effect really exists or not, examining data for several days after magnetic storms. On account of the ambiguities due to difficulties to determine the undisturbed state of the geomagnetic field and also the smallness of the later part of storm-time variation, they found no reliable indication that μ differs from unity.

Adams and Green[11] investigated experimentally the influence of hydrostatic pressure on the critical temperature of magnetization for ferromagnetic materials. Five ferromagnetic metals were used in their experiment. Deter-

11) L. H. ADAMS and J. W. GREEN, *Terr. Mag.*, 36 (1931), 161.

minations were made at pressures up to 2000 *atmospheres* for iron and magnetite, 2200 *atmospheres* for nickel, 2600 *atmospheres* for nickel-steel, and in the case of meteorite, 3600 *atmospheres*. The results indicate that pressure has practically no-effect. However, the Curie point seems likely to decrease slightly with increasing pressure in the case of nickel-steel and meteoric iron.

An estimate of the increase of the Curie point of nickel and nickel-iron alloys was made by Slater[12] from a theoretical stand-point in which he applied Clapeyron's equation. According to him, the rate of increase for nickel with pressure must be of the order of 5×10^{-5} *degree per atmosphere*, while, in the case of nickel-iron alloys, it would become less as more iron is added. It becomes zero with 70% nickel and negative for alloys with more iron. Assuming that the temperature and the pressure amount respectively to the order of 5000 *degrees* and 10^6 *atmospheres* near the centre of the earth, it is required that the increasing rate of the Curie point must reach to the order of 5×10^{-3} *degree per atmosphere* for the ferromagnetism in that region. This magnitude is as much as one hundred times larger than the magnitude expected from the theory even for pure nickel. For this reason, it is implausible that ferromagnetic state is kept inside the earth.

Although the physical and geophysical studies mentioned above are not always applicable to real earth, it is noticeable that they are all disadvantageous to the assumption that the earth's interior could be ferromagnetic. Then, at the present stage of investigation, we naturally consider that μ does not materially differ from unity in the earth together with the impossibility of any ferromagnetic explanation of the earth's permanent magnetization.

From this point of view, the writer assumes $\mu=1$ throughout the investigations which will be developed in this paper.

CHAPTER II. ELECTROMAGNETIC INDUCTION BY DAILY VARIATION (so-called S_q-field).

1. The influence of the ocean.

In promoting the study of the electromagnetic induction in the earth, we meet a dfficulty. This is the possibility that the electric currents induced in the oceans may have an appreciable effect on the magnetic variations as often discussed by researchers of the present problem. In sea-water, whose electrical conductivity is about one hundred times larger than that of rock and soil, the induced electric currents in the ocean will become appreciably intense, especially

12) J. C. SLATER, *Phys. Rev.*, 58 (1940), 54.

in a deep sea. In order to investigate the influence of the oceans, Chapman and Whitehead[4] assumed that the earth is surrounded by a sea of uniform depth. According to them, the internal part of the transient magnetic field must be considerably affected by the presence of a comparatively shallow sea. As they assumed that the sea is spread all over the earth, their conclusion shows the limit to which the transient magnetic field is likely to be affected by the existing irregular distribution of land and sea. On account of the complicated ocean boundaries, the actual currents in the ocean, however, will be much less intense than it would be in such an idealized sea mentioned above. Then it is desirable to examine the extent to which the large oceans shield the inner part of the earth from the outer field. From this point of view, the writer will estimate below to what extent S_q may be affected by the presence of a sea bounded by two meridians $\pi/2$ apart in longitude.

The intensity of electric field $\overrightarrow{E}$ and the magnetic induction $\overrightarrow{B}$ in a conductor satisfy the equation

$$\mathrm{rot}\,\overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}, \qquad \text{(1.1)}$$

when the phenomena are assumed to be quasistationary. Now we regard our sea as a thin conductor whose thickness and specific conductivity are respectively D and σ. When the thickness is so thin that the problem can be treated as a two-dimensional one, it is obviously sufficient for us to take into consideration only the component of rot $\overrightarrow{E}$ and $\overrightarrow{B}$ which are normal to the surface. Then, considering Ohm's law, we get

$$(\mathrm{rot}\,\overrightarrow{i}\,)_r = -D\sigma\frac{\partial B_r}{\partial t} \qquad \text{(1.2)}$$

where $\overrightarrow{i}$ is electric current density and suffix r denotes the normal to the surface.

As $\mathrm{div}\,\overrightarrow{i}=0$, we can define a current-function J which gives in polar coordinate

$$i_\theta = \frac{\partial J}{a\sin\theta\partial\phi}, \quad i_\phi = -\frac{\partial J}{a\partial\theta} \qquad \text{(1.3)}$$

where a denotes the radius of this spherical shell.

Substituting (1.3) in (1.2), we get

$$\frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right\}J = a^2D\sigma\frac{\partial B_r}{\partial t} \qquad \text{(1.4)}$$

When J can be expressed by a series of the type

$$J=\sum_n\sum_m K_n^m S_n^m \quad \text{...(1.5)}$$

where S_n^m denotes spherical surface harmonic and K_n^m is a function of time, (1.4) becomes by use of well known recurrence formulae

$$-\sum_n\sum_m K_n^m(t)n(n+1)S_n^m=a^2D\sigma\,\frac{\partial B_r}{\partial t}\,. \quad \text{.............................(1.6)}$$

Further, we assume that K_n^m can be expanded into a Fourier series of the form

$$K_n^m(t)=\sum_s K_{n,m}^{\,s}e^{ispt}\,, \quad \text{...(1.7)}$$

whence we get from (1.6)

$$-\sum_{s}\sum_n\sum_m n(n+1)K_{n,m}^{\,s}S_n^m e^{ispt}=a^2D\sigma\,\frac{\partial B_r}{\partial t} \quad \text{...........................(1.8)}$$

If we can expand the righthand-side of this equation into a series of $S_n^m e^{ispt}$, we can determine $K_{n,m}^s$ by comparing the coefficients of the corresponding harmonics.

We only deal, in this Section, with main harmonics of S_q in which magnetic flux varies so slowly that we can neglect the influence of self-induction. Then, $\frac{\partial B_r}{\partial t}$ in (1.8) can be approximately regarded as the flux-change through unit area which orignates outside of the conductor.

As already shown by Chapman[3] the equinoctial mean of S_q can be approximately expressible as

$$B_r=\sum_s P_{s+1}^s(a_{s+1}^s\cos sp\tau+b_{s+1}^s\sin sp\tau) \quad \text{....................(1.9)}$$

where τ denotes local time. Taking (1.9) as the inducing field, we shall estimate the magnitude and direction of electric currents induced in a thin conducting shell bounded by two meridians, $\phi=0$ and $\phi=\pi/2$ on a sphere.

The normal component of the electric currents must vanish at the boundaries, namely $i_\phi=0$ at $\phi=0$ and $\phi=\pi/2$. This condition is satisfied by preferring $P_n^{2\nu}\sin 2\nu\phi$ instead of S_n^m. Then (1.8) becomes

$$-\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\nu=0}^{\leq n/2} n(n+1)K_{n,\nu}^{\,s}\,e^{ispt}P_n^{2\nu}\sin 2\nu\phi=a^2D\sigma\,\frac{\partial B_r}{\partial t}\,. \quad \text{..........(1.10)}$$

Under certain conditions, a function $f(\theta,\phi)$, defined between $\phi=0$ and

$\phi=\pi/2$, can be expanded in the next form

$$f(\theta,\phi)=\sum_{n=1}^{\infty}\sum_{\nu=0}^{\leqq n/2}\gamma_{n,\nu}P_n^{2\nu}\sin 2\nu\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.11)$$

where

$$\gamma_{n,\nu}=\frac{2n+1}{\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{\pi/2}f(\theta,\phi)\,P_n^{2\nu}\sin 2\nu\phi\sin\theta\, d\theta\, d\phi \quad \ldots\ldots\ldots\ldots(1.12)$$

From (1.9) we get

$$\frac{\partial B_r}{\partial t}=\frac{\partial}{\partial t}\sum_s P_{s+1}^s\{a_{s+1}^s\cos s(pt+\phi)+b_{s+1}^s\sin s(pt+\phi)\}$$

or in complex expression

$$=ip\sum_s se^{ispt}P_{s+1}^s\{(a_{s+1}^s\cos s\phi+b_{s+1}^s\sin s\phi)$$
$$+i(a_{s+1}^s\sin s\phi-b_{s+1}^s\cos s\phi) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.13)$$

where t denotes $\phi=0$ meridian time.

Using (1.11) and (1.12), (1.13) becomes

$$\frac{\partial B_r}{\partial t}=ip\sum_s\sum_n\sum_\nu(a_{s+1}^s-ib_{s+1}^s)\zeta_{n,\nu}^s e^{ispt}P_n^{2\nu}\sin 2\nu\phi \quad \ldots\ldots\ldots\ldots(1.14)$$

where

$$\zeta_{n,\nu}^s=\frac{2n+1}{\pi}\int_0^{\pi}P_{s+1}^s P_n^{2\nu}\sin\theta d\theta\int_0^{\pi/2}e^{is\phi}\sin 2\nu\phi d\phi \ . \quad \ldots\ldots\ldots\ldots(1.15)$$

Substituting (1.15) in (1.10) and then comparing the corresponding coefficients of both sides, we get

$$K_{n,\nu}^s=\frac{a^2D\sigma ps}{in(n+1)}(a_{s+1}^s-ib_{s+1}^s)\zeta_{n,\nu}^s \ . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.16)$$

The current-function becomes

$$J=\frac{a^2D\sigma p}{i}\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\nu=0}^{\leqq n/2}\frac{s}{n(n+1)}(a_{s+1}^s-ib_{s+1}^s)\zeta_{n,\nu}^s e^{ispt}P_n^{2\nu}\sin 2\nu\phi \quad (1.17)$$

after all. Then we get current-intensity from (1.3) and (1.17).

$\zeta_{n,\nu}^s$'s can be easily obtained from (1.15). Their numerical values for $s=1$, 2, 3, $n=2, 3, \ldots, 7$ and $\nu=1, 2, 3$ are given in Table I.

Table I $\zeta_{n,\nu}^{s}$

	ν \ n	2	3	4	5	6	7
$s=1$	1	0	$0.0817(1+i)$	0	$0.0088(1+i)$	0	$0.0019(1+i)$
	2	——	——	0	$0.0147(1-i)$	0	$0.0041(1-i)$
	3	——	——	——	——	0	$0.0052(1+i)$
$s=2$	1	0	$0.167i$	0	0	0	0
	2	——	——	0	0.0432	0	0.0110
	3	——	——	——	——	0	0
$s=3$	1	0	$-0.0278(1-i)$	0	$0.0022(1-i)$		
	2	——	——	0	$0.0051(1-i)$		

Then the first three harmonics of the real part of J become

$$J=a^2D\sigma p(C_1\cos pt+S_1\sin pt+C_2\cos 2pt \\ +S_2\sin 2pt+C_3\cos 3pt+S_3\sin 3pt) \quad \text{.........(1.18)}$$

where

$$\left.\begin{aligned}
C_1=&(a_2^1-b_2^1)\,(0.0817\,P_3^2+0.0088\,P_5^2+0.0019\,P_7^2+\ldots)\sin 2\phi \\
&-(a_2^1+b_2^1)\,(0.0147\,P_5^4+0.0042\,P_7^4+\ldots)\sin 4\phi+(a_2^1-b_2^1) \\
&(0.0052\,P_7^6+\ldots)\sin 6\phi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
S_1=&(a_2^1+b_2^1)\,(0.0817\,P_3^2+0.0088\,P_5^2+0.0019\,P_7^2+\ldots)\sin 2\phi \\
&+(a_2^1-b_2^1)\,(0.0147\,P_5^4+0.0042\,P_7^4+\ldots)\sin 4\phi+(a_2^1+b_2^1) \\
&(0.0052\,P_7^6+\ldots)\sin 6\phi+ \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
C_2=&0.167\,a_3^2\,P_3^2\sin 2\phi-b_3^2\,(0.0432\,P_5^4+0.0110\,P_7^4+\ldots) \\
&\sin 4\phi+ \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
S_2=&0.167\,b_3^2\,P_3^2\sin 2\phi+a_3^2\,(0.0432\,P_5^4+0.0110\,P_7^4+\ldots) \\
&\sin 4\phi+ \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
C_3=&(a_4^3+b_4^3)\,\{(0.0278\,P_3^2-0.0022\,P_5^2+\ldots)\sin 2\phi \\
&-(0.0051\,P_5^4+\ldots)\sin 4\phi+\ldots\} \;\ldots\ldots\ldots\ldots\ldots\ldots, \\
S_3=&-(a_4^3-b_4^3)\,\{(0.0278\,P_3^2-0.0022\,P_5^2+\ldots)\sin 2\phi \\
&-(0.0051\,P_5^4+\ldots)\sin 4\phi+\ldots\} \;\ldots\ldots\ldots\ldots\ldots\ldots.
\end{aligned}\right\} \quad \text{...... (1.19)}$$

The lines on which C_1, S_1,...... are constant give the current-line that corresponds to the current-flows varying with t as $\cos pt$, $\sin pt$,...... respectively. For illustration, current-systems which correspond to C_1, S_1, C_2 and S_2 are shown in Figs. 1, 2, 3 and 4 respectively. The numerical values of a_{s+1}^s's and b_{s+1}^s's are taken from Chapman's analysis[3] of S_q, 1905 (sun-spot maximum). According to his analysis, the equinoctial mean of the harmonic coefficients

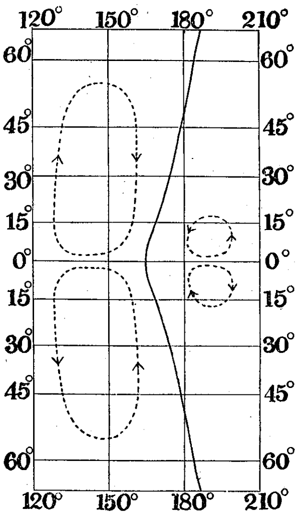

Fig. 1. The current-systems for $\cos pt$.

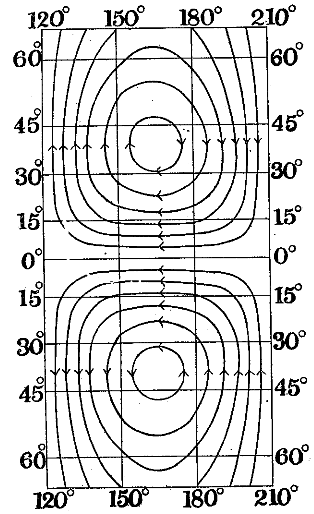

Fig. 2. The current-systems for $\sin pt$.

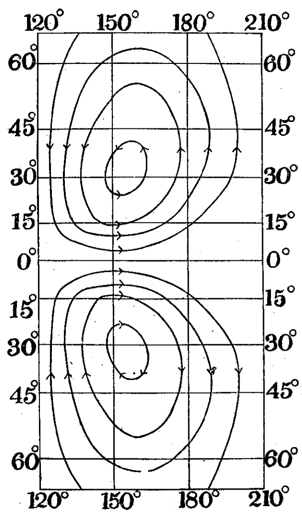

Fig. 3. The current-systems for $\cos 2pt$.

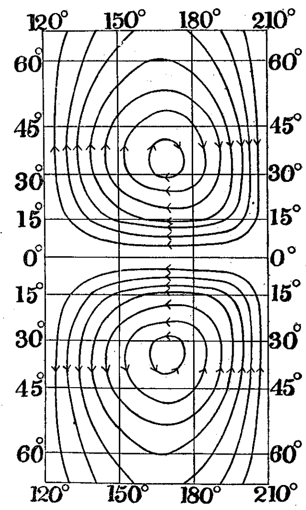

Fig. 4. The current-systems for $\sin 2pt$.

are given in Table II[13]. The time-origin of his analysis was taken at Greenwich midnight. When the west boundary of our ocean is assumed to be 120° E meridian which agrees roughly with that of Pacific Ocean, the harmonic coefficients referred to the 120° E meridian time or $\phi=0$ meridian time can be easily deduced from those values in Table II. These are given in Table III.

Table II. Equinoctial mean of the coef. for 1905. (after S. Chapman).

s	a_{s+1}^{s}	b_{s+1}^{s}
1	7.1 γ	$-2.4\,\gamma$
2	-6.6	0.1
3	4.4	-1.5
4	1.7	0.9

Table III. Reduced coef.

s	a_{s+1}^{s}	b_{s+1}^{s}
1	$-5.6\,\gamma$	$-5.0\,\gamma$
2	3.2	-5.8
3	4.4	-1.5
4	-0.8	-1.9

Table IV. The amount of eletcric current flowing between two adjacent lines.

D	ΔI
100 *m*	120 *amp.*
1000	1200
5000	6000

In order to estimate the intensity of the current-flow, we take $a=6400$ *km*, $\sigma=4\times 10^{-11}$ *emu*[14] and $p=2\pi/(60\times 60\times 24)$. Then the amount of the current-flow between two adjacent current-lines is given in Table IV in accordance with the depth of sea.

These electric currents produce magnetic fields whose potential at $P(r_0, \theta_0, \phi_0)$ is given by

$$W=a^2\int_{\theta=0}^{\pi}\int_{\phi=0}^{\pi/2} J(\theta,\phi)\,\frac{r_0\cos\gamma-a}{R^3}\sin\theta d\theta\, d\phi\,, \qquad (1.20)$$

where γ is defined by

$$\cos\gamma=\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)\,.$$

Meanwhile, we get out of the earth $(r_0>a)$

$$1/R=(1/r_0)\sum_k (a/r_0)^k P_k(\cos\gamma)\,.$$

Then differentiating with respect to a, we obtain

$$(r_0\cos\gamma-a)/R^3=(1/a^2)\sum_k k(a/r_0)^{k+1} P_k(\cos\gamma)\,.$$

13) According to the more recent analysis which will be referred to in § 2, these values differ slightly from those given in Table II. It is sufficient, however, to take Chapman's values to discuss the order of the influence of the oceans.

