2. *On the Theory of Elastic Waves in Granular Substance. I.*

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Elastic waves are propagated in sand with only 1/3 of their velocity in rock. We tried here to explain this property by a simple structural model. Though many problems are left untouched, we at least succeeded to gain some fundamental formulæ.

1. *One-dimensional model.* Consider a large number of elastic spheres of radius r and mass m pressing each other on a straight-line. In equilibrium, each sphere suffers equal pressure from both sides due to elastic deformation, which we could write $P(\alpha)$, where $\alpha=2r-a$ and $a=X_{n+1}-X_n$, X_n being the coordinate of the center of gravity of the nth sphere. Owing to small displacements, X_n changes to X_n+x_n. If we write $\Delta\alpha_n$ and ΔP_n for the variations between nth and $(n+1)$th spheres,

$$-\Delta\alpha_n=x_{n+1}-x_n, \qquad (1)$$

$$\Delta P_n=P_n-P=k\Delta\alpha_n, \quad (k=dP/d\alpha). \qquad (2)$$

Neglecting higher orders, we may consider that the center of gravity of the sphere always coincides with its geometrical center. Then we can write down the equation of motion of the n th sphere

$$m\ddot{x}_n=-P_n+P_{n+1}=-k(\Delta\alpha_n-\Delta\alpha_{n+1})$$
$$=-k\{(x_n-x_{n+1})-(x_{n-1}-x_n)\}. \qquad (3)$$

Under Born-von Kármán's boundary condition, (3) is easily solved, putting

$$x_n=A\ \exp\{2\pi i(\sigma X_n-\nu t)\}, \qquad (4)$$

where σ means wave number, that is, inverse of wave length. In our case σa must be very small, so that we can neglect its higher orders. Substituting (4) in (3), $m\ddot{x}_n+4\pi^2a^2\sigma^2kx_n=0$.

Hence for the velocity $\quad v_0=\nu/\sigma=a\sqrt{(k/m)}=2r\sqrt{(k/m)} \qquad (5)$

To determine k, we could adopt the formula gained statically by Hertz[1].

$$P=(3\pi\vartheta)^{-1}(2r)^{\frac{1}{2}}\alpha^{\frac{3}{2}}, \qquad \vartheta=(\lambda+2\mu)/4\pi\mu(\lambda+\mu) \quad \text{.............(6)}$$

Hence
$$k=dP/d\alpha=(1/\pi\vartheta)\sqrt{(r\alpha/2)} \quad \text{.............(7)}$$

Putting (7) in (5), if ρ means the density,

$$v_0^2=(3/\pi^2\vartheta\rho)\sqrt{(\alpha/2r)}. \quad \text{.............(8)}$$

When $\lambda=\mu$, (8) is expressed as

$$v_0/v_p \doteqdot 0.921(\alpha/2r)^{\frac{1}{4}}, \quad \text{.............(9)}$$

where
$$v_p^2=(\lambda+2\mu)/\rho=3\mu/\rho.$$

If we eliminate α from (6) and (9), we know v_0 is proportional to $P^{1/6}$.

Similar results were already gained by K. Iida[2].

2. *Three-dimensional model.* Imagine that centers of spheres form a space lattice in equilibrium, and pressures are the same at all contact points.

Let vector R_0 indicate the center of a sphere in question. If n is the number of its neighboring spheres, vectors connecting their centers R_j with the center R_0 are expressed as

$$\pm a\mathfrak{t}_j=R_{\pm j}-R_0, \quad |\mathfrak{t}_j|=1, \quad (j=1,2,\ldots\ldots n/2) \quad \text{.............(1)}$$

Assuming that the direction of the pressure coincides with $\mathfrak{t}_j$,

$$\Delta P_j=\mathfrak{t}_j\cdot k\,\Delta\alpha_j \doteqdot -k(\mathfrak{r}_j-\mathfrak{r}_0,\ \mathfrak{t}_j)\,\mathfrak{t}_j. \quad \text{.............(2)}$$

where $\mathfrak{r}_j$ is the displacement of the center of the jth neighboring sphere.

Neglecting other forces we get the equation of motion

$$m\ddot{\mathfrak{r}}_0=k\sum_{\pm j}(\mathfrak{r}_j-\mathfrak{r}_0,\ \mathfrak{t}_j)\mathfrak{t}_j=k\sum_j(\mathfrak{r}_j+\mathfrak{r}_{-j}-2\mathfrak{r}_0,\ \mathfrak{t}_j)\mathfrak{t}_j. \quad \text{.............(3)}$$

We put
$$\mathfrak{r}_j=\mathfrak{A}\exp\{2\pi i(\sigma R_j-\nu t)\}, \quad \text{.............(4)}$$

where $\nu/|\sigma|=v$ and $\sigma=|\sigma|s$, $|s|=1$ means the wave number vector.

Now $|\sigma|$ being very small, (3) becomes

$$m\ddot{\mathfrak{r}}_0+4\pi^2a^2\sigma k\sum(s\mathfrak{t}_j)^2(\mathfrak{r}_0\mathfrak{t}_j)\mathfrak{t}_j=0, \quad \text{.............(5)}$$

or
$$(v/v_0)^2\mathfrak{A}-\sum(\mathfrak{A}\mathfrak{t}_j)(s\mathfrak{t}_j)^2\mathfrak{t}_j=0. \quad \text{.............(6)}$$

From (6) secular equation for $(v/v_0)^2$ may be immediately obtained. We can solve this third order equation assuming lattice structure. But this is uninteresting, for, although its three roots depend on the propa-

(1) LOVE, "*Theory of Elasticity.*" p. 200.
(2) K. IIDA, *Bull. E. R. I.*, **16** (1938) 131, **17** (1939) 782.

gating direction, we practically observe only two sorts of velocities—longitudinal and transversal, which should be the consequence of irregular overlapping of so many fundamental lattices considered above. We will deal with this problem in the next section.

Here, treating approximately as a continuous substance, we will determine apparent elastic constants of our model.

Modifying (5) into

$$m\ddot{\mathfrak{r}}_0 - ka^2\sigma^2\nu^{-2}\sum(s\mathfrak{t}_j)^2(\ddot{\mathfrak{r}}_0\mathfrak{t}_j)\mathfrak{t}_j = 0,$$

We get by integration

$$m(\dot{\mathfrak{r}}_0)^2 - ka^2v^{-2}\sum(s\mathfrak{t}_j)^2(\dot{\mathfrak{r}}_0\mathfrak{t}_j)^2 = 0. \qquad (7)$$

If ε is the energy per one sphere $(=m\dot{\mathfrak{r}}_0^2/2)$, and the components of $\mathfrak{r}_0$, s and $\mathfrak{t}_j$ are (u_1, u_2, u_3), (s_1, s_2, s_3) and (t_{j1}, t_{j2}, t_{j3}) respectively, (7) becomes

$$2\varepsilon/m = (v_0/v)^2\sum(s_1t_{1j} + s_2t_{2j} + s_3t_{3j})^2(\dot{u}_1t_{1j} + \dot{u}_2t_{2j} + \dot{u}_3t_{3j})^2. \qquad (8)$$

Considering (4), we can under continuity assumption define strain components $x_1 \ldots\ldots x_6$ as follows

$$\left.\begin{aligned} &x_1 = -s_1\dot{u}_1/v, \quad x_2 = -s_2\dot{u}_2/v, \ \ldots\ldots\ldots \\ &x_4 = -s_2\dot{u}_3/v - s_3\dot{u}_2/v, \ldots\ldots\ldots\ldots \end{aligned}\right\} \qquad (9)$$

Hence (8) becomes

$$2\varepsilon/mv_0^2 = \sum(x_1t_{j1}^2 + x_2t_{j2}^2 + x_3t_{j3}^2 + x_4t_{j2}t_{j3} + x_5t_{j3}t_{j1} + x_6t_{j1}t_{j2})^2. \qquad (10)$$

If ρ^* is apparent density, ρ^*/m means the number of spheres per unit volume. Then according to the theory of elasticity

$$2\varepsilon\rho^*/m = (x_1c_1 + x_2c_2 + x_4c_4 + x_4c_4 + x_5c_5 + x_6c_6)^2, \qquad (11)$$

where the right side is written symbolically—c_ic_j means elastic constant c_{ij} if developed.