14) CHAPMAN and BARTELS, *Geomagnetism* p. 424.

Further, considering the relation

$$P_k(\cos\gamma)=\sum_{l=0}^{k} P_k^l(\cos\theta)\, P_k^l(\cos\theta_0)\cos l\,(\phi-\phi_0)\,,$$

the potential is given by the next expression

$$W=\sum_k\sum_l k(a/r_0)^{k+1} P_k^l(\cos\theta_0)\int_{\theta=0}^{\pi}\int_{\phi=0}^{\pi/2} J(\theta,\phi)\, P_k^l(\cos\theta)$$
$$\sin\theta\cos l\,(\phi-\phi_0)\, d\theta\, d\phi\,. \qquad \text{(1.21)}$$

Then we can express the magnetic field on the earth's surface with sum of spherical surface harmonics. This procedure, however, is so troublesome that we can not practically get the whole distribution of the magnetic force over the earth. Besides, as our ocean, which is bounded by two meridians, is merely an idealized model, it is more practical to make an estimate approximately of the order of the field-intensity than try hard to get mathematical exactness.

Taking Fig. 2, for example, we see that the electric current-flow is fairly parallel and uniform at the point on the equator of the centre of our ocean. Then, to a fair degree of approximation, the magnetic force will be the same as that due to an infinite horizontal plane uniform current-sheet with the same direction and intensity of current-flow. Let this current-intensity be j in electromagnetic unit, the horizontal magnetic force at this point will be $2\pi j$, northwards. Giving 120, 1200 and 6000 *amperes* per 5 *degrees* of latitude to j as was shown in Table IV, we get respectively 0.1, 1.4 and 7.0 *gammas*.

By means of a similar method, we can also estimate the order of the magnetic field at the boundaries. Near the middle latitude, the magnetic field will be approximately the same with that due to a half-infinite horizontal plane uniform current-sheet with intensity j. The horizontal magnetic field produced by this current-sheet is given by πj. Then, giving the same values to j, we get respectively 0.0, 1.0 and 5.0 *gammas*, eastwards at the west boundary in the northern hemisphere.

Thus, in the same way, we can estimate the horizontal magnetic force due to current-flows shown in Figs. 1, 3 and 4. The results in the northern hemisphere are tabulated in Table V in which we do not show 8-hourly component and higher harmonics, for their magnitudes are small compared with those of 24- and 12-hourly ones.

As to the vertical force, we can easily see that it will be affected in the main at the centres of eddies and will not be influenced on the equator at all. As to the quantitative estimate, it being rather complicated in this case, we cease to discuss in detail.

Tabel V. Magnetic force produced by the induced currents in the ocean.

	Depth	C_1	S_2	C_2	S_1
North component at the middle point on the equator	100 m	0.0 γ	0.1 γ	−0.1 γ	0.1 γ
	1000	0.0	1.4	−1.2	1.4
	5000	0.0	7.0	−5.8	7.0
West component at the middle latitude on the west boundary of the ocean	100	0.0	0.0	0.0	0.0
	1000	0.2	1.0	−0.8	1.0
	5000	1.0	5.0	−4.2	5.0

As was mentioned above, the horizontal magnetic force produced by these oceanic currents amounts to about 10 *gammas* at the middle point of the ocean whose electrical conductivity and depth are assumed to be 4×10^{-11} *emu* and 5000 m respectively. At the boundaries of the ocean, it amounts, in the middle latitude, to several *gammas* adopting the same value to the depth. Near the continent, however, the magnetic force may be smaller than this value because the sea becomes shallower.

After all, we presume that the influence of the ocean will become appreciable near the centre of the ocean, while, at the boundaries between land and sea, the effect of the ocean amounts to only a few *gammas* or less. This will be apt to be hidden by local irregularities of the external part of S_q.

As the distribution of land and sea on the real earth is very complicated, we can only discuss the possibility of the ocean-effect in regard to such an idealized and simple model as has been treated in this Section. But it is obvious that S_q may not be affected so much by the ocean that the field due to induced currents in the ocean occupies its main part. Probably, the magnetic field produced by the oceanic currents may amount to the order of one tenth or less of S_q except near the centre of the ocean. Besides, it depends upon universal time, while the external part of S_q seems to depend upon local time mainly. Then, on the average, the internal part of S_q obtained from spherical harmonic analysis in which it is assumed that S_q depends on local time can be considered, to a fair degree of approximation, as that is produced by the induced currents flowing in the deeper part of the earth.

Although, it is smaller compared with the whole amplitude of S_q, the possible ocean-effects, amounting to a few *gammas*, are not so small that we can not detect them at all if we could eliminate the fields which depend solely on local time and other irregularities. It will be worth while, then, to make an effort to study whether such influences are really existing or not. However, the exact elimination of the superposing fields may be almost impossible unfortunately because the actual distribution of land and sea is very much complicated. Thus we can, in the following paragraph, only make an attempt to abstract very roughly the influence of the ocean.

As shown in several figures in this Section, the induced currents in the ocean will produce magnetic fields which are almost the same in intensity and opposite in direction at the middle point of the both boundaries of the ocean. Besides, it is notable that the fields depend on universal time and not on local time at all. If two magnetic observatories, whose latitudes are the same are situated on both sides of the ocean, the daily variation of the magnetic declination will be affected equally in their magnitudes and oppositely in their directions. Then, assuming the other part of the variation depends on only local time and neglecting local irregularities, the daily variation of the declination may be expressed as follows;

$$\left.\begin{aligned} D_W &= F(t)+f(\tau) \\ D_E &= F(t)-f(\tau) \end{aligned}\right\} \quad \cdots\cdots(1.22)$$

where suffices W and E denote respectively the quantities belonging to the west and east-side observatories. t and τ are respectively local and universal time.

If we take the local time at the west-side observatory as the universal time, (1.22) becomes

$$D_W = F(t)+f(t)\,, \quad \cdots\cdots(1.23\text{ a})$$

$$D_E = F(t)-f(t-\alpha)\,, \quad \cdots\cdots(1.23\text{ b})$$

where α is the difference of the local time between the two station. Subtracting (1.23 b) from (1.23 a), we get

$$D_W - D_E = f(t)+f(t-\alpha)\,. \quad \cdots\cdots(1.24)$$

When we assume that $D_W - D_E$ and $f(t)$ can be expanded into Fourier series of the types

$$D_W - D_E = \sum_n (A_n \cos nt + B_n \sin nt)\,, \quad \cdots\cdots(1.25)$$

$$f(t) = \sum_n (a_n \cos nt + b_n \sin nt)\,, \quad \cdots\cdots(1.26)$$

the righthand-side of (1.24) becomes

$$\sum_n \{a_n(1+\cos n\alpha) - b_n \sin n\alpha\}\cos nt + \sum_n \{a_n \sin n\alpha + b_n(1+\cos n\alpha)\}\sin nt \quad \cdots\cdots(1.27)$$

Then equating the corresponding coefficients, we get

$$\left.\begin{aligned} a_n(1+\cos n\alpha)-b_n\sin n\alpha=A_n\,,\\ a_n\sin n\alpha-b_n(1+\cos n\alpha)=B_n\,. \end{aligned}\right\} \qquad\text{(1.28)}$$

Solving (1.28), it becomes

$$\left.\begin{aligned} a_n&=\frac{A_n(1+\cos n\alpha)+B_n\sin n\alpha}{(1+\cos n\alpha)^2+\sin^2 n\alpha}\,,\\ b_n&=\frac{-A_n\sin n\alpha+B_n(1+\cos n\alpha)}{(1+\cos n\alpha)^2+\sin^2 n\alpha}\,. \end{aligned}\right\} \qquad\text{(1.29)}$$

As α is known, we can get a_n and b_n from A_n and B_n. Then we are able to get $f(t)$, that is the part varies with universal time.

When we discuss the effect of Pacific Ocean, Kakioka (36.2° in latitude and 140.2° in longitude) and Tucson (32.2° in latitude and 249.2° in longitude) may approximately suffice the condition above mentioned. The harmonic coefficients of magnetic declination (+ denotes the west declination) averaged through the all international quiet days in 1924 are given in Table VI.

Table VI. Fourier coef. of declination for 1924.

	c_0	c_1	s_1	c_2	s_2
Kakioka	0.01′	−0.56′	−0.79′	−0.80′	0.70′
Tucson	0.00′	−0.25′	−1.13′	−0.35′	1.30′

Then making the difference, we get

$$A_1=0.31', \quad B_1=0.34', \quad A_2=-0.45', \quad B_2=-0.60'.$$

Taking into consideration that α is nearly 16 hours, we obtain from (1.29)

$$a_1=-0.45', \quad b_1=0.44', \quad a_2=-0.75', \quad b_2=0.09'.$$

In order to compare these coefficients to the estimate which has been done at the first part of this Section, we rewrite these coefficients referred to 120° E meridian time. Then it becomes

$$a_1=-0.33', \quad b_1=0.54', \quad a_2=-0.60', \quad b_2=0.46'.$$

If we assume that the horizontal intensity of the permanent magnetic field of the earth at both stations amounts to 0.28 *Gauss*, the 24- and 12-hourly components of f can be written respectively in the next form.

$$5.2\sin(t-2.0) \text{ and } 6.2\sin(2t-3.5)\,, \quad (\text{Unit in } \gamma) \qquad\text{(1.30)}$$

Meanwhile, from the estimate with respect to an idealized ocean bounded by two meridians $\pi/2$ apart in longitude, it is informed that they may become respectively

$$4.4\sin(t+0.7) \text{ and } 5.2\sin(2t-2.7), \quad (\text{Unit in } \gamma) \quad \ldots\ldots\ldots\ldots(1.31)$$

taking 4000 *m* as the depth of the ocean. This estimate was made for the year 1905. As the magnetic activities in 1905 and 1924[15] are of the same order, we may compare (1.30) to (1.31). The agreement between the theoretical estimate and that obtained from the observation seems good, especially the amplitude ratio of 24-hourly component to 12-hourly one. We must not, however, forget that we dare to make several assumptions fearlessly both in theory and in the treatment of the actual data. Accordingly, it is dangerous to conclude that the actual effect of Pacific Ocean is just of the type as expressed in (1 30) or (1.31). They are only rough approximations.

In short, according to the writer's opinion, there must be the influences of the presence of the ocean, such as Pacific Ocean, upon S_q. But they may not be so considerable that they will be almost eliminated in spherical harmonic analysis on the average assuming that S_q depends solely on local time. On the other hand, under suitable assumptions we can detect the ocean-effect though it is rather small and less reliable.

The ocean-effect on the other variations in the earth's magnetic field will be discussed later.

2. Spherical harmonic analyses of S_q which were done up to this time and their relations to the uniform core model of the earth.

As was briefly referred to in the Introduction of this paper, Chapman[3] was the first one who analyzed S_q, using data obtained from well-distributed observatories. The results of his analysis has been hitherto used by himself, Price and Lahiri, and Terada to investigate the cause of S_q and also to infer the electrical state of the earth's interior. In order to explain the relation between the external and the internal parts of S_q, both in amplitude and phase of main harmonics, Chapman took a model of the earth in which a conducting core whose electrical conductivity amounts to 3.6×10^{-13} *emu* is surrounded by non-conducting layer whose thickness is about 250 *km*. So far as we assume that the distribution of conductivity is uniform, this simple model is the best one. We shall call this "uniform core model" hereafter. We shall also call the uniform core model having the values determined by Chapman the "Chapman's model".

15) The annual means of the u-measure of magnetic activity are 0.85 and 0.74 for 1905 and 1924 respectively.

After Chapman's analysis in which he used the magnetic data for 1902 and 1905 obtained from 21 stations, however, the number of magnetic observatories increased remarkably during th's 1/4 century. Using the data of the Second Polar Year, 1932~1933, Hasegawa[16] made an analysis of S_q. On the other hand, the same analysis was executed by Benkova[17] with respect to the summer of 1933. As these analyses were based upon the observed data from observatories amounting to several tens in number and distributed widely all over the world, these two analyses may be more complete and reliable than Chapman's old one.

The amplitude ratio and phase difference between the external and the internal parts of main harmonics of their analyses are shown in Table VII

Table VII. The results of the spherical harmonic analyses.

n	m	Chapman		Hasegawa		Benkova	
		e/i	$\epsilon-\iota$	e/i	$\epsilon-\iota$	e/i	$\epsilon-\iota$
2	1	2.8	$-13°$	2.30	$-9°$	2.34	$-5°$
3	2	2.2	$-18°$	2.43	$-10°$	2.30	$-5°$
4	3	2.5	$-21°$	2.25	$-14°$	——	——
5	4	2.7	$-23°$	——	——	——	——
1	1	2.5	$-7°$	1.98	$-2°$	2.15	$3°$

together with that of Chapman's study.[18] According to Nagata, the numerical values of the conductivity and q (ratio of radius of the conducting core to that of the earth) in uniform core model which are consistent with both of amplitude ratio and phase difference are given in Table VIII. As also pointed out by Nagata, the discrepancies of σ and q between those which are expected from Chapman's analysis and those obtained from recent two analyses are considerable. Especially, it is remarkable that the values of σ and q obtained from each harmonic agree fairly well in Hasegawa's and Benkova's analyses, while, in Chapman's one, they agree not so well with each other as in these two

16) M. HASEGAWA, Read at the Annual Meeting (1943) of the Physico-Mathematical Society of Japan.

17) N. P. BENKOVA, *Terr. Mag.*, 45 (1940), 425.

18) This comparison was made by Dr. T. Nagata in connexion with comment on Chapman's analysis. Read at the Dec. (1946) Meeting of the Geophysical Institute, Tokyo Imperial University.

Table VIII. σ and q of uniform core model for each analysis.

n	m	Chapman		Hasegawa		Benkova	
		$\sigma(emu)$	q	$\sigma(emu)$	q	$\sigma(emu)$	q
2	1	8.8×10^{-13}	0.920	1.7×10^{-12}	0.943	5.1×10^{-12}	0.933
3	2	4.1×10^{-13}	0.968	1.3×10^{-12}	0.940	5.2×10^{-12}	0.936
4	3	3.4×10^{-13}	0.961	0.71×10^{-12}	0.961	——	——

analyses. With the superiority of data in mind, it will be natural to prefer the results of Benkova's analysis in discussing electromagnetic induction by S_q in so far as a new analysis will be done in the future. Hence, we get $\sigma=5\times10^{-12}$ *emu* and the thickness of the non-conducting layer is about 400 *km* namely, the core must be about ten times conductive and its radius must become slightly smaller than was estimated at the first time by Chapman.

3. Electromagnetic induction in the earth's model consisting of core, mantle and superficial non-conducting layer.

Electromagnetic induction by periodic field in a uniformly conducting sphere was fully discussed by Chapman, Whitehead, and Price[19]. Its application to the earth was outlined in Section 2 of this Chapter.

As mentioned in the Introduction of this paper, we have seismological evidences[20] that there must be a distinguishable discontinuity at the depth of 2900 *km* in the earth. As this discontinuity is so notable that we must presume that physical conditions of substances may differ in- and outside this boundary, it will be worth while to attempt to find how far the electrical property differs in both regions. We do not take into consideration the other discontinuities reported from researches on seismic waves such as at the depth of 1200 *km* and 60 *km* because they are comparatively small compared with that of the boundary just mentioned. In short, the purpose of this section is to construct an electrical model of the earth which is in accord with the earth's structure inferred from seismic waves.

Mathematical theory of electromagnetic induction in a sphere which has an inner core covered by outside layer of different conductivity is easily constructed in the same way as in a uniformly conducting sphere.

In conducting region, we have vector potential $\vec{A}$, which satisfies the relation

19) Mathematics of the electromagnetic induction in a uniformly conducting sphere is summarized in Chapman and Bartels' well known book "*Geomagnetism*" Vol. II, p. 732.