Comparing (10) with (11), we get

$$c_{\kappa\iota} = \rho^*v_0^2\sum_t\tau_\kappa\tau_\iota \qquad (12)$$

where
$$\tau_1 = t_{t1}^2, \quad \tau_2 = t_{t2}^2, \quad \tau_3 = t_{t3}^2,$$
$$\tau_4 = t_{t2}t_{t3}, \quad \tau_5 = t_{t3}t_{t1}, \quad \tau_6 = t_{t1}t_{t2}.$$

We know immediately that Cauchy's relations are satisfied, which is also natural seeing that the assumption of central force is made.

From (12) important formulæ are gained. Thus

$$c_{12} + c_{13} + c_{23} = c_{44} + c_{55} + c_{66},$$
$$c_{11} + c_{22} + c_{33} = (n/2)\rho^*v_0^2 - 2(c_{12} + c_{13} + c_{23}), \qquad (13)$$

3. *Elastic constants of an aggregate of aeolotropic bodies, when regarded as if the whole were an isotrope.* Suppose an aggregate of many small bodies of one kind. Their crystal axes are arranged in all directions, and in their average we should like to treat the aggregate as an isotrope.

We write strain components x_i and elastic constants c_{ij} in matrices

$$x=\begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix}, \qquad C=\begin{pmatrix} c_{11}\ldots\ldots c_{16} \\ \ddots \\ c_{63} \end{pmatrix}. \qquad\qquad (1)$$

Then the energy per unit volume may be expressed as

$$2E=x^*Cx, \qquad\qquad (2)$$

where x^* means the transposed matrix of x.

The direction of each crystal axis determines the rotation r from fixed position, so we write E^r and x^r in (2) to show their rotated position. If $V(r)$ means the representation of rotation group in terms of x, we may write

$$x^r=V(r)x. \qquad\qquad (3)$$

Then $$2E^r=x^{*r}Cx^r=x^*(V^*(r)CV(r))x. \qquad\qquad (4)$$

We define the energy E_0 of the resulted isotrope as the average of all E^r, that is

$$2E^0=x^*C^0x=2(1/\Omega)\int E^r d\Omega. \qquad\qquad (5)$$

where Ω means the volume of the parameter space of rotation r.

From (5) and (6)

$$C^0=(1/\Omega)\int V^*(r)CV(r)d\Omega. \qquad\qquad (6)$$

Now our problem becomes clear—being given C, to determine C^0.

To make use of the theory of representations, we must first decompose the matrix $V(r)$. This will be done by transforming $V(r)$ with a constant matrix T

$$U(r)=T^{-1}V(r)T, \qquad U^*(r)=U(r)^{-1}. \qquad\qquad (7)$$

If we put $$C^{0T}=T^*C^0T, \qquad C^T=T^*CT, \qquad\qquad (8)$$

we get from (6) and (7)

$$C^{0T}=(1/\Omega)\int U(r)^*C^TU(r)d\Omega. \qquad\qquad (9)$$

As x is a symmetric tensor of rank 2, $U(r)$ must be decomposed into two irreducible parts of 1st and 5th order. Then, from the well known lemma of Schur concerning irreducible representations, we know at last that (9) is a diagonal matrix, and its diagonal elements are

$$c_{11}^{0T}=c_{11}^{T},\quad 5c_{jj}^{0T}=\{c_{22}^{T}+c_{33}^{T}+c_{44}^{T}+c_{55}^{T}+c_{66}^{T}\}\ (j\geqq 2). \qquad (10)$$

Thus we must only know T to determine C^0.