20) *e.g.* B. GUTENBERG, *ZS. f. Geophysik*, 3 (1927), 371.

$$\nabla^2 \vec{A} = 4\pi\sigma \frac{\partial \vec{A}}{\partial t} \quad \text{......(3.1)}$$

for the phenomenon is assumed to be quasistationary. Here, magnetic permeability μ is assumed to be unity for the reason mentioned in Chapter I. The typical term of the solution of (3.1) can be written in polar coordinate

$$\vec{A}_n^m = af_n(t, \rho)[\vec{r}\,\mathrm{grad}\, S_n^m], \quad (\rho = r/a) \quad \text{......(3.2)}$$

where a and S_n^m denote respectively radius of the sphere in question and spherical surface harmonic. f_n is a solution of the differential equation of the next type

$$\frac{\partial}{\partial\rho}\left(\rho^2 \frac{\partial f_n}{\partial\rho}\right) = \left\{n(n+1) + 4\pi a^2\sigma\rho^2 \frac{\partial}{\partial t}\right\} f_n \,. \quad \text{......(3.3)}$$

As the r, θ and ϕ components of $\vec{A}_n^m$ in (3.2) become respectively

$$0, \quad -af_n \frac{\partial S_n^m}{\sin\theta\partial\phi}, \quad af_n \frac{\partial S_n^m}{\partial\theta}, \quad \text{......(3.4)}$$

we get the components of the magnetic field as follows;

$$\left.\begin{aligned} H_r &= -\frac{1}{\rho} n(n+1) f_n S_n^m, \\ H_\theta &= -\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f_n)\frac{\partial S_n^m}{\partial\theta}, \\ H_\phi &= -\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f_n)\frac{\partial S_n^m}{\sin\theta\partial\phi}. \end{aligned}\right\}^{21)} \quad \text{......(3.5)}$$

On the other hand, there is, in non-conducting medium, magnetic potential whose typical term can be written in the type

$$W_n^m = a\{e_n(t)\rho^n + i_n(t)\rho^{-n-1}\} S_n^m, \quad \text{......(3.6)}$$

by which we get the components of the magnetic field as follows[22];

21) In the case of $\mu = 1$

$$\vec{H} = \mathrm{rot}\,\vec{A}$$

or writing the components, we easily get

$$H_r = \frac{\partial\,(A_\phi \sin\theta)}{r\sin\theta\,\partial\theta} - \frac{\partial A_\theta}{r\sin\theta\,\partial\phi},$$
$$H_\theta = \frac{\partial A_r}{r\sin\theta\,\partial\phi} - \frac{\partial(rA_\phi)}{r\partial r},$$
$$H_\phi = \frac{\partial(rA_\theta)}{r\partial r} - \frac{\partial A_r}{r\partial\theta}.$$

22) $\vec{H} = -\mathrm{grad}\; W$

$$\left.\begin{aligned} H_r &= -\{ne_n\rho^{n-1}-(n+1)\boldsymbol{i}_n\rho^{-n-2}\}S_n^m\,, \\ H_\theta &= -(e_n\rho^{n-1}+\boldsymbol{i}_n\rho^{-n-2})\frac{\partial S_n^m}{\partial\theta}\,, \\ H_\phi &= -(e_n\rho^{n-1}+\boldsymbol{i}_n\rho^{-n-2})\frac{\partial S_n^m}{\sin\theta\partial\phi}\,. \end{aligned}\right\} \qquad \text{(3.7)}$$

In our present model of the earth, as shown in Fig. 5, the continuity of the magnetic field must be satisfied at every boundary. We get, then, from (3.5) and (3.7)

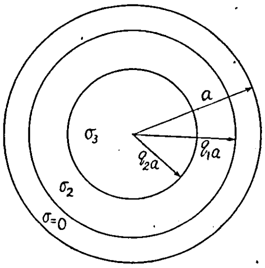

Fig. 5. Schematic view of the earth's model.

$$\left.\begin{aligned} & n(n+1)f_{2,n}(t,q_1)=ne_nq_1^n-(n+1)\boldsymbol{i}_nq_1^{-n-1}\,, \\ & q_1^{-1}f_{2,n}(t,q_2)+\left[\frac{\partial}{\partial\rho}f_{2,n}(t,\rho)\right]_{\rho=q_1}=e_nq_1^n+\boldsymbol{i}_nq_1^{-n-1}\,, \\ & f_{2,n}(t,q_2)=f_{3,n}(t,q_2)\,, \\ & q_2^{-1}f_{2,n}(t,q_2)+\left[\frac{\partial}{\partial\rho}f_{2,n}(t,\rho)\right]_{\rho=q_2}=q_2^{-1}f_{3,n}(t,q_2)+\left[\frac{\partial}{\partial\rho}f_{3,n}(t,\rho)\right]_{\rho=q_2}\,, \end{aligned}\right\} \qquad \text{(3.8)}$$

where

$$f_{2,n}(t,\rho)=C_2\rho^nF_{2,n}(t,\rho)+D_2\rho^{-n-1}G_{2,n}(t,\rho) \qquad \text{(3.9)}$$

and

$$f_{3,n}(t,\rho)=C_3\rho^nF_{3,n}(t,\rho)\,.\ ^{23)} \qquad \text{(3.10)}$$

In these expressions, C_2, D_2 and C_3 denote functions of t, while $F_{\nu,n}(t,\rho)$ and $G_{\nu,n}(t,\rho)$ are the solutions of the equations

$$\left\{\frac{\partial^2}{\partial\rho^2}+\frac{2(n+1)}{\rho}\frac{\partial}{\partial\rho}-4\pi a^2\sigma_\nu\frac{\partial}{\partial t}\right\}f_{\nu,n}=0\,,\quad(\nu=2,3) \qquad \text{(3.11a)}$$

or writing in operational form

$$\left\{\frac{d^2}{d\rho^2}+\frac{2(n+1)}{\rho}\frac{d}{d\rho}-k_\nu^2a^2\right\}f_{\nu,n}(p,\rho)=0\,, \qquad \text{(3.11b)}$$

23) The coefficient of $G_{3,n}(t,\rho)$ is taken to be zero because the field must remain finite at the centre.

where, putting $\partial/\partial t = p$

$$k_\nu^2 = 4\pi\sigma_\nu p\,. \qquad \text{(3.12)}$$

Solving (3.12), we obtain putting $x = k_\nu \rho$

$$F_n(x) = 1 + \frac{x^2}{2(2n+3)} + \frac{x^4}{2\cdot 4\cdot(2n+3)(2n+5)} + \dots\,, \qquad \text{(3.13)}$$

$$G_n(x) = e^{-x}\left\{1 + x + \frac{2(n-1)}{2!(2n-1)}x^2 + \frac{2^2(n-2)(n-1)}{3!(2n-2)(2n-1)}x^3 + \dots\right\}, \qquad \text{(3.14)}$$

for $x < 1$ and further

$$\frac{F_{n-1}(x)}{F_n(x)} = \frac{x}{2n+1}\left\{1 + \frac{n}{x} + \frac{n(n+1)}{2x^2} + \frac{n(n+1)}{2x^3} \dots\right\}, \qquad \text{(3.15)}$$

$$\frac{G_{n+1}(x)}{G_n(x)} = \frac{x}{2n+1}\left\{1 + \frac{n+1}{x} + \frac{n(n+1)}{2x^2} - \frac{n(n+1)}{2x^3} \dots\right\}, \qquad \text{(3.16)}$$

for large values of the real part of x. The nature of F_n and G_n were fully investigated in the book "Geomagnetism"[24]. Solving the simultaneous equations (3.8) we get the relations between $e_n(t)$, $i_n(t)$, $C_2(t)$, $D_2(t)$ and $C_3(t)$ by means of operators in the forms

$$i_n(t) = I(p)e_n(t)\,, \qquad \text{(3.17)}$$

$$C_2(t) = \overline{C}_2(p)e_n(t)\,, \qquad \text{(3.18)}$$

$$D_2(t) = \overline{D}_2(p)e_n(t)\,, \qquad \text{(3.19)}$$

$$C_3(t) = \overline{C}_3(p)e_n(t)\,, \qquad \text{(3.20)}$$

in which

$$I(p) = -q_1^{2n+1}\frac{n}{n+1}\frac{\left\{1 - \frac{F_{n-1}(2,1)}{F_n(2,1)}\right\}\left\{1 - \frac{F_{n-1}(3,2)}{F_n(3,2)} - \frac{G_{n+1}(2,2)}{G_n(2,2)}\right\} + \tau^{2n+1}\left\{\frac{F_{n-1}(3,2)}{F_n(3,2)} - \frac{F_{n-1}(2,2)}{F_n(2,2)}\right\}\frac{F_n(2,2)}{F_n(2,1)}\frac{G_n(2,1)}{G_n(2,2)}}{\frac{F_{n-1}(2,1)}{F_n(2,1)}\left\{1 - \frac{F_{n-1}(3,2)}{F_n(3,2)} - \frac{G_{n+1}(2,2)}{G_n(2,2)}\right\} + \tau^{2n+1}\left\{1 - \frac{G_{n+1}(2,1)}{G_n(2,1)}\right\}\left\{\frac{F_{n-1}(3,2)}{F_n(3,2)} - \frac{F_{n-1}(2,2)}{F_n(2,2)}\right\}\frac{F_n(2,2)}{F_n(2,1)}\frac{G_n(2,1)}{G_n(2,2)}}\,, \qquad \text{(3.21)}$$

$$\overline{C}_2(p) = \frac{1}{n+1}\frac{1}{F_n(2,1)}\frac{1 - \frac{F_{n-1}(3,2)}{F_n(3,2)} - \frac{G_{n+1}(2,2)}{G_n(2,2)}}{\frac{F_{n-1}(2,1)}{F_n(2,1)}\left\{1 - \frac{F_{n-1}(3,2)}{F_n(3,2)} - \frac{G_{n+1}(2,2)}{G_n(2,2)}\right\} + \tau^{2n+1}\left\{1 - \frac{G_{n+1}(2,1)}{G_n(2,1)}\right\}\left\{\frac{F_{n-1}(3,2)}{F_n(3,2)} - \frac{F_{n-1}(2,2)}{F_n(2,2)}\right\}\frac{F_n(2,2)}{F_n(2,1)}\frac{G_n(2,1)}{G_n(2,2)}}\,, \qquad \text{(3.22)}$$

$$\overline{D}_2(p) = q_2^{2n+1}\frac{1}{n+1}\frac{1}{G_n(2,2)}\frac{F_n(2,2)}{F_n(2,1)}\frac{\frac{F_{n-1}(3,2)}{F_n(3,2)} - \frac{F_{n-1}(2,2)}{F_n(2,2)}}{\frac{F_{n-1}(2,1)}{F_n(2,1)}\left\{1 - \frac{F_{n-1}(3,2)}{F_n(3,2)} - \frac{G_{n+1}(2,2)}{G_n(2,2)}\right\} + \tau^{2n+1}\left\{1 - \frac{G_{n+1}(2,1)}{G_n(2,1)}\right\}\left\{\frac{F_{n-1}(3,2)}{F_n(3,2)} - \frac{F_{n-1}(2,2)}{F_n(2,2)}\right\}\frac{F_n(2,2)}{F_n(2,1)}\frac{G_n(2,1)}{G_n(2,2)}}\,, \qquad \text{(3.23)}$$

24) CHAPMAN and BARTELS, *loc. cit.* Vol. II, p. 738.

$$\bar{C}_3(p)=\frac{1}{n+1}\frac{1}{F_n(3,2)}\frac{F_n(2,2)}{F_n(2,1)}\frac{1-\dfrac{F_{n-1}(2,2)}{F_n(2,2)}-\dfrac{G_{n+1}(2,2)}{G_n(2,2)}}{\dfrac{F_{n-1}(2,1)}{F_n(2,1)}\left\{1-\dfrac{F_{n-1}(3,2)}{F_n(3,2)}-\dfrac{G_{n+1}(2,2)}{G_n(2,2)}\right\}+\tau^{2n+1}\left\{1-\dfrac{G_{n+1}(2,1)}{G_n(2,1)}\right\}\left\{\dfrac{F_{n-1}(3,2)}{F_n(3,2)}-\dfrac{F_{n-1}(2,2)}{F_n(2,2)}\right\}\dfrac{F_n(2,2)}{F_n(2,1)}\dfrac{G_n(2,1)}{G_n(2,2)}}. \tag{3.24}$$

For the sake of simplicity, $F(k_\nu q_\mu a)$ and $G(k_\nu q_\mu a)$ are written respectively as $F(\nu,\mu)$ and $G(\nu,\mu)$ in the above expression. We also write τ in place of q_2/q_1.

When the inducing field is periodic, the induced field becomes also periodic. In the case of period $2\pi/\alpha$, it is easily seen that the amplitude ratio i/e and the phase difference $\iota-\epsilon$ of the field at the surface $(r=a)$ are given by

$$i/e=\mathrm{mod}\, I(i\alpha)\,, \qquad \iota-\epsilon=\arg I(i\alpha) \tag{3.25}$$

with $i\alpha$ in place of p.
In the case of S_q, in so far as it depends solely on local time, we get

$$\alpha=2\pi m/(60\times 60\times 24) \tag{3.26}$$

measuring t in seconds. m is the order of S_n^m.
We write

$$k_\nu^2 q_\mu^2 a^2=2i\beta_{\nu,\mu}^2\,, \qquad \beta_{\nu,\mu}^2=\frac{4\pi^2\sigma_\nu m q_\mu^2 a^2}{24\times 60\times 60}\,, \tag{3.27}$$

so that $\beta_{\nu,\mu}^2$ is real and positive. Then it becomes

$$k_\nu q_\mu a=\beta_{\nu,\mu}(1+i)\,, \qquad 1/k_\nu q_\mu a=(1-i)/2\beta_{\nu,\mu}\,. \tag{3.28}$$

As the lowest value of $\beta_{\nu,\mu}$ is about 7 in the case of daily variation, we are able to calculate approximately F_{n-1}/F_n and G_{n+1}/G_n by use of (3.15) and (3.16).
Writing

$$\frac{F_{n-1}(\nu,\mu)}{F_n(\nu,\mu)}=A_{\nu\mu}+iB_{\nu\mu}\,, \qquad \frac{G_{n+1}(\nu,\mu)}{G_n(\nu,\mu)}=C_{\nu\mu}+iD_{\nu\mu}\,, \tag{3.29}$$

where $A_{\nu\mu}$, $B_{\nu\mu}$, $C_{\nu\mu}$ and $D_{\nu\mu}$ are given by

$$\left.\begin{aligned}
A_{\nu\mu}&=\frac{\beta_{\nu\mu}}{2n+1}\left\{1+\frac{n}{\beta_{\nu\mu}}+\frac{n(n+1)}{4\beta_{\nu\mu}^2}\cdots\cdots\right\},\\
B_{\nu\mu}&=\frac{\beta_{\nu\mu}}{2n+1}\left\{1-\frac{n(n+1)}{4\beta_{\nu\mu}^2}-\frac{n(n+1)}{4\beta_{\nu\mu}^3}\cdots\right\},\\
C_{\nu\mu}&=\frac{\beta_{\nu\mu}}{2n+1}\left\{1+\frac{n+1}{\beta_{\nu\mu}}+\frac{n(n+1)}{4\beta_{\nu\mu}^2}\cdots\cdots\right\},\\
D_{\nu\mu}&=\frac{\beta_{\nu\mu}}{2n+1}\left\{1-\frac{n(n+1)}{4\beta_{\nu\mu}^2}+\frac{n(n+1)}{4\beta_{\nu\mu}^3}\cdots\right\},
\end{aligned}\right\} \tag{3.30}$$

On the other hand, it is known that F_n and G_n become approximately

$$F_n(x)=\frac{(2n+1)!}{2^n n!}\frac{e^x}{x^n}\rho_n(x),\qquad {}^{25)}$$

$$G_n(x)=-\frac{2^n(n-1)!}{(2n-1)!}x^{n+1}e^{-x}\rho_n(x),$$

where

$$\rho_n(x)=\frac{1}{2x}-\frac{n(n+1)}{(2x)^2}+\frac{(n-1)n(n+1)(n+2)}{2!(2x)^3}-\cdots\cdots.$$

Then we obtain approximately

$$\frac{F_n(x)}{G_n(x)}=\frac{(2n+1)!(2n-1)!}{2^{2n}n!(n-1)!}\frac{e^{2x}}{x^{2n+1}}$$

for large values of the real part of x.
Then it becomes

$$\frac{F_n(2,2)}{G_n(2,2)}\frac{G_n(2,1)}{F_n(2,1)}=E+iF \qquad\cdots\cdots(3.31)$$

where

$$\left.\begin{aligned}E&=(\beta_{21}/\beta_{22})^{2n+1}e^{2(\beta_{22}-\beta_{21})}\cos 2(\beta_{22}-\beta_{21}),\\ F&=(\beta_{21}/\beta_{22})^{2n+1}e^{2(\beta_{22}-\beta_{21})}\sin 2(\beta_{22}-\beta_{21}).\end{aligned}\right\}\qquad\cdots\cdots(3.32)$$

Finally, we obtain

$$I(i\alpha)=\frac{n}{n+1}q_1^{2n+1}\frac{\xi_2+i\eta_2}{\xi_1+i\eta_1}\qquad\cdots\cdots(3.33)$$

where

$$\left.\begin{aligned}\xi_1=&A_{21}(1-A_{32}-C_{22})+B_{21}(B_{32}+D_{22})\\ &+\tau^{2n+1}[E\{(1-C_{21})(A_{32}-A_{22})+D_{21}(B_{32}-B_{22})\}\\ &-F\{(1-C_{21})(B_{32}-B_{22})-D_{21}(A_{32}-A_{22})\}],\\ \eta_1=&B_{21}(1-A_{32}-C_{22})-A_{21}(B_{32}+D_{22})\\ &+\tau^{2n+1}[E\{(1-C_{21})(B_{32}-B_{22})-D_{21}(A_{32}-A_{22})\}\\ &+F\{(1-C_{21})(A_{32}-A_{22})+D_{21}(B_{32}-B_{22})\}],\\ \xi_2=&-(1-A_{21})(1-A_{32}-C_{22})+B_{21}(B_{32}+D_{22})\\ &-\tau^{2n+1}\{E(A_{32}-A_{22})-F(B_{32}-B_{22})\},\\ \eta_2=&B_{21}(1-A_{32}-C_{22})+(1-A_{21})(B_{32}+D_{22})\\ &-\tau^{2n+1}\{E(B_{32}-B_{22})+F(A_{32}-A_{22})\}.\end{aligned}\right\}\qquad\cdots\cdots(3.34)$$

25) CHAPMAN and BARTELS, *loc. cit* p. 739.

Now using ξ_1, η_1, ξ_2 and η_2 given in (3.33), we obtain from (3.25)

$$e/i=q_1^{-2n-1}\frac{n+1}{n}\sqrt{\frac{\xi_1^2+\eta_1^2}{\xi_2^2+\eta_2^2}}, \qquad \epsilon-\iota=\tan^{-1}\frac{\eta_1}{\xi_1}-\tan^{-1}\frac{\eta_2}{\xi_2} \quad \cdots\cdots\cdots(3.35)$$

Thus we can estimate the amplitude ratio and the phase difference which correspond to the given electrical conductivity in the mantle and the core of our model of the earth.

The writer applies the above mentioned theory to the actual geomagnetic problem in two cases. The first is based on Chapman's analysis by which the electrical state of the earth's interior was hitherto discussed. The second is on the basis of Benkova's analysis which seems more reliable than the former. According to Chapman's analysis which is given in Table VII, it is believed that P_3^2 is the most reliable because its coefficients remain almost constant in every analysis for different years and seasons as shown in his original paper. Then, we shall attempt to determine σ_2 and σ_3 consistent with the amplitude ratio and the phase difference for $n=3$ and $m=2$ in the first place.

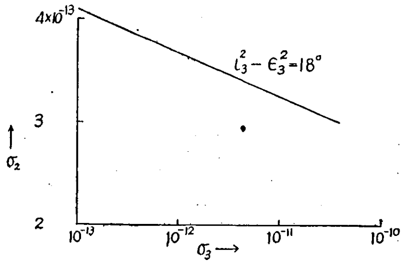

Fig. 6. Combination of σ_2 and σ_3 by which we obtain $\iota_3^2-\epsilon_3^2=18°$.

As q_1 does not differ very much from unity[26], the combination of σ_2 and σ_3 by which we get approximately $\iota_3^2-\epsilon_3^2=18°$ is calculated from (3.35). As appears in Fig. 6, σ_2 is more effective than σ_3. In this calculation, q_2 is taken to be 0.540 for the inner

Table IX.

σ_2		3.8×10^{-13}	3.4×10^{-13}	3.0×10^{-13}	2.6×10^{-13}	Chapman's
n	m \ σ_3	3×10^{-13}	3×10^{-12}	3×10^{-11}	3×10^{-10}	Analysis
2	1	—	—	18°	—	13°
3	2	18°	18°	18°	18°	18°
4	3	18°	19°	21°	22°	21°
5	4	19°	20°	22°	24°	23°

26) In such a case, the phase difference will not practically affected by the small difference of the numerical value of q_1.

boundary of the mantle is of the depth of 2900 *km*. Phase differences of main harmonics are given in Table IX for the combinations of σ_2 and σ_3, where q_1 is assumed to be nearly unity. Comparing these values to Chapman's analysis, we find that the combination $\sigma_2=3.0\times10^{-13}$ *emu*, $\sigma_3=3\times10^{-11}$ *emu* is the best. Then, taking this combination, amplitude ratio of main harmonics are calculated for several values of q_1 as appeared in Table X. As is easily

Table X.

n	m \ q_1	0.985	0.980	0.975	0.970	Chapman's Analysis
2	1	2.3	2.3	2.4	2.4	2.8
3	2	2.1	2.1	2.2	2.3	2.2
4	3	2.0	2.1	2.2	2.3	2.5
5	4	2.0	2.1	2.3	2.4	2.7

seen in this Table, we prefer $q_1=0.97$ to let the calculation approximately agree with the results of the actual analysis. Ultimately, it is determined that the conductivity amounts to 3×10^{-11} *emu* in the core, 3.0×10^{-13} *emu* in the mantle and the thickness of the non-conducting layer is 200 *km* as the most probable conductivity-distribution in the earth.

As already mentioned in Section 2 in this Chapter, the newly made analyses showed us that Chapman's analysis might be somewhat rough. Then, to make the same determination of conductivity-distribution on the basis of Benkova's analysis will be worth while. Although only the coefficients of P_2^1 and P_3^2 are obtained by Benkova, we can get the most probable conductivity-distribution in the same way just mentioned. Thus, the electrical conductivity is determined to be 5.0×10^{-11} *emu* in the core and also 5×10^{-12} *emu* in the mantle, while q_1 comes to be 0.94, that is, the thickness of the outermost non-conducting layer must be about 400 *km*. The comparison of the calculation to the actual analysis is shown in Table XI.

Table XI.

n	m	e_n^m/i_n^m Cal.	e_n^m/i_n^m Obs.	$\epsilon_n^m-\iota_n^m$ Cal.	$\epsilon_n^m-\iota_n^m$ Obs.
2	1	2.29	2.34	$-5.0°$	$-5°$
3	2	2.26	2.30	$-4.9°$	$-5°$

4. The distribution of the induced currents in the earth.

Now, let us investigate how deep the electric currents induced dy S_q penetrate into the earth.

As the relation between the current-density $\overrightarrow{c}$ and the vector potential $\overrightarrow{A}$ is given by

$$\overrightarrow{c}=-\sigma\frac{\partial \overrightarrow{A}}{\partial t},\qquad\text{(4.1)}$$

we get from (3.2)

$$\overrightarrow{c}{}_n^m=-\sigma a\frac{\partial f_n}{\partial t}[\overrightarrow{r}\,\mathrm{grad}\,S_n^m]\qquad\text{(4.2)}$$

corresponding to the harmonic whose degree and order are respectively n and m.

Considering (3.9), (3.10), (3.18) and (3.20), we can write the expressions of current-density for the mantle and the core in operational forms such as

$$\overrightarrow{c}{}_{n,2}^m=-\sigma_2ap\{\overline{C}_2(p)\rho^nF_n(k_2\rho a)+\overline{D}_2(p)\rho^{-n-1}G_n(k_2\rho a)\}e_n(t)[\overrightarrow{r}\,\mathrm{grad}\,S_n^m],\ldots\text{(4.3)}$$

$$\overrightarrow{c}{}_{n,3}^m=-\sigma_3ap\overline{C}_3(p)\rho^nF_n(k_3\rho a)e_n(t)[\overrightarrow{r}\,\mathrm{grad}\,S_n^m],\qquad\text{(4.4)}$$

where $\overline{C}_2(p)$, $\overline{D}_2(p)$ and $\overline{C}_3(p)$ are given by (3.22), (3.23) and (3.24). Putting $p=i\alpha$, we obtain current intensity for any value of ρ by taking the moduli of $\overrightarrow{c}{}_{n,2}^m$ an1 $\overrightarrow{c}{}_{n,3}^m$ which correspond respectively to the mantle and the core. As the calculation can be easily performed in the same way with that of mod $I(i\alpha)$ in Section 3, we shall only describe the results, neglecting the detail of the numerical calculation.

As determined in the foregoing Section of this Chapter, we take respectively 5.0×10^{-12} and 5×10^{-11} *emu* for the electrical conductivity in the mantle and the core on the basis of Benkova's analysis. The distribution of the current-intensity in such an earth's model is shown in Fig. 7 in arbitrary scale. The rate of decrease of the current intensity seems so rapid that it becomes practically zero at $\rho=0.7$, that is, at the depth of 2000 *km*. For this reason, the currents induced in the core are very small compared with those induced in the mantle. Accordingly, the contribution of the currents flowing

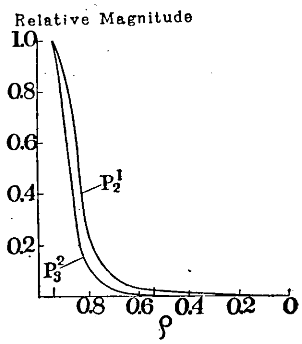

Fig. 7. The distribution of the induced currents in the earth.

in the core to the magnetic field at the earth's surface is very small. Thus, we must not forget that the determined conductivity of the core is rather ambiguous. The more accurate determination may only be effected by investigating much slower variation than S_q.

Indeed, considering that the contribution of the currents flowing in a thin shell whose radius is ρa to the magnetic field at the surface is proportional to $\rho^{n+2}|\vec{c}|$ for the harmonic whose degree is n, we obtain the contribution of the currents of any depth as shown in Fig. 8 for 24-hourly and 12-hourly components of S_q, where it is easily seen that 90% of the field are due to the currents flowing in the upper part of the mantle, from the surface of the mantle to the depth of 1400 *km* for 24-hourly component and 960 *km* for 12-hourly component. These estimates, however, are not quite accurate because the phase of the currents differ for different depths. But, as already pointed out by Chapman and Price[8], the phase difference is slight, especially in high conducting sphere. Then, to a fair degree of approximation, we conclude that the magnetic field at the surface must be mainly due to the currents in the upper part of the mantle.

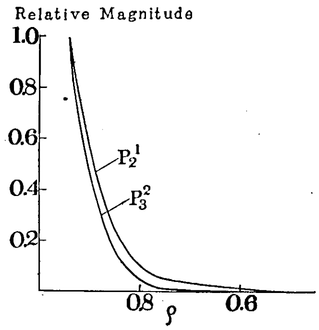

Fig. 8. The contribution of the currents to the magnetic field at the earth's surface.

Summarizing the results obtained in the present Chapter, we may say that the earth is composed of the outermost non-conducting layer, about 400 *km* in thickness, the mantle whose electrical conductivity amounts to 5.0×10^{-12} *emu*, and, as is well known from seismology, the core in which the conductivity seems likely to amount as much as ten times or more compared with that of the mantle. Though the determination of the conductivity in the core is rough, the results seem to be in accord with that of Lahiri and Price[10] in which the increase of the conductivity with increasing depth was decisively reported. As to the more accurate determination of the conductivity in the deeper part of the earth, we must treat electromagnetic induction by much slower variations with induced currents which penetrate very deeply.

CHAPTER III. ELECTROMAGNETIC INDUCTION BY SOLAR DAILY DISTURBANCE VARIATION (so-called S_D -field).

In a series of papers, Chapman[5), 27), 28)] studied the average characteristics of magnetic disturbances or storms, deriving possible atmospheric current-systems for them. According to him, magnetic disturbances can be separated into storm-time (so-called D_{st}) and solar daily disturbance variation (so-called S_D). As already mentioned in Introduction, D_{st} was hitherto subjected to spherical harmonic analysis[4)] with applications to the investigation on the earth's electrical state[8), 10)]. On the other hand, S_D was not yet used for the problem in question.

On the basis of magnetic data collected by himself, Chapman proposed the atmospheric electric current-systems for S_D as shown in Fig. 9. Though the

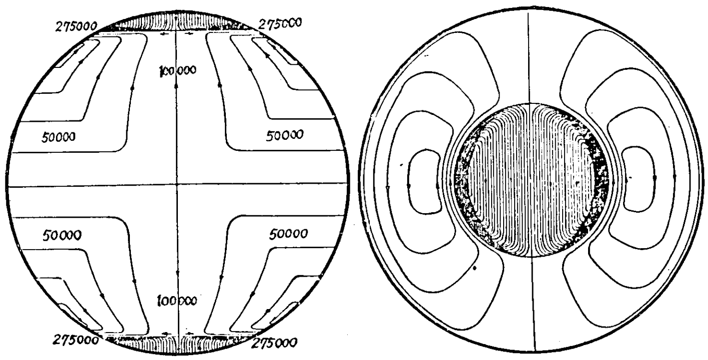

Fig. 9. Idealized current-systems in amperes for S_D, assuming magnetic and geographic axes coincident. Left: View from equator at noon meridian, Right: View from North Pole (After S. Chapman)

current-systems were derived for an ideal earth having its geomagnetic and geographic axes coincident, it is remarkable that the electric currents concentrate so intensely above the auroral zones that the currents above the polar caps and the low and middle latitude seem to be mere leak from the auroral zones. The characteristics of S_D current-systems was also theoretically confirmed by the writer[29)] on the stand-point of "dynamo" -theory as well as the theory of bay-type disturbance.

27) S. CHAPMAN, *Proc. Roy. Soc.* A, **115** (1927) 242.
28) S. CHAPMAN, *Terr. Mag.*, **40** (1935) 349.
29) T. RIKITAKE, *Rep. Ionos. Res. Japan*, 2 (1948), 57.

Since the feature of the current-systems for S_D differ very much from that for S on quiet days or S_q, it will be of interest and useful to check our foregoing studies by discussing the electromagnetic induction by S_D. However, we are obliged to treat our problem rather unsatisfactorily on account of the shortage of available data. So far as the writer could get, the most utilizable data were given by Vestine[30] who showed, as illustrated in Fig. 10, the vari-

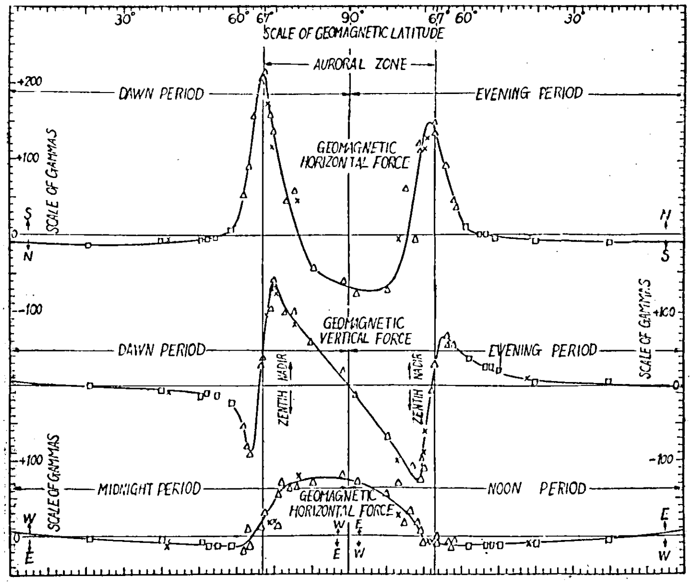

Fig. 10. Variations with latitude of maxima and minima of geomagnetic components of S_D for international disturbed minus quiet days. (After E. H. Vestine)

ations with latitudes found in the maximum magnitudes of the components of S_D for dawn, noon, evening and midnight periods on the basis of data for the Second International Polar Year and auxiliary for various epochs from 1911 to 1933. With the aid of the smoothed curves drawn among the points given by observation, we can analyze in like manner with S_q. The separation of the magnetic potential into the external and the internal parts will be done only in the case of 24-hourly component because the variations of shorter period seem very small. Discussions on the phase difference between two parts will be neglected because the data are not satisfactory for this purpose.

When we assume that the variation depends solely on local time, the magnetic potential is generally expressed by

30) E. H. VESTINE, *Terr. Mag.*, 43 (1938), 261.

$$W=\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left\{\left(e_{n,a}^{m}\frac{r^n}{R^{n-1}}+i_{n,a}^{m}\frac{R^{n+2}}{r^{n+1}}\right)\cos mt\right.$$
$$\left.+\left(e_{n,b}^{m}\frac{r^n}{R^{n-1}}+i_{n,b}^{m}\frac{R^{n+2}}{r^{n+1}}\right)\sin mt\right\}P_n^m(\cos\theta) \quad \ldots\ldots\ldots\ldots\ldots(1)$$

in polar coordinate, where R denotes the earth's radius. As is well known, e_n^m's and i_n^m's are respectively the coefficients of the harmonics whose origin exist outside and within the earth. Then the components of the magnetic field at the earth's surface can be expressed as follows;

$$\left.\begin{array}{lll}\text{North-component}: & X=\sum\limits_{n}\sum\limits_{m} n\{(e_{n,a}^{m}+i_{n,a}^{m})\cos mt \\ & \qquad +(e_{n,b}^{m}+i_{n,b}^{m})\sin mt\}\,X_n^m\,, \\ \text{East-} \quad ,, & : \; Y=\sum\limits_{n}\sum\limits_{m} n\{-(e_{n,a}^{m}+i_{n,a}^{m})\sin mt \\ & \qquad +(e_{n,b}^{m}+i_{n,b}^{m})\cos mt\}\,Y_n^m\,, \\ \text{Downward-} \quad ,, & : \; Z=\sum\limits_{n}\sum\limits_{m}\{(ne_{n,a}^{m}-\overline{n+1}i_{n,a}^{m})\cos mt \\ & \qquad +(ne_{n,b}^{m}-\overline{n+1}i_{n,b}^{m})\sin mt\}\,P_n^m\,. \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

where

$$X_n^m=\frac{1}{n}\frac{dP_n^m}{d\theta}, \qquad Y_n^m=\frac{m}{n\sin\theta}P_n^m, \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

as defined by Schmidt[31].

On the other hand, we get by means of spherical harmonic analysis

$$\left.\begin{array}{lll}\text{North-component}: & X=\sum\limits_{n}\sum\limits_{m}(a_n^m\cos mt+b_n^m\sin mt)\,X_n^m\,, \\ \text{East-} \quad ,, & : \; Y=\sum\limits_{n}\sum\limits_{m}(a_n^{m\prime}\cos mt+b_n^{m\prime}\sin mt)\,Y_n^m\,, \\ \text{Downward-} \quad ,, & : \; Z=\sum\limits_{n}\sum\limits_{m}(\tilde{a}_n^m\cos mt+\tilde{b}_n^m\sin mt)\,P_n^m\,. \end{array}\right\} \quad \ldots\ldots\ldots(4)$$

Equating, then, the corresponding term of X and Z in (3) and (4), we have

$$n(e_{n,a}^{m}+i_{n,a}^{m})=a_n^m\,, \qquad n(e_{n,b}^{m}+i_{n,b}^{m})=b_n^m\,,$$
$$ne_{n,a}^{m}-\overline{n+1}\,i_{n,a}^{m}=\tilde{a}_n^m\,, \qquad ne_{n,b}^{m}-\overline{n+1}\,i_{n,a}^{m}=\tilde{b}_n^m\,.$$

Solving this we obtain

$$e_{n,a}^{m}=\frac{(n+1)\,a_n^m+n\tilde{a}_n^m}{n(2n+1)}, \qquad i_{n,a}^{m}=\frac{a_n^m-\tilde{a}_n^m}{2n+1}, \text{ etc.} \ldots\ldots\ldots\ldots\ldots(5)$$

The same determination can be also done from Y and Z in (3) and (4).

31) AD. SCHMIDT, *Tafeln der Normierten Kugelfunktionen*, (1935).

Thus we can separate the external and internal parts of the magnetic potential from the combination of X, Z or Y, Z. In practice, the combination X and Z is taken in this Chapter.