In the space of homogeneous polynomials, solid spherical harmonics are the bases of irreducible representation of rotation group. To the representation of 5th order corresponds spherical function of order 2. Adding to them an invariant $(x^2+y^2+z^2)$, T will mean the transformation from these six functions to the components of the symmetric tensor, x^2, y^2, z^2, $2yz$, $2zx$, $2xy$. To calculate T is then an easy task.

Finally we calculated c_{jj}^{T} from the second equation of (8), and putting them in (10), we got c_{jj}^{0T}. Determination of C^0 was the consequent result of the first equation of (8) written in the following form

$$C^0=(T^*)^{-1}C^{0T}T^{-1}. \qquad (11)$$

The obtained C^0 is a symmetric matrix, and the components except the followings, are all zeros.

$$c_{11}^0=c_{22}^0=c_{33}^0=(3/15)(c_{11}+c_{22}+c_{33})+(2/15)(c_{23}+c_{31}+c_{12})+(4/15)(c_{44}+c_{55}+c_{66}),$$

$$c_{23}^0=c_{31}^0=c_{12}^0=(1/15)(\quad ,, \quad)+(4/15)(\quad ,, \quad)-(2/15)(\quad ,, \quad),$$

$$c_{44}^0=c_{55}^0=c_{66}^0=(1/2)(c_{11}^0-c_{12}^0). \qquad (12)$$

From this result C^0 can be interpreted as the tensor of elastic constants of an isotrope, as we expected.

Application of this section is not restricted to the granular structure only, but here we will not go further.

4. *Conclusions.* Applying the last section to our case, that is, if we substitute 2-(13) in 3-(12), we get immediately

$$c_{12}^0=(n/30)\rho^* v_0^2,\qquad c_{11}^0=(n/10)\rho^* v_0^2=3c_{12}^0 \qquad (1)$$

c_{11}^0 and c_{12}^0 depend only on the number of neighboring spheres, and not on lattice structure. This is a remarkable fact. (Equation (1) could also be gained by directly averaging 2. (3).)

Then the velocities of elastic waves are

$$v_p^*=\surd(c_{11}^0/\rho^*)=v_0\surd(n/10),\quad v_s^*=\surd\{(c_{11}^0-c_{12}^0)/2\rho^*\}=v_p^*/\surd 3. \quad (2)$$

These are the desired results.

To consult with experimental data, a slight deformation should be made on the expression of v_0. We define statical pressure P_0 as the force per unit area acting perpendicular on the plane imagined in the granular aggregate. Consider a polyhedron wrapping a sphere made up of tangential planes at contact points, and replace its surface area with that of the equivoluminal sphere. Then P_0 may be expressed as

$$P_0=(nP/4\pi r^2)(\rho^*/\rho)^{2/3} \quad \text{(3)}$$

From (2) and (3)

$$v_p^*=0.930\,\varepsilon_n(P_0/\rho)^{1/6}\cdot v_p^{2/3},\quad \varepsilon_n=(n/10)^{1/3}(\rho^*/\rho)^{-1/9} \quad \text{(4)}$$

Values of ε_n are shown in Table I. We compared the results with K. Iida's experiments[3], assuming temporarily $P_0=\rho^*gh$, the coincidence was somewhat satisfactory with sand but not so good in case of lead and rubber, and the remarkable change due to porosity $(=1-\rho^*/\rho)$ could not be well explained.

An example of the numerical calculation is in Tab. II, which we think proves qualitative correctness of our theory.

For the results in this paper, so also for the faults, if any, one of the authors, Takahashi, is responsible.

Table I. Values of ε_n.

n	6	8	10
ρ^*/ρ	.52	.60	.70
ε_n	.90	.90	1.04

Values of ρ^*/ρ due to L. C. Graton and H. J. Frazer are quoted in Iida's paper.

Table II. Wave-velocities in a Sand, when $\rho=2.7$ g/cm^3, $v_p=5000$ m/sec. and $\varepsilon_n=1$.

p_0 bar	.001	.01	1	100	1000
v_p^* m/sec	157	230	497	1070	1570
v_s^* ,,	91	133	287	618	907

(3) K. Iida, *loc. cit.*