In the case of S_D, b_n^m and $\tilde{b}_n^m$ are taken to be zero with suitable preferrence of the time origin, that is, it becomes, on the average, near 18 h of local time according to Vestine. Besides, as the 24-hourly component is the most predominant, we assume $m=1$. In that case we get from (4)

$$X=\sum_{n=1}^{\infty} a_n^1 X_n^1 \cos t\,, \quad \text{.........................(6)}$$

$$Z=\sum_{n=1}^{\infty} \tilde{a}_n^1 P_n^1 \cos t\,. \quad \text{..........................(7)}$$

On account of the orthogonality of $P_n{}^1$, the coefficients of (7) are given by

$$\tilde{a}_n^1=\frac{2n+1}{4\pi}\int_0^{\pi} Z(\theta)\, P_n^1 \sin\theta\, d\theta^{32)}. \quad \text{......................(8)}$$

To get $a_n{}^1$ from (6), we consider the expansion

$$X\sin\theta=\sum_{n=1}^{\infty} c_n^1 P_n^1 \cos t\,.$$

Meanwhile, it becomes by the well known recurrence formula

$$X\sin\theta=\sum_{n=1}^{\infty}\frac{a_n^1}{n}\sin\theta\frac{dP_n^1}{d\theta}\cos t$$

$$=\sum_{n=1}^{\infty} a_n^1\frac{n+1}{2n+1}\left\{\frac{\sqrt{(n+1)^2-1}}{n+1}P_{n+1}^1-\frac{\sqrt{n^2-1}}{n}P_{n-1}^1\right\}\cos t\,.$$

Then, as is in the case of Z, multiplying both side of this equation by $P_s{}^1 \sin\theta$ and integrating from $\theta=0$ to $\theta=\pi$, we get

$$-a_2^1\frac{3}{5}\frac{\sqrt{3}}{2}=c_1^1\,,\quad a_2^1\frac{3}{5}\frac{\sqrt{8}}{3}-a_4^1\frac{5}{9}\frac{\sqrt{15}}{4}=c_3^1\,,$$

$$a_4^1\frac{5}{9}\frac{\sqrt{24}}{5}-a_6^1\frac{7}{13}\frac{\sqrt{35}}{6}=c_5^1\,,\ \text{........}\qquad \text{.........(9a)}$$

or conversely

$$a_2^1=-1.92c_1^1\,,\quad a_4^1=-1.86\,(1.09c_1^1+c_3^1)\,,$$

$$a_6^1=-1.88\,(1.10c_1^1+1.01c_3^1+c_5^1)\ \text{......}\qquad\text{......(9b)}$$

---

32) $$\int_0^{\pi} P_j{}^l P_k{}^l \sin\theta\, d\theta=\frac{4}{2k+1}\quad\text{for } j=k$$
$$=\ 0\qquad\text{for } j\neq k.$$

where, as is easily gotten,

$$c_n^1 = \frac{2n+1}{4\pi}\int_0^\pi X(\theta)\, P_n^1 \sin^2\theta d\theta \,. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Here, the terms $n=2\nu$ ($\nu=1, 2, ..$) are only taken into consideration because the current-systems concerned here are approximately symmetric with respect to the equator.

Applying the above mentioned theory to the mean values, X and Z obtained from Vestine's smoothed curves, we have

$$\left.\begin{array}{llll} a_2^1=13\,, & a_4^1=-12\,, & a_6^1=-46 & \\ \tilde{a}_2^1=6.2\,; & \tilde{a}_4^1=-38\,, & \tilde{a}_6^1=-27 & \end{array} (\text{unit in } \pi\gamma)\right\} \ldots\ldots(11)$$

for the first three harmonics.

Then, calculating e_n^1 and i_n^1 by (5) and (11), we obtain the amplitude ratio e_n^1/i_n^1, which is given in Table XII together with that expected in Chapman's and Benkova's models[33] of the earth described in the preceding Chapter.

Table XII. The ratio of the external parts to the internal parts for three harmonics obtained from Vestine's data and expected from the theory of induction within Chapman's and Benkova's models.

n	m	Vestine	Chapman	Benkova
2	1	3.7	2.5	2.2
4	1	2.3	3.2	2.5
6	1	4.4	4.8	3.2

33) In the case of the uniform core model, the amplitude ratio is given by

$$e_m^m/i_m^m = \frac{n+1}{n} q^{-2n-1} \sqrt{\frac{A^2+B^2}{(A-1)^2+B^2}}\,,$$

where

$$A = \frac{\beta}{2n+1}\left\{1+\frac{n}{\beta}+\frac{n(n+1)}{4\beta^2}\ldots\ldots\ldots\ldots\ldots\right\},$$

$$B = \frac{\beta}{2n+1}\left\{1-\frac{n(n+1)}{4\beta^2}-\frac{n(n+1)}{4\beta^3}\ldots\ldots\right\}.$$

In these expressions, β is calculated from

$$\beta^2 = \frac{4\pi^2\sigma m q^2 a^2}{24\times 60\times 60}$$

where σ, q, a denote respectively the specific electrical conductivity, the ratio of the radius of the core to that of the earth and radius of the earth.

Although the agreement between the values obtained from the actual data and the calculated ones based on the uniform core model is not quite well, it can be said approximately that the uniform core model is also roughly consistent with the electromagnetic induction by S_D, provided its conductivity amounts to the order of $10^{-12} \sim 10^{-13}$ *emu* and is covered by non-conducting layer of a few hundred *kilometers* in thickness besides.

It is desirable, however, that a more accurate discussion will be made in the future with the aid of more complete data of S_D.

CHAPTER IV. ELECTROMAGNETIC INDUCTION BY STORM-TIME VARIATION (so-called D_{st}-field).

The theory of induction by aperiodic fields in a uniformly conducting sphere was developed by Price[6), 7)], being applied to the induction by storm-time variation or D_{st}-field by Chapman and Price[8)]. Lahiri and Price[10)] also studied the induction in a non-uniform sphere with applications to the conductivity-distribution in the earth which is compatible with both S_q and D_{st}.

According to these studies, the conductivity in the core obtained from the investigations on D_{st} became larger than that of S_q as a whole, that is, it amounts to the order of 4×10^{-12} *emu*. Taking into consideration that the induced currents penetrate deeper in the case of D_{st} than of S_q, the conductivity must increase considerably with increasing depth as shown in the study of non-uniform core.

However, their discussions on the discrepancies between the results of S_q and D_{st} are all based upon the well known Chapman's analysis[3)] of S_q, which has better be criticized on the stand-point of recent investigations. As already mentioned in Section 2, Chapter II, we have, at the present stage of investigation, reasons to believe that the analyses carried out by Hasegawa and Benkova are more reliable than Chapman's one. Then, it will be worth-while to treat the problem on the basis of newly made analyses.

Since the conductivity in the uniformly conducting core must amount to about 5×10^{-12} *emu* using Benkova's results, as Nagata[34)] pointed out, the discrepancies between the results obtained from observations and those of calculation on the basis of Chapman's model ($\sigma=3.6\times10^{-13}$ *emu*, $q=0.97$) is almost overcome leaving a small difference between the actual analysis and the calculation. For this reason, it is roughly satisfactory for us, as a first approximation, to take a uniformly conducting sphere in which the conductivity amounts to 5×10^{-12} *emu* being covered by non-conducting layer of 400 *km* in

34) T. NAGATA, Read at the Dec. Meeting of the Geophysical Institute, Tokyo Imperial University, 1946.

its thickness as the possible model of the earth compatible not only with S_q but also with D_{st}.

Next, taking into consideration that the conductivity in the core seems likely to be about ten times larger than in the mantle in the case of S_q, we shall discuss the electromagnetic induction by D_{st} in the earth's model described in Chapter II.

According to Chapman and Whitehead[4], the magnetic potential of magnetic storms in the low and middle latitude is expressible on the average by

$$W=a\sum_{n=1,3,5}\{e_n(t)(r/a)^n+i_n(t)(a/r)^{n+1}\}P_n(\cos\theta) \quad \text{..................(1)}$$

where a denotes the radius of the earth. Among these harmonics, the term $n=1$ is the most important. Then the discussion will be limitted to the case $n=1$. The variations of e_1 and i_1 with time were given by Chapman and Price[8] as reproduced in Fig. 11.

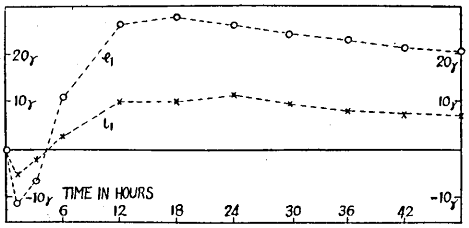

Fig. 11. (After S. Chapman and A. T. Price).

As has been done by Lahiri and Price[10], e_1 is expressed by

$$e_1(t)=Ae^{-\omega t}H(t) \text{ or} \begin{cases} =Ae^{-\omega t} & (t>0) \\ =0 & (t<0) \end{cases} \quad \text{..................(2)}$$

after 18 hours passed from the beginning neglecting the initial phase, where

$$A=28\gamma, \quad \omega=3.2\times10^{-6}\,sec^{-1}.$$

$H(t)$ denotes the Heaviside unit function. Writing in operational form, we get

$$e_1(t)=\frac{Ap}{p+\omega}H(t). \quad \text{..................(3)}$$

Then it becomes by (3.17) of Chapter II

$$i_1(t)=\frac{Ap\,I(p)}{p+\omega}H(t) \quad \text{..................(4)}$$

where, as easily gotten from (3.21) of Chapter II, $I(p)$ is given by

$$I(p)=-\frac{q_1^3}{2}\frac{\left\{1-\frac{F_0(2,1)}{F_1(2,1)}\right\}\left\{1-\frac{F_0(3,2)}{F_1(3,2)}-\frac{G_2(2,2)}{G_1(2,2)}\right\}+\tau^3\left\{\frac{F_0(3,2)}{F_1(3,2)}-\frac{F_0(2,2)}{F_1(2,2)}\right\}\frac{F_1(2,2)}{F_1(2,1)}\frac{G_1(2,1)}{G_1(2,2)}}{\frac{F_0(2,1)}{F_1(2,1)}\left\{1-\frac{F_0(3,2)}{F_1(3,2)}-\frac{G_2(2,2)}{G_1(2,2)}\right\}+\tau^3\left\{1-\frac{G_2(2,1)}{G_1(2,1)}\right\}\left\{\frac{F_0(3,2)}{F_1(3,2)}-\frac{F_0(2,2)}{F_1(2,2)}\right\}\frac{F_1(2,2)}{F_1(2,1)}\frac{G_1(2,1)}{G_1(2,2)}} \tag{5}$$

Considering that the mantle shields the core and consequently the induced currents in the mantle contribute to the magnetic field at the surface in the main, the second terms of the numerator and the denominator of the righthand-side of (5) must be small compared with the first terms[35].
Solving the operational equation (4), we get

$$i_1(t)=\frac{A}{2\pi i}\int_L e^{pt}\frac{I(p)}{p+\omega}\,dp\ , \tag{6}$$

where the path of integration L is taken to be Bromwich's one having all singularities on its lefthand-side in the complex plane. As to the singularities, it is difficult to get them rigorously because (5) is very complicated. However, we may assume as a first approximation that they are the same with those of the uniform core model in which the conductivity is equal to that of the mantle because the contribution of the induced current in the core seems very small compared with that in the mantle. Under such assumption, we shall calculate $i_1(t)$.

As is obvious from the definition of F_n and G_n, since it becomes in the case of $n=1$ that

$$\left.\begin{aligned}
F_0(k_\nu q_\mu a)&=\frac{\sinh(k_\nu q_\mu a)}{k_\nu q_\mu a}\,,\\
F_1(k_\nu q_\mu a)&=\frac{3}{(k_\nu q_\mu a)^2}\left\{\cosh(k_\nu q_\mu a)-\frac{\sinh(k_\nu q_\mu a)}{k_\nu q_\mu a}\right\},\\
G_1(k_\nu q_\mu a)&=e^{-k_\nu q_\mu a}(1+k_\nu q_\mu a)\,,\\
G_2(k_\nu q_\mu a)&=e^{-k_\nu q_\mu a}\left\{1+k_\nu q_\mu a+\frac{(k_\nu q_\mu a)^2}{3}\right\},
\end{aligned}\right\} \tag{7}$$

the poles are approximately given by the roots of

$$\sinh(k_2 q_1 a)=0 \qquad (k_2^2=4\pi\sigma_2 p) \tag{8}$$

as well as $-\omega$. Then, denoting the successive pole by $-\alpha_s$, we get

$$\alpha_s=\left(\frac{\pi s}{\kappa_{21}}\right)^2,\ s=1,2,3,\ldots\ldots \tag{9}$$

35) Really, we get the expression for uniformly conducting sphere neglecting the second terms.

where

$$\kappa_{21}^2 = 4\pi\sigma_2 q_1^2 a^2 \,. \qquad \cdots\cdots(10)$$

Then taking into consideration that the integral along the circumference of the semi-circle vanishes at infinity, the contour-integration shown in Fig. 12 gives when the radius of the semi-circle becomes infinitely large

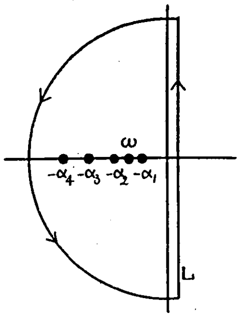

Fig. 12.

$$i_1(t) = A\Big[e^{-\omega t} I(-\omega) + \frac{3q_1^3}{\kappa_{21}^2} \sum_{s=1}^{\infty} \frac{e^{-\alpha_s t}}{\alpha_s - \omega} \{1 + \tau^3 \Phi(-\alpha_s)\}\Big] \qquad \cdots\cdots(11)$$

where

$$\Phi(p) = \frac{F_1(2,2)\left\{\dfrac{F_0(3,2)}{F_1(3,2)} - \dfrac{F_0(2,2)}{F_1(2,2)}\right\}\dfrac{G_1(2,1)}{G_1(2,2)}}{\{F_1(2,1) - F_0(2,1)\}\left\{1 - \dfrac{F_0(3,2)}{F_1(3,2)} - \dfrac{G_2(2,1)}{G_1(2,2)}\right\}} \,. \qquad \cdots\cdots(12)$$

The calculated values of $i_1(t)$ are shown in Fig. 13 together with those based on Benkova's model of the earth. The results of the analysis are also illustrated there. The small discrepancies between the observation and the calculation based on the uniform core model improved by introducing the core in which the conductivity is about ten times larger than the mantle.

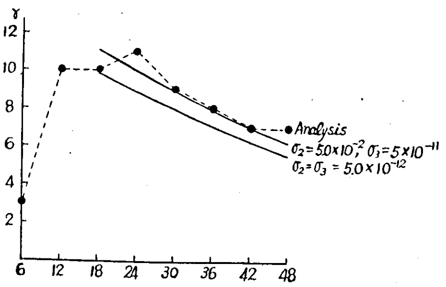

Fig. 13. Comparison of the theoretical values to the analysis.

Although the calculation carried out in this Chapter is rather rough, we may say that the electrical model of the earth obtained from the study on S_q is also compatible with the electromagnetic induction by storm-time variation.

According to Chapman and Price[10], the core in which $\sigma = 4.4\times10^{-12}$ *emu* $q = 0.94$ appears to be about right for the explanation of D_{st}, while the size and conductivity in the present model fairly agree with these values. Taking into consideration that the inner core is shielded by the outer layer, then, the distribution of the electric currents in the mantle will be almost the same with that obtained by Chapman and Price. Hence we cease to study the penetration of the induced currents.

CHAPTER V. ELECTROMAGNETIC INDUCTION BY BAY-TYPE DISTURBANCE.

1. Morphology of bay-type disturbance.

In addition to the variations in the earth's magnetic field which were studied in the foregoing Chapters, we have a kind of magnetic disturbance called bay-type disturbance. Because the form of the magnetographic record resembles bay in a sea-coast. In this Chapter, the electromagnetic induction by bay-type disturbance will be investigated in the same way with the treatments of the former Chapters.

According to the studies which were carried out up to this time, it was established that the bay-type disturbance occurs nearly at the same time all over the world durating $1\sim5$ *hours*. Its maximum amplitude often reaches to about 100 *gammas* in the middle latitudes. Among these investigations[36,37,38,39,40] on the bay-type disturbance, Hatakeyama's[41] investigation is one of the most excellent. In the first place, examining a large number of magnetograms obtained at Toyohara Magnetic Observatory in Sagalien, he determined both directions and magnitudes of disturbing magnetic forces and consequently those of electric currents, which, being distributed in the upper atmosphere of the earth, are considered to be the primary origin of the disturbance, and tried to obtain general features of the electric current-systems in the upper atmosphere. Hatakeyama's further studies on the world-wide character of this disturbance also concluded that such electric current-systems must exist. The idealized current-systems obtained by him seems to be fixed to the sun while the electric currents are concentrating above the auroral zones as shown in Fig. 14 which is reproduced from his original paper. We can easily see that these current-systems agree with those of S_D as already shown in Fig. 9 which was obtained by Chapman. Meanwhile, the present writer[29] mathematically studied the mechanism of occurrence of the bay-type disturbance and S_D on the basis of "dynamo"-action in the ionosphere where the air is electric-

36) L. STEINER, *Terr. Mag.*, 26 (1921), 1.

37) LUBIGER, *Diss. Göttingen* (1924).

38) E. WIECHERT, *Mitt. Geophys. Warte Gr. Raum, Königsberg*, Nr. 22 (1934).

39) K. BIRKELAND, *The Norwegian Aurora Polaris Expedition*, 1902-1903, I and II (1908).

40) A. G. MCNISH, *Trans. Edinburgh Meeting Internat. Union Geod. Geophys. Ass. Terr. Mag. Electr. Bull.*, 10 (1937).

41) H. HATAKEYAMA, *Geophys. Mag.*, 11 (1937), 1 and 12 (1938), 15.

ally conducting. In that study, it was assumed that the conductivity above the auroral zones is very high compared with that of the middle and low latitudes

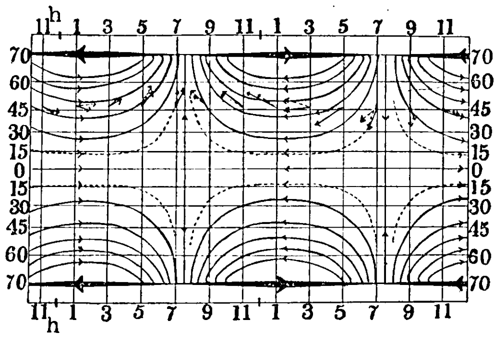

Fig. 14. Current-arrows at Toyohara and current-systems for bay-type disturbance. Broken arrows represent the disturbing forces at t_2 and thick arrows represent those at t_3. t_2 and t_3 are defined in the next Section.
(After H. Hatakeyama)

with Störmer-Birkelands' theory in mind. According to the results of the writer's theoretical investigation, the current-function R_D which corresponds to the current-systems in question is given by

$$R_D = KCk_1^1 C_5 \cot\theta \sin(t+\alpha_1^1)$$

in the middle and low latitudes, where K, C, $k_1{}^1$, C_5 and $\alpha_1{}^1$ are constants. θ and t denote the colatitude and the local time respectively.

As the height of the electric currents is very small compared with the radius of the earth, the components of the magnetic forces at the earth's surface which are produced by these electric current-systems, vary, to a fair degree of approximation, with θ as $\frac{d}{d\theta}\cot\theta$, $\frac{\cot\theta}{\sin\theta}$ and $\cot\theta$ in X, Y and Z respectively. Thus the magnetic field increases remarkably with increase of latitude. Then, if we take the usual spherical harmonics which are defined in the region including $\theta=0$ and $\theta=\pi$, we must take a large number of higher harmonics to express the distribution of these magnetic fields. This procedure seems to be too troublesome and less reliable to analyze the actual data except the case in which the distribution is given everywhere all over the earth. Actually the mean disturbing forces of the bay-type disturbance of Feb. 24, 1933,[42] are such as shown in Figs. 15 and 16, where, assuming that

42) The data used in this Chapter are all taken from Hatakeyama's paper,

the components of the disturbing forces vary as the sine and cosine of longitude on the same latitude, Y and Z are respectively reduced on $\phi=0°$ and $\phi=90°$ meridian with respect to the geomagnetic coordinate.

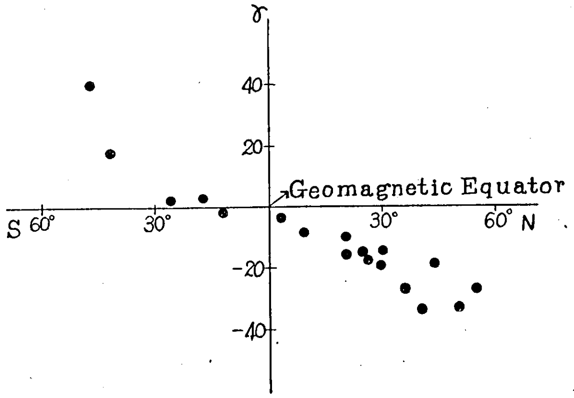

Fig. 15. The east component of the mean disturbing force of bay-type disturbance, Feb. 24, 1933. (Reduced on $\phi=0°$ meridian)

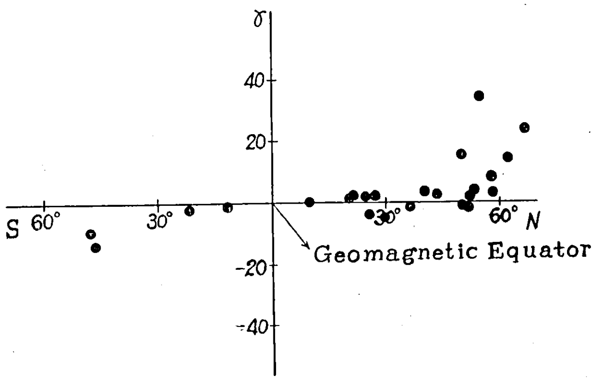

Fig. 16. The vertical component of the mean disturbing force of the bay-type disturbance, Feb. 24, 1933. (Reduced on $\phi=90°$ meridian)

2. The analysis of bay-type disturbances.

In order to analyze Hatakeyama's data, we shall introduce a system of functions which satisfy

$$\frac{d^2U}{d\theta^2}+\cot\theta\frac{dU}{d\theta}+\left\{n(n+1)-\frac{m^2}{\sin^2\theta}\right\}U=0 \quad\text{.................(2.1a)}$$

in the region excluding both the north and south poles.
As is well known[43], (2.1 a) or

$$(1-\mu^2)\frac{d^2U}{d\mu^2}-2\mu\frac{dU}{d\mu}+\left\{n(n+1)-\frac{m^2}{1-\mu^2}\right\}U=0 \quad (\mu=\cos\theta) \quad\text{...(2.1b)}$$

are satisfied by $P_n^m(\mu)$ or $Q_n^m(\mu)$ provided $\mu^2<1$, where

$$P_n^m=(1-\mu^2)^{m/2}\frac{d^mP_n}{d\mu^m},\quad Q_n^m=(1-\mu^2)^{m/2}\frac{d^mQ_n}{d\mu^m}.$$

In like manner with the usual spherical harmonic analysis, we can generally express the magnetic potential of the disturbance as follows;

$$\begin{aligned} W=\sum_n\sum_m\Bigg\{&\left(e_{n,a}^m\frac{r^n}{R^{n-1}}+i_{n,a}^m\frac{R^{n+2}}{r^{n+1}}\right)\cos m\phi \\ &+\left(e_{n,b}^m\frac{r^n}{R^{n-1}}+i_{n,b}^m\frac{R^{n+2}}{r^{n+1}}\right)\sin m\phi\Bigg\}P_n^m(\cos\theta) \\ +\sum_n\sum_m\Bigg\{&\left(\tilde{e}_{n,a}^m\frac{r^n}{R^{n-1}}+\tilde{i}_{n,a}^m\frac{R^{n+2}}{r^{n+1}}\right)\cos m\phi \\ &+\left(\tilde{e}_{n,b}^m\frac{r^n}{R^{n-1}}+\tilde{i}_{n,b}^m\frac{R^{n+2}}{r^{n+1}}\right)\sin m\phi\Bigg\}Q_n^m(\cos\theta) \quad\text{......(2.2)}\end{aligned}$$

where R denotes the radius of the earth and the coordinates are taken to be geomagnetic. In the case of the bay-type disturbance, however, as the magnetic potential of the disturbing forces are nearly symmetric with respect to the geomagnetic equator as shown in Fig. 14, it is approximately sufficient for the present analysis to take into consideration only the terms which are odd functions of μ. Then, we only deal with P_n^m's where $n+m$ is odd and Q_n^m's where $n+m$ is even. Besides, as the disturbing forces seem to vary as the sine or cosine of longitude on the same latitude, the greatest part of the magnetic potential must be expressible with the sum of $P_n^1\frac{\cos\phi}{\sin\phi}$ and $Q_n^1\frac{\cos\phi}{\sin\phi}$.
Meanwhile, we see in Figs. 15 and 16 that the east and vertical components both reduced on the meridian on which they become maximum increase mono-

43) e.g. E. W. HOBSON, "*The Theory of Spherical and Ellipsoidal Harmonics,*" (1931).

tonously with latitude changing the sign approximately at the equator. This fact suggests that the term including Q_1^1 [44] occupies the main part of the potential function of the disturbing forces. Thus, so far as we discuss the magnetic disturbance in the low and middle latitudes such as from 0° to 45°, we may assume without serious danger that the magnetic potential can be expressed by

$$W=\left\{\left(e_a r+i_a\frac{R^3}{r^2}\right)\cos\phi+\left(e_b r+i_b\frac{R^3}{r^2}\right)\sin\phi\right\}Q_1^1. \qquad (2.3)$$

Q_1^1 and $Q_1^1/\sin\theta$ are shown graphically in Fig. 10. From (2.3), we get the magnetic forces at the surface ($r=R$) as follows;

$$\left.\begin{aligned} X&=\{(e_a+i_a)\cos\phi+(e_b+i_b)\sin\phi\}\frac{dQ_1^1}{d\theta},\\ Y&=\{-(e_a+i_a)\sin\phi+(e_b+i_b)\cos\phi\}\frac{Q_1^1}{\sin\theta},\\ Z&=\{(e_a-2i_a)\cos\phi+(e_b-2i_b)\sin\phi\}\,Q_1^1. \end{aligned}\right\}\ldots(2.4)$$

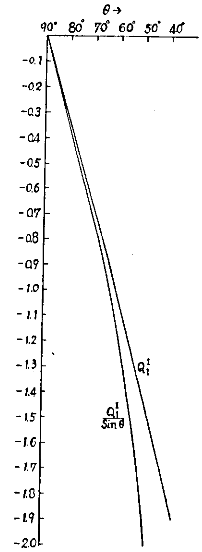

Fig. 17.

Then we can determine e_a+i_a and e_b+i_b from the observed data of X or Y and also e_a-2i_a and e_b-2i_b from that of Z by means of the method of least squares. Combining these coefficients we can calculate e_a, i_a, e_b and i_b. Thus the separation of the disturbing forces into the external and internal parts is over.

As to the application of the method described above to the actual problem, the writer used the data published by Hatakeyama who divided the course of a bay-type disturbance into six parts of equal time-interval, say t_0t_1, t_1t_2, ..., t_5t_6 determining the deviation of each component from the normal curve at t_1, t_2, ..., t_5. He collected copies of magnetograms sent from many observatories (about 30 in number) distributed widely over the earth. The determination of the disturbing forces was made in the case of the bay-type disturbances of Aug. 28, Sept. 23, 1932, Feb. 24 and Apr. 10, 1933 as reproduced in Tables XIII, XIV, XV and XVI.

Selecting the data of 16 stations whose geomagnetic latitudes are all less than 45°, the writer determined e_a, i_a, e_b and i_b from the combination of Y and Z. For example, they are tabulated in Table XVII in the case of the

44) $Q_1^1=-2(1-\mu^2)^{1/2}\mu\left(1+\frac{2}{3}\mu^2+\frac{3}{5}\mu^4+\ldots\ldots\right).$

Table XIII. Disturbing forces during the bay-type disturbance of Aug. 28, 1932. (After H. Hatakeyama)

Station	Geomag-netic latitude	Geomag-netic longitude	2 *h* 10 *m*			2 *h* 43.5 *m*			3 *h* 17 *m*		
			α	ΔH	ΔZ	α	ΔH	ΔZ	α	ΔH	ΔZ
Toyohara	36.°4	206.°5	S 4°W	109γ	−2γ	S23°W	101γ	−2γ	S38°W	43γ	+1γ
Kakioka	26. 0	206. 0	S26 W	53	−34	S26 W	53	−22	S38 W	33	− 3
Tsingtau	24. 7	188. 3	S12 W	63	− 6	S23 W	50	− 6	S54 W	31	− 4
Lu-Kia-Pang	20. 0	189. 1	S13 W	67	—	S20 W	54	—	S50 W	33	—
Antipolo	3. 3	189. 8	S 7 E	35	−11	S 7 E	23	− 8	S29 W	21	− 5
Kuyper	−17. 4	175. 7	S	34	−14	S34 E	14	− 3	S45 E	6	+ 3
Watheroo	−41. 8	185. 6	S32 E	80	+ 5	S56 E	74	+19	S60 E	24	+ 5
Toolangi	−46. 7	220. 8	S15 E	119	−28	S27 E	96	−31	S23 E	52	−19
Christchurch	−48. 0	252. 6	S 2 E	89	−11	S 7 E	46	−15	S 5 E	37	− 6
Apia	−16. 0	260. 2	S 7 E	62	—	S 9 E	59	—	S19 E	31	—
Honolulu	21. 0	266. 5	S11 E	60	+ 2	S 8 E	61	+ 1	S 7 E	35	+ 3
Sitka	60. 0	275. 4	N27 E	174	+189	N63 E	68	+121	S81 E	20	+73
Agincourt	55. 0	347. 0	N28 E	126	−59	S75 E	99	−33	S62 E	60	− 7
Cheltenham	50. 1	350. 5	N38 E	134	−55	S35 E	76	−28	S50 E	68	−25
Tucson	40. 4	312. 2	S82 E	146	+14	S51 E	45	+ 5	S 3 E	50	+ 2
Teoloyucan	29. 6	326. 8	N87 E	77	+ 7	S31 E	31	− 1	S22 E	48	−11
San Juan	29. 9	3. 4	N 3 E	52	−11	N79 E	5	− 6	S43 E	22	− 8
Huancayo	−0. 6	353. 8	N 4 W	51	− 6	S35 W	40	−14	S 9 W	44	− 6
Dehra Dun	20. 5	149. 9	S11 W	26	− 2	N14 W	8	+10	N41 W	9	+ 4
Alibag	9. 5	143. 6	S17 E	17	+ 3	N51 E	13	+10	N30 E	8	+ 7
Helwan	27. 2	106. 4	N45 E	50	−10	N 1 W	54	−14	N11 W	32	− 4
Ebro	43. 9	79. 7	N 8 E	49	−20	N29 E	63	−16	N34 E	37	+ 1
Coimbra	45. 0	70. 2	—	—	—	—	—	—	—	—	—
Val-Joyeux	51. 3	84. 5	N38 W	46	−22	N20 E	59	−22	N41 E	38	−16
De Bilt	53. 8	89. 4	N21 W	45	−49	N41 E	69	−63	N55 E	47	−39
Seddin	52. 4	97. 0	N54 W	56	−44	N14 E	64	−52	N24 E	45	−34
Swider	50. 6	104. 6	N60 W	66	+27	N 6 W	61	+25	N 1 W	39	+20
Lovö	58. 1	105. 8	S63 W	112	−190	N55 W	28	−149	N 5 E	33	−61
Eskdalemuir	58. 5	82. 9	S17 W	27	−140	N56 E	37	−122	N50 E	34	−71
Lerwick	62. 6	88. 6	S 7 W	398	−270	S10 W	174	−250	S 6 E	62	−135
Tromsö	67. 1	116. 7	S37 W	248	+711	S26 W	743	+77	S25 W	248	−291
Vassouras	−11. 9	23. 9	N 8 W	57	—	N63 W	25	—	S55 W	21	—
Mauritius	−26. 6	122. 4	N64 E	30	+ 5	N54 E	44	+ 8	N58 E	32	+ 9

(Each element for 1 *h* 03 *m* and 4 *h* 24 *m* was assumed to be undisturbed.)

Table XIV. Disturbing forces during the bay-type disturbance of Sept. 23, 1932. (After H. Hatakeyama)

Station	19 h 48 m			20 h 18 m			20 h 48 m			21 h 18 m		
	α	ΔH	ΔZ	α	ΔH	ΔZ	α	ΔH	ΔZ	α	ΔH	ΔZ
Toyohara	S28°W	15γ	+1γ	S34°W	18γ	+2γ	S54°W	32γ	+2γ	S63°W	33γ	+3γ
Kakioka	S56 W	7	− 2	S52 W	11	− 3	S73 W	24	− 2	S79 W	20	+ 3
Tsingtau	S76 W	8	− 1	S37 W	10	− 2	S75 W	27	− 2	N85 W	25	0
Lu-Kia-Pang	S81 W	6	0	S45 W	11	− 1	S73 W	24	− 3	N70 W	23	− 3
Antipolo	—	0	+ 2	S	9	0	S63 W	11	+ 2	N73 W	10	0
Kuyper	N60 E	14	+ 2	N83 E	8	+ 3	S59 E	6	0	N43 E	18	+10
Watheroo	S83 E	42	+25	S86 E	31	+20	S80 E	35	+17	N89 E	50	+34
Toolangi	S18 E	25	− 5	S46 E	30	−13	S31 E	26	−10	S48 E	36	−14
Christchurch	S67 W	15	+ 4	S48 W	24	+ 2	S49 W	20	+ 2	S24 W	32	+ 1
Apia	S	6	—	S20 W	12	—	S 6 W	19	—	S21 W	17	—
Honolulu	S81 W	12	− 6	S40 W	8	− 7	S49 W	23	−15	S39 W	19	−10
Sitka	N27 W	30	+15	N63 E	27	+ 6	N31 W	12	+13	N76 E	4	+ 7
Agincourt	S41 W	28	+ 6	N18 W	26	+ 4	S32 W	21	+16	S63 E	20	+15
Cheltenham	S28 W	26	+ 2	N 3 E	17	+ 6	S 8 W	27	+ 6	S57 E	30	+14
Tucson	S77 W	9	+ 1	S59 W	6	+ 3	S13 W	13	+ 2	S19 W	18	+ 2
Teoloyucan	S	8	+ 2	S16 W	7	+ 1	S18 W	22	+ 4	S12 W	24	+ 3
San Juan	S 5 W	11	+ 1	S18 E	6	− 1	S11 E	20	+ 2	S45 E	18	− 2
Huancayo	S15 W	30	+ 1	S 9 E	12	+ 2	S 3 W	39	+ 1	S 8 W	21	0
Dehra Dun	N28 W	28	+ 5	N60 W	8	+ 1	N29 W	37	+ 9	N22 W	43	+ 5
Alibag	N	25	− 4	N 4 W	13	0	N11 W	26	− 7	N 6 W	41	+ 2
Helwan	N12 E	30	−13	N34 W	19	− 2	N 9 E	52	−19	N 6 E	51	−11
Ebro	S89 E	53	−19	N16 E	18	−10	N60 E	82	−32	N34 E	50	−11
Val-Joyeux	S79 E	65	+ 1	N 2 E	24	0	N67 E	71	− 9	N43 E	50	− 5
De Bilt	N80 E	101	−15	S83 E	23	+ 2	N59 E	91	−30	N68 E	51	−14
Seddin	N80 E	83	− 5	N80 W	18	0	N42 E	96	−25	N33 E	60	−14
Swider	N64 E	72	− 8	N12 W	14	+ 6	N28 E	74	− 3	N37 E	59	+ 5
Lovö	—	—	—	W	1	−15	N42 E	57	−61	N74 E	59	−41
Eskdalemuir	W	26	+ 9	N18 W	13	− 9	N53 E	100	−27	S84 E	30	−27
Lerwick	S64 E	100	−22	S14 W	54	−49	S73 E	75	−131	S39 E	59	−106
Tromsö	N20 E	90	0	S68 W	57	+97	S42 E	39	+189	S 1 E	177	+144
Vassouras	S12 E	14	—	S	7	—	S 4 E	16	—	W	6	—
Mauritius	N12 E	24	− 2	N22 W	11	+ 1	N 9 E	37	0	N12 E	40	+ 3

(Each element for 19 h 18 m and 22 h 18 m was assumed to be undisturbed.)

Table XV. Disturbing forces during the bay-type disturbance of Feb. 24, 1933. (After H. Hatakeyama)

Station	11 *h* 36 *m*			11 *h* 51 *m*			12 *h* 06 *m*			12 *h* 21 *m*		
	α	ΔH	ΔZ	α	ΔH	ΔZ	α	ΔH	ΔZ	α	ΔH	ΔZ
Toyohara	N80°E	36γ	0γ	N75°E	38γ	+1γ	N40°E	35γ	+1γ	N67°E	15γ	+1γ
Kakioka	N69 E	25	− 1	N44 E	37	+ 8	N43 E	25	+ 3	N63 E	11	− 3
Tsingtau	S75 E	24	+ 2	N58 E	32	0	N58 E	22	− 2	N67 E	8	− 1
Zô-se	S70 E	20	− 1	N67 E	30	− 4	N66 E	22	− 2	N80 E	11	− 1
Antipolo	S51 E	6	+ 2	N36 E	14	+ 6	N49 E	11	+ 2	N67 E	8	− 2
Kuyper	S	15	− 6	N63 W	5	− 8	N18 E	6	− 4	N45 E	6	− 2
Watheroo	S48 E	30	−30	N73 W	30	−18	N61 W	13	− 7	N	2	+ 1
Toolangi	S86 W	66	+ 1	N48 W	45	+15	N37 W	21	+13	N	2	+ 7
Christchurch	N44 W	81	+22	N25 W	47	+ 7	N33 W	20	+ 2	N68 W	5	+ 2
Apia	N51 W	28	—	N12 W	25	—	N12 W	14	—	N45 W	6	—
Honolulu	N 7 W	33	+ 3	N20 W	33	− 5	N30 W	20	− 6	N16 W	7	− 1
Sitka	S69 W	126	−78	S46 W	144	−121	S18 W	93	−118	S11 W	43	−76
Agincourt	S39 W	77	− 8	S48 W	88	− 8	S40 W	42	− 9	S45 W	14	− 6
Cheltenham	S58 W	45	− 3	S57 W	68	− 7	S69 W	39	+ 7	S75 W	20	− 7
Tucson	N59 W	51	− 1	N67 W	48	− 3	N70 W	23	− 4	N79 W	10	− 3
Teoloyucan	N60 W	28	+ 4	N69 W	30	+ 3	N67 W	21	+ 2	N74 W	7	+ 1
San Juan	W	18	+ 3	S82 W	22	+ 3	W	18	+ 1	N81 W	12	0
Huancayo	S18 W	13	+ 3	S34 W	19	+ 5	S42 W	16	+ 5	S39 W	19	+ 4
Dehra Dun	S11 E	32	− 2	S39 E	28	0	S61 E	21	+ 2	S45 E	6	+ 4
Alibag	S14 E	21	+ 3	S26 E	30	+ 3	S43 E	21	0	S41 E	9	− 6
Helwan	S 7 W	24	+ 9	S 2 E	28	+ 8	S13 E	13	0	E	2	− 3
Ebro	S18 W	22	+ 5	S13 W	37	+ 9	S 9 W	22	0	S15 E	11	− 4
Coimbra	S 7 W	26	—	S 7 W	43	—	S 8 W	27	—	S 9 E	13	—
Val-Joyeux	S 9 W	26	+ 1	S 9 W	37	0	S16 W	22	− 3	S10 W	11	− 3
De Bilt	S 4 W	43	+ 8	S 8 W	41	+ 4	S21 W	19	0	S84 E	9	+ 5
Seddin	S18 W	32	+ 6	S11 W	48	+ 4	S12 W	25	0	S14 W	8	− 1
Swider	S17 W	30	+ 5	S11 W	38	0	S11 W	20	− 6	S	5	− 3
Lovö	S19 W	37	− 2	S16 W	41	+ 8	S18 W	23	+15	S49 W	11	+12
Eskdalemuir	—	—	+ 1	—	—	+ 2	S27 W	18	+ 7	S 9 E	6	+ 2
Lerwick	S14 W	33	+ 7	S18 W	44	+20	S19 W	24	+17	S33 W	17	+13
Tromsö	N43 E	26	+22	N55 E	111	+28	N28 E	59	+23	S16 W	29	+14
Vassouras	S15 W	11	− 2	S20 W	15	− 2	S40 W	8	+ 1	S58 W	9	+ 2
Mauritius	S 2 E	24	+ 1	S 3 E	20	− 2	S 5 E	11	− 2	S	3	− 3

(Each element for 11 *h* 21 *m* and 12 *h* 51 *m* was assumed to be undisturbed.)

Table XVI. Disturbing forces during the bay-type disturbance of Apr. 10, 1933. (After H. Hatakeyama)

Station	15 *h* 00 *m*			15 *h* 18 *m*			15 *h* 36 *m*		
	α	ΔH	ΔZ	α	ΔH	ΔZ	α	ΔH	ΔZ
Toyohara	N31°W	35γ	−7γ	N36°W	17γ	−4γ	N16°W	7γ	−2γ
Kakioka	N12 W	25	+14	N22 W	18	+ 9	N	11	+ 3
Tsingtau	N	32	− 1	N15 W	23	0	N20 E	12	0
Zô-Se	N 2 E	28	− 5	N 5 W	22	− 2	N	11	0
Antipolo	N 3 W	23	+ 7	N10 W	23	− 2	N14 E	12	− 3
Kuyper	N 9 W	33	+17	N 7 W	24	+27	N	13	+16
Watheroo	N19 W	40	+10	N14 W	26	+12	N17 W	14	+ 6
Toolangi	N33 E	45	+ 8	N43 E	34	+ 7	N28 E	17	+ 5
Christchurch	N77 E	28	− 5	N82 E	22	− 1	N70 E	12	+ 2
Honolulu	N76 W	8	− 2	N59 W	6	+ 1	N56 W	4	0
Sitka	S43 W	48	+10	S55 W	40	+ 6	S48 W	13	− 2
Agincourt	S24 W	17	0	S13 E	21	0	S38 E	11	+ 1
Cheltenham	S24 W	17	− 3	S16 E	19	− 4	S24 E	12	− 2
Tucson	S 4 E	15	+ 4	S27 E	16	+ 3	S45 E	8	+ 2
Teoloyucan	S16 W	7	− 2	S22 E	5	− 2	S45 E	3	− 1
San Juan	S 8 W	7	− 1	S11 W	5	− 2	S45 E	1	− 2
Huancayo	S 6 E	9	+ 1	N25 E	19	+ 2	N21 E	22	+ 1
Dehra Dun	N37 E	26	+10	N18 E	26	+ 7	N 5 E	11	− 3
Alibag	N63 E	7	+ 2	N16 E	19	0	N17 E	14	+ 1
Helwan	S73 E	17	+ 2	N70 E	20	− 4	N49 E	11	− 2
Ebro	S25 E	17	+ 3	S72 E	13	− 6	N39 E	6	− 8
Coimbra	S11 E	20	—	E	7	—	N34 E	4	—
Val-Joyeux	S32 E	26	+ 1	S73 E	17	+ 1	N45 E	10	+ 2
De Bilt	S44 E	35	+ 1	N63 E	30	− 5	N34 E	19	+ 1
Seddin	S33 E	30	+ 2	N80 E	34	− 1	N37 E	20	+ 2
Swider	S46 E	30	− 2	N72 E	36	−11	N34 E	18	− 5
Lovö	S29 E	35	+ 4	N81 E	46	+20	N27 E	28	+12
Eskdalemuir	S24 E	32	+ 4	N47 E	21	+ 5	N27 E	22	+ 5
Lerwick	S12 E	29	+ 8	N75 E	23	+ 9	N41 E	33	+ 5
Tromsö	N59 E	108	0	N83 E	70	−18	N19 W	21	−41
Vassouras	S51 E	14	+ 1	S60 E	14	+ 2	N75 E	11	+ 3
Mauritius	S84 W	10	− 3	N61 W	10	− 3	N45 W	6	− 1

(Each element for 14 *h* 24 *m* and 16 *h* 12 *m* was assumed to be undisturbed.)

Table XVII. The coef. of the magnetic potential of the disturbing force in the case of the bay-type disturbance of Feb. 24, 1933. (Unit : γ)

G. M. T.		e_a	i_a	e_b	i_b
11 *h*	21 *m*	0	0	0	0
11	36	-2.3 ± 2.2	-0.2 ± 1.3	12.9 ± 1.9	6.9 ± 1.2
11	51	-1.8 ± 2.4	-0.7 ± 1.4	15.9 ± 3.1	8.0 ± 1.3
12	06	-2.0 ± 1.6	-0.6 ± 0.9	11.9 ± 1.4	5.8 ± 0.9
12	21	-2.4 ± 1.2	-0.5 ± 0.7	6.8 ± 1.1	2.7 ± 0.7
12	51	0	0	0	0

bay-type disturbance of Feb. 24, 1933. As shown in this Table, the probable errors of the determined coefficients amount to 1~3 *gammas*. However, a part of these apparent errors may be attributed to the method of the analysis in which we did not take into consideration higher harmonics of the magnetic potential. Thus owing to the smallness of these apparent errors, we may say that the greatest part of the magnetic potential is given by the term including $Q_1^1 \cos\phi$ and $Q_1^1 \sin\phi$.

These coefficients are shown graphically in Fig. 18.

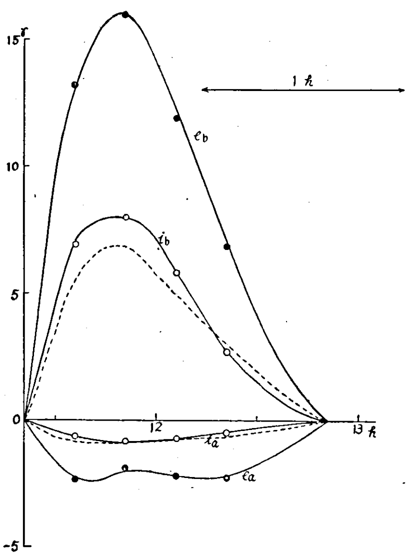

Fig. 18. The coefficients of the magnetic potential in the case of the bay-type disturbance on Feb. 24, 1933. Broken line shows the theoretical value for i which will be obtained in Section 3.

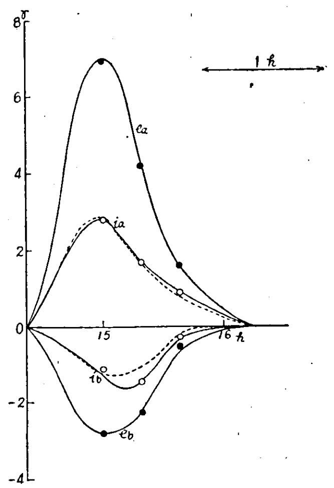

Fig. 19. The coefficients of the magnetic potential in the case of the bay-type disturbance on Apr. 10, 1933. Broken line shows the theoretical value for i which will be obtained in Section 3.

The same determination was made in each bay-type disturbance mentioned above being illustrated in Figs. 19, 20, and 21.

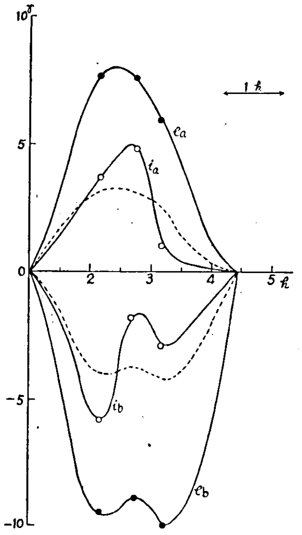

Fig. 20. The coefficients of the magnetic potential in the case of the bay-type disturbance on Aug. 28, 1932. Broken line shows the theoretical value for i which will be obtained in Section 3.

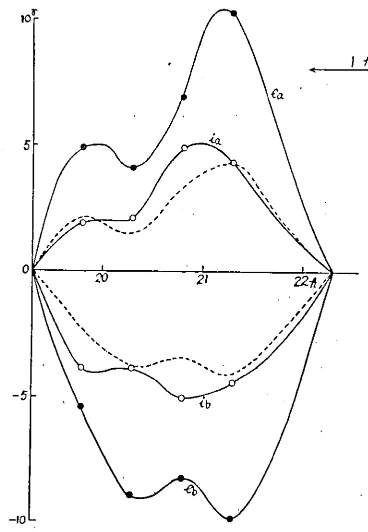

Fig. 21. The coefficients of the magnetic potential in the case of the bay-type disturbance on Sept. 23, 1932. Broken line shows the theoretical value for i which will be obtained in Section 3.

3. The relation to the electrical state of the earth's interior.

Now, it is of interest to study to what extent the theory of electromagnetic induction is applicable for the relation between the external and the internal parts of the bay-type disturbance. As the durations of the bay-type disturbances range from one to several hours, shorter than the daily variation, and besides the mode of variation is quite different from the daily variation on quiet days, the investigation will become a good check to the induction-theory.

If we take the uniform core model in which $\sigma=5\times10^{-12}\, emu$ and $q=0.94$ as already informed by the study on S_q, the internal part of the magnetic potential must be related to the external one by[6]

$$i_n(t)=\frac{d}{dt}\int_0^t e_n(t-u)\,h(u)\,du=e_n(t)\,h(0)+\int_0^t e_n(t-u)\,h'(u)\,du \quad \ldots(3.1)$$

where $h(t)$ corresponds to $i_n(t)$ when $e_n(t)=0\,(t<0)$ and $=1\,(t<0)$.

When $n=1$, it becomes

$$h(t)=\frac{1}{2\pi i}\int_L e^{pt} I(p)\frac{dp}{p}\,, \quad \text{.................................(3.2)}$$

where L is Bromwich's path of integration and

$$I(p)=\frac{q^3}{2}\left\{1-\frac{F_1(\sqrt{4\pi\sigma p}qa)}{F_0(\sqrt{4\pi\sigma p}qa)}\right\}. \quad \text{.............................(3.3)}$$

F_n was already defined in Chapter II. We get from (3.2)

$$h(t)=\frac{3q^3}{\pi^2}\sum_{s=1}^{\infty}\frac{1}{s^2}\exp\left(-\frac{\pi s^2}{4\sigma q^2a^2}t\right) \quad \text{.......................(3.4)}$$

in like manner with the study on the induction by D_{st}. When t is small, $h(t)$ can be written as

$$h(t)=\frac{q^3}{2}\left(1-\frac{3}{\pi qa}\sqrt{\frac{t}{\sigma}}+\frac{3t}{4\pi\sigma q^2a^2}\right), \quad \text{.................(3.5)}$$

as studied by Price[6)].

In the case of Benkova's uniform core model, $h(t)$ decreses very slowly as shown in Fig. 22. The mean rate of decrese during a few *hours* from the beginning amounts to only $0.015\times\frac{q^3}{2}$ *per hour* or $4.1\times10^{-6}\times\frac{q^3}{2}$ *per second.* Then neglecting the second term of the right-hand-side of (3.1), we get to a high degree of approximation

$$i_1(t)=\frac{q^3}{2}e_1(t)=0.415\,e_1(t)\,. \quad \text{...........(3.6)}$$

Applying (3.6), the broken lines in Figs. 18, 19, 20 and 21 indicate the changes in the internal parts which are expected on the stand-point of induction-theory based on Benkova's uniform core model.

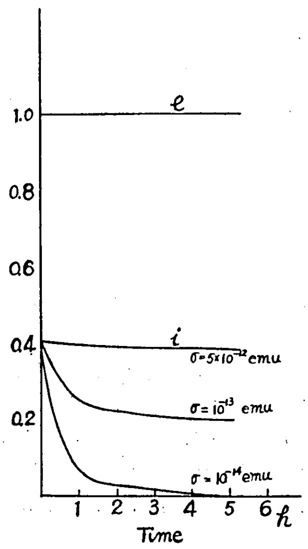

Fig. 22. The change in the induced field with the lapse of time. ($e=0$ for $t<0$, $e=1$ for $t>0$)

The results of the analyses roughly agree with those of the theory when we take into consideration the accuracy of the determination of the coefficients. Thus, we may conclude Benkova's model is approximately compatible with the induction by bay-type disturbance.

As shown in the figures mentioned above, however, it seems that the theoretical curves are used to be lower than those of the analysis. If this were true, it might become better to give larger values to q or to consider that the outer layer is somewhat conducting. Next, we shall determine the radius of the core on the basis of the present analysis of this Chapter.

The ratio of i to e[45] when e becomes maximum are given in Table XVIII with respect to the examples of the bay-type disturbance concerned in this Chapter. As shown in this Table, its most probable value becomes 2.2 ± 0.3. The probable error seems rather large because we have only four examples. If the conductivity in the core is smaller than the order of 10^{-12} *emu*, $h(t)$ decays rapidly as also shown in Fig. 22. Then $i(t)$ also decays rapidly. Thus the conductivity must not be materially smaller than the order of 10^{-12} *emu* in order to explain the facts, as shown in Figs. 18, 19, 20 and 21, that e/i is approximately equal to 2 and the mode of time-change of e and i resemble each other. For these reasons, we may say that the conductivity in the core amounts to the order of 10^{-12} *emu* or more. In that case, we can determine q or the radius of the core with the aid of e/i obtained just before. By (3.6) we have

Table XVIII. The ratios at maximum variation.

Date	e/i
Feb. 24, 1933	2.0
Apr. 10, „	2.5
Aug. 28, 1932	1.7
Sept. 23, „	2.1

$$q = \sqrt[3]{2e/i}\ .$$

Then the most probable value of q becomes 0.96, say, the thickness of the non-conducting layer amounts to about 260 *km*. However, it is desirable, in the future, to determine the conductivity in the core and the thickness of the non-conducting layer separately with the aid of more accurate analysis.

Although the determination is somewhat rough, it is interesting that the scale of the core obtained from bay-type disturbances is likely to differ slightly from that obtained in the case of S_q.

As to the influence of the ocean on the electromagnetic induction by bay-type disturbance, it will not be large by the same reason in the case of induction by S_q. Then we may neglect it.

4. The distribution of the electric currents in the earth.

We concluded in Section 3 of this Chapter that the electrical state of the earth's interior, inferred from the study on the electromagnetic induction by

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45) In order to avoid errors due to the smallness of the coefficient, we have better to use $e = \sqrt{e_a^2 + e_b^2}$ and $i = \sqrt{i_a^2 + i_b^2}$ in place of e_a, i_a, e_b and i_b themselves.

bay-type disturbances, nearly agrees with the uniform core model based on Benkova's analysis of S_q. Now in the present Section, we are going to study how deep the induced currents penetrate into the earth. As already investigated in the case of S_q, the estimate of the induced currents can be made as follows.

According to the study on the electromagnetic induction within the uniformly conducting sphere, the current-density $\vec{c}_n^m$ which corresponds to the spherical harmonics S_n^m is given by

$$\vec{c}_n^m = -\sigma a\,\rho\, C(p)\, p\, F_n(k\rho a)\, e_n(t)\, [\vec{r}\,\mathrm{grad}\, S_n^m]^{46)}\,, \qquad (4.1)$$

whereas the notations in this expression are the same with those in Section 4, Chapter II, except $C(p)$. Since the case including Q_1^1 is the most important in the case of the bay-type disturbance, it is sufficient to take into consideration only

$$\vec{c}_1^1 = -\sigma a\,\rho\, C(p)\, p\, F_1(k\rho a)\, e(t)\, [\vec{r}\,\mathrm{grad} Q_1^1 {\cos \atop \sin} \phi]\,, \qquad (4.2)$$

where

$$C(p) = \frac{1}{2}\,\frac{1}{F_0(kqa)}^{47)}\,. \qquad (4.3)$$

In order to study the radial distribution of the electric currents in the core, we deal with the coefficient of $[\vec{r}$ grad $Q_1^1 {\cos \atop \sin} \phi]$, that is

$$c(t) = -\frac{\sigma a \rho}{2}\, p\, \frac{F_1(k\rho a)}{F_0(kqa)}\, e(t)^{48)}\,. \qquad (4.4)$$

Solving the operational equation (4.4), then, we get the behavior of the change in $c(t)$ with the lapse of time together with the radial distance or ρ (its ratio to the radius of the earth). Owing to the complexity of $e(t)$ obtained, as shown in several figures in Section 2, from the actual analyses, it will be not only troublesome but unpractical to study each example. We shall here discuss an idealized case. Assuming that

$$e(t) = 0 \ (0 > t)\,, \quad e(t) = Ate^{-\omega t}\ (0 < t) \qquad (4.5)$$

where $\omega = 0.278 \times 10^{-3}\, sec^{-1}$,

we may express the general tendency of the change in the external field as shown in Fig. 23. Rewriting (4.5) in operational form we have

46) $k^2 = 4\pi\sigma p$.

47) CHAPMAN and BARTELS, *loc. cit.* Vol. II, p. 741.

48) $F_1(x) = \frac{3}{x^2}\left(\cosh x - \frac{\sinh x}{x}\right)$, $F_0(x) = \frac{\sinh x}{x}$.

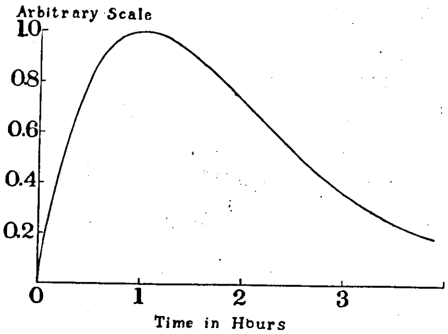

Fig. 23. Idealized change in inducing field of bay-type disturbance.

$$e(t)=\frac{Ap}{(p+\omega)^2}H(t)\ , \quad \cdots\cdots(4.6)$$

where $H(t)$ denotes Heaviside's unit function. Substituting (4.6) into (4.4), we get

$$c(t)=-A\frac{\sigma a\rho}{2}\frac{F_1(k\rho a)}{F_0(kqa)}\frac{p}{(p+\omega)^2}H(t)\ . \quad \cdots\cdots(4.7)$$

According to the operational calculus, the solution of (4.7) becomes

$$c(t)=-A\frac{\sigma a\rho}{4\pi i}\int_L\frac{F_1(k\rho a)}{F_0(kqa)}\frac{p}{(p+\omega)^2}e^{pt}dp\ , \quad \cdots\cdots(4.8)$$

where L is Bromwich's path of integration. Taking into consideration that the poles are given by $-\omega$ (double pole) and the roots of

$$\sinh\sqrt{4\pi\sigma p}\ qa=0$$

except zero, namely, $-\alpha_s=-\dfrac{\pi s^2}{4\sigma q^2a^2}$ $(s=1,2,3,\ldots\ldots)$, we can integrate (4.8) getting

$$c(t)=-A\frac{\sigma a\rho}{2}\left[\left\{(1+pt)\frac{F_1(k\rho a)}{F_0(kqa)}\right.\right.$$

$$\left.\left.+p\frac{d}{dp}\frac{F_1(k\rho a)}{F_0(kqa)}\right\}_{p=-\omega}e^{-\omega t}-\sum_{s=1}^{\infty}\left\{\frac{F_1(k\rho a)}{F_0(kqa)}\right\}_{p=-\alpha_s}\frac{\alpha_s}{\omega-\alpha_s}e^{-\alpha_s t}\right]\ . \quad \cdots\cdots(4.9)$$

The series in (4.9), however, converges rather slowly for small value of t. Then we shall obtain a suitable expression for small value of t as follows. Using Bessel function, F_n is expressed by

$$F_n(x)=\left(\frac{1}{2}ix\right)^{-n-\frac{1}{2}}\Gamma\left(n+\frac{3}{2}\right)J_{n+\frac{1}{2}}(ix)^{49)}$$

49) CHAPMAN and BARTELS, *loc. cit.* p. 738.

or

$$F_n(x)=\left(\frac{2}{x}\right)^{n+\frac{1}{2}}\Gamma\left(n+\frac{3}{2}\right)I_{n+\frac{1}{2}}(x). \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.10)$$

Meanwhile with the aid of well known asymptotic expansion

$$I_\nu(x)\simeq\sqrt{\frac{1}{2\pi x}}\,e^x\left\{1-\frac{4\nu^2-1^2}{1!\,8x}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!\,8x^2}-\cdots\right\},$$

we get for large value of x

$$F_0(x)\simeq e^x/2x\ ,\quad F_1(x)\simeq 3e^x/2x^2 \text{ [50]} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.11)$$

Thus substituting (4.11) into (4.7), we get

$$c(t)\simeq -A\,\frac{3\sqrt{\sigma}\,q}{4\sqrt{\pi}\,\rho}\,(1/\sqrt{p})\,e^{-(q-\rho)\sqrt{4\pi\sigma p}\,a}\,H(t) \quad\ldots\ldots\ldots(4.12)$$

for large value of p. As well known in the operational calculus, the solution of an operational equation when $p\to\infty$ corresponds to the one when $t\to0$. Then we get an expression of current-density for small value of t solving (4.12) as follows;

$$c(t)=-A\,\frac{3\sqrt{\sigma}\,q}{4\sqrt{\pi}\,\rho}\left[2\sqrt{\frac{t}{\pi}}\,e^{-\frac{(q-\rho)^2\pi\sigma a^2}{t}}\right.$$
$$\left.-(q-\rho)\sqrt{4\pi\sigma}\,a\left\{1-\text{erf.}\sqrt{\frac{(q-\rho)^2\pi\sigma a^2}{t}}\right\}\right]^{51)} \quad\ldots\ldots\ldots(4.13)$$

As to the change in the current density with increase of the depth, we can calculate from (4.9) or (4.13). For instance, the change in the current-density is shown in Fig. 24 in arbitrary scale for $t=3.6\times10^3$ *sec*, that is, after one *hour* past from the beginning and at that time e becoming maximum in our idealized bay-type disturbance. The current-density decreases very rapidly with increase of the depth. Indeed, it becomes less than one *percent* of its

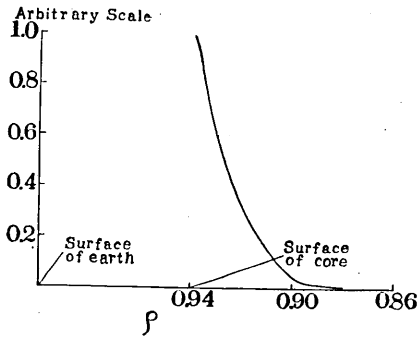

Fig. 24. The distribution of the induced current in the earth.

50) $\Gamma(3/2)=\sqrt{\pi}/2$, $\Gamma(5/2)=3\sqrt{\pi}/2^2$.

51) H. JEFFREYS, *Operational Method in Mathematical Physics.*, p. 112.
$$\frac{1}{a\sqrt{p}}\,e^{-ax\sqrt{p}}=\frac{2}{a}\left(\frac{t}{\pi}\right)^{1/2}e^{-a^2x^2/4t}-x\left(1-erf.\frac{ax}{2\sqrt{t}}\right).$$

amount at the surface of the core at the depth of a few hundred *kilometers* beneath the surface of the core. As already mentioned in the study on the current-distribution in the case of S_q, the effective values of the magnetic fields observed at the surface caused by the currents flowing in a thin shell whose radius is ρa are proportional to $\rho^{n+2}|\overrightarrow{c}|$ for the harmonic whose degree is n, the contribution of the currents flowing at any depth is shown by almost the same curve in Fig. 24 because ρ does not differ materially in the effective range of the depth. Then we know from this curve that 90% of the magnetic field are caused by the induced currents flowing in the regions from $\rho=0.94$ to $\rho=0.88$, namely from the surface of the core to the depth of 1200 *km* beneath the surface of the earth.

Then the conductivity determined in this Chapter seems to correspond to that of the upper part of the core because the contribution of the currents more than 1200 *km* below is almost negligible.

Summing up the investigations in this Chapter, we may say that the electrical state of the earth's interior which is inferred from the electromagnetic induction by S_q is confirmed also by the induction by bay-type disturbances in which the distribution of the disturbing forces over the earth differ very much from that of S_q. Since bay-type disturbances generally end within several *hours*, the induced currents penetrate into the earth more shallowly than in the case of S_q. Accordingly, the electrical state which is determined by studying bay-type disturbances seems to correspond to the shallower region compared with that obtained from the study of 24- or 12-hourly component of S_q.

Although bay-type disturbances are only analyzed with respect to the low and middle latitudes in this study, it is hoped, however, that an analysis will be made including the polar regions with the aid of abundant data which will be accumulated in the future.

(To be continued)

5. 地球内の電磁感應およびその地球内部の電氣的性質との關係 第1報 (1)

地震研究所 力武常次

A. Schuster が地球磁場の日變化を分析し，その大半は地球外部に原因を有するが，一部は地球の内部に原因を有することを發見して以來，地球内部に原因を有する部分は，地球外部に原因を有する部分の變化に伴つて地球内部に誘導される電流に因る磁場として説明出來ることが多くの學者によつて示され，從つて地球内部の電磁氣的性質を或程度推測出來るようになつた．しかしながら，地表に於ける觀測のみでは地球内部の性質を一義的に決定することは困難であり，過去に於ては主として次の立場がとられている．即ち地球の一番外側は絶緣層であつてその内側に一樣な電氣傳導度を有する核があるとして，外部および内部磁場の關係を説明し得るように外層の厚さおよび核の電氣的性質を決定するものである．

Chapman によつて爲された日變化の研究によれば，この導電核の電氣傳導度 σ は 3.6×10^{-13} *cmu* であり，絶緣層は 250 *km* の厚さとなる．また Chapman, Whitehead および Price 等の磁氣嵐の主相 (D_{st}) の研究では，上とほとんど同じ大きさの核に對して $\sigma=4.4\times10^{-12}$ *cmu* となり，S_q と D_{st} の喰違いを説明するために，S_q に比して緩漫な D_{st} によつて誘起されは電流は S_q のそれにくらべて核の内部迄浸徹し，見掛け上求められた σ はより深い部分の電氣傳導度を表わすものと考えられていた．このように σ の分布は必ずしも一樣ではないのであるから，Lahiri と Price は σ が r^{-m} (r は地球中心よりの距離) に比例して變化するようなモデルについて考察し，S_q と D_{st} の双方に對して成立つ σ の分布を求めた．また σ の違う數層よりなるモデルについては寺田一彦氏によつて研究された．

しかしながら，Lahiri と Price の研究に於ては m が相當變化しても，地表に於ける内外磁場の關係はあまり變化せず，地球内部の電氣的性質をさらに詳細に吟味する爲にはもつとはやい變化からおそい變化迄を取扱つて種々の深さについて研究することが望ましい．

筆者は本報文に於て出來るだけ多くの種類の地磁氣の變化を取扱うべく，使用し得る資料を集めたが現在のところ，靜穩日日變化 (S_q)，擾亂日日變化 (S_D)，磁氣嵐の主相 (D_{st})，灣型變化，太陽面爆發に伴う變化および磁氣嵐の急始について不充分ではあるが研究を行うことが出來た．このほかに地磁氣の脈動のような短週期の變化も存在するのであるが，地球上の廣い地域に分布している觀測資料を得ることが出來ないので他の機會にゆずることにする．

本報文に於て取扱つた地磁氣の變化は，2〜3分程度の短い變化から2〜3日程度の稍々長い時間にわたる變化を含んでおり，從來諸家に依つて爲された研究とならべて考察すると，地球内部の電氣的性質が或程度判明するように思われる．なお諸種の地磁氣變化が海特に大洋の存在のためにうける影響をも考察した．

第1章に於て地球内部の透磁率に關して從來行われた地球物理學的および物理學的考察を綜合すると，現在の學問では地球内部で透磁率が1より甚だしく大きいとは考えられないことを述べてある．これは地磁氣の永久磁場の説明として永久帶磁説が多くの難點をもつていることに對應する．したがつて，本報文を通じて透磁率は1に等しいとして取扱つた．

第2章は靜穩日日變化 (S_q) による電磁感應であつて，はじめの部分に於て太平洋のような大洋中に誘導される電流がどのような大きさの磁場をつくるかを調べるために，90° はなれた二つ

の子午線によつてかこまれた等深の海を考えて，S_q がどの程度の影響をうけるか計算した（海の影響については從來地球全体を覆う一樣な深さの海についてのみ研究が行われている）．計算の結果は大洋の中心部を除いて振幅は S_q の $1/10$ 以下となり，變化は世界時のみにより地方時には無關係であるので，變化が地方時のみに關係するとして行つた球函數分析に於ては海洋の影響は平均として小さくなる．したがつてこの場合求められた內部磁場は大体地球內部を流れる電流によるものと考えてよい．なお適當な假定のもとに，太平洋の影響を出してみると理論値とオーダーに於て一致する値が得られた．

次に比較的最近行われた S_q に關する長谷川萬吉博士および Benkova の球函數分析の結果を使用して地球內部の電氣的性質を求めた．絕緣層および電氣傳導度一定の電導核よりなる地球については既に永田武博士によつて決定されているが，その結果は Chapman の結果と異り，核の σ が $5\times10^{-12}\,emu$ 程度外層の厚さは $400\,km$ 程度でなければならない．筆者は地震波の研究で確立されているように，地下 $2900\,km$ に於て急激な不連續があることから，電氣的性質にも不連續があるものとして，內核，中間層および絕緣外層よりなるモデルに就てしらべたが，結果は永田博士の與えた結果にほゞ一致する．內核に於ては一應 $\sigma=5\times10^{-11}\,emu$ という値を得たが，誘導電流はほとんど中間層內で減衰するので此の値はあまり信用することが出來ない．

第３章に於ては S_D の電磁感應をしらべたが，資料の不足から S_q の研究によつて得られた地球の電氣的モデルが S_D の內外磁場の相互關係を大略說明出來るという結果を得たにとどまる．

第４章は D_{st} の電磁感應の研究である．長谷川博士や Benkova の資料をもととして永田博士の決定した地球の電氣的モデルは Chapman の D_{st} に關する分析結果をほゞ說明出來，Chapman を中心とする一群の學者によつて取上げられたような S_q と D_{st} との喰違いは大して重要でなくなる．筆者は第２章の後半で示したように地球內核をより導電的であるとすれば，觀測と理論とのよりよい一致を期待出來ることを D_{st} について近似的に示した．しかし，S_q の場合と同じく內核に迄浸徹する電流は極めて僅かであるから，あまり嚴密な議論をすることは全く無意味である．

第５章に於ては，畠山久尙博士の集めた資料にもとずいて，地磁氣灣型變化の電磁感應を研究した．畠山博士が經驗的に求めまた筆者によつてダイナモ說にもとずいて理論的に得られた灣型變化の電流系を念頭において，磁場のポテンシャルの球函數展開のうち最も重要な $Q_{1\sin}^{1\cos}\phi$ の係數を決定し，さらに外および內側に原因を有するポテンシャルの係數を求めた．決定は一つの灣型變化を通じて３または４度なされた．この係數の大きさの關係から，絕緣層におゝわれた導電核中では σ が少くとも $10^{-12}\,emu$ 程度で絕緣層の厚さは $300\,km$ 程度でなければならないことになる．誘導電流の分布を求めてみると導電核の表面からはかつて $2\sim300\,km$ の深さの所ではほとんど零になることがわかる．　　（未完）