1. *Mathematical Study of the Propagation of Waves upon Stratified Medium.* (1)

By Yasuo SATÔ,

Earthquake Research Institute.

(Read Dec. 16, 1947.—Received March 30, 1949.)

INTRODUCTION.

Since the celebrated work of H. Lamb[1], the generation and propagation of waves upon the surface of elastic body is studied by a number of eminent authors[2]. They are, however, so skillful—consequently intricate—that no one could apply their results to a more complicated problem such as the propagation of waves upon stratified medium. Hence, it seems to the author that it is not useless to state a simple way, if not an exact way, of calculation applicable to practical cases.

PART I. PRELIMINARY ANALYSIS.

§ 1. Asymptotic expansion of a definite integral $\int_a^b F(\xi) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} \xi x d\xi$.

A form of definite integral shown above often appears in the problem of elastic waves. Here we shall show a general and convenient way of getting the asymptotic expansion of such an integral. We know already that this integral tends to zero when x becomes large without limit. (Riemann-Lebesgue's lemma[3]). Use notations

$$I(x) = I_c(x) + iIs(x) = \int_a^b F(\xi) \exp(i\xi x) d\xi, \qquad \Phi(\xi) = F(\xi) \cos \xi x$$

When x is sufficiently large, $\cos \xi x$ generally varies far more rapidly than the function $F(\xi)$, therefore $\Phi(\xi)$ may be regarded as a cos-curve whose amplitude is represented by $F(\xi)$, and the integral $I_c(x)$ is the algebraic sum of the area between the curve $\Phi(\xi)$ and ξ-axis. On the other hand, since two neighbouring area elements are nearly equal and have different signs, they cansel with each other, and the values of integral is determined only by the two limits of integral and points where $F'(\xi)$ becomes infinity. The following calculations are lead by the above idea.

(1) H. LAMB, *Phil. Trans* **A** 203, 1 (1904) 1.
(2) H. NAKANO, *Jap. Journ. Astro. Geophys.* **2** (1925) 1.
K. SEZAWA and G. NISIMURA, *Bull. Earthq. Res. Inst.* **6** (1929) 1, 7 (1929) 41.
T. SAKAI, *Geophys. Mag.* **14** (1936) 582.
(3) WHITTAKER and WATSON, *Modern Analysis* p. 172.

§ 2. When $F(\xi)$ is differentiable at the closed domain $[a, b]$.

We assume as $0 \leqq a < b$, and $z = \xi x$. Then, by partial integration

$$I_c(x) = \int_{ax}^{bx} F\left(\frac{z}{x}\right)\frac{1}{x}dz = \frac{1}{x}\left[F\left(\frac{z}{x}\right)\sin z\right]_{ax}^{bx} - \frac{1}{x}\int_{ax}^{bx} F'\left(\frac{z}{x}\right)\sin z \frac{1}{x}dz$$

$$= x^{-1}[F(b)\sin bx - F(a)\sin ax] + x^{-2}[F'(b)\cos bx - F'(a)\cos ax]$$

$$+ x^{-3}[F''(b) \;,, \; + F''(a) \;,, \;] + x^{-4}[F'''(b) \;,, \; - F'''(a) \;,, \;] + \cdots \quad (2\cdot1)$$

Similarly

$$I_s(x) = x^{-1}[-F(b)\cos bx + F(a)\cos ax] + x^{-2}[F'(b)\sin bx - F'(a)\sin ax]$$

$$+ x^{-3}[F''(b) \;,, \; - F''(a) \;,, \;] + x^{-4}[-F'''(b) \;,, \; + F'''(a) \;,, \;] + \cdots \quad (1\cdot2)$$

§ 3 When $F(\xi)$ contains $\sqrt{\xi^2 - a^2}$.

We assume that $F(\xi)$ can be expanded at $\xi = a$ as a power series of $(\xi - a)^{\frac{1}{2}}$ that begins by the term $\cdot (\xi - a)^{-\frac{1}{2}}$. We use the following notations.

$$\begin{cases} F(\xi) = F_{-1}(a)\frac{1}{2}(\xi-a)^{\frac{1}{2}} + F^0(\xi) \\ \qquad F^0(\xi) \equiv F_0(a) + F_1(a)(\xi-a)^{\frac{1}{2}} + F_2(a)(\xi-a) + \cdots\cdots\cdots\cdots \quad (3\cdot1) \\ F^{0\prime}(\xi) = F_1(a)\cdot\frac{1}{2}(\xi-a)^{-\frac{1}{2}} + F^1(\xi) \\ \qquad F^1(\xi) = F_2(a) + F_3(a) - \frac{3}{2}(\xi-a)^{\frac{1}{2}} + F_4(a)\cdot 2(\xi-a) + \cdots\cdots \text{ etc.} \end{cases}$$

$$F^k(a) = k\,!\; F_k(a)$$

Denote $C + iS = \int_{ax}^{bx}(z - ax)^{-\frac{1}{2}}\exp(iz)dz$,

then we have, using Fresnel's integrals; $C\{\quad\}$ and $S\{\quad\}$[4)]

$$C + iS = \sqrt{(2\pi)}\cdot\exp(iax)C\{(b-a)x\} + i\sqrt{(2\pi)}\exp(iax)S\{(b-a)x\}$$

$$= \sqrt{\pi}\cdot \quad \exp i(ax + \pi/4) + x^{-\frac{1}{2}}(b-a)^{-\frac{1}{2}}\exp i(bx - \pi/2) + \cdots \quad (3\cdot2)$$

From (3·1)

$$I_c(x) = x^{-\frac{1}{2}}F_{-1}(a)\int_{ax}^{bx}(z-ax)^{-\frac{1}{2}}\cos z\,dz + x^{-1}\int_{ax}^{bx}F^0\left(\frac{z}{x}\right)\cos z\,dz \quad \cdots (3\cdot3)$$

Since the first term is calculated in (3·2), applying partial integration

(4) TERAZAWA, *Sûgakugairon* p. 189.

to the second term,

2nd term

$$= x^{-1}\left[F^0\left(\frac{z}{2}\right)\sin z\right]_{ax}^{bx} - x^{-1}\int\left[F_1(a)\cdot\frac{1}{2}\left(\frac{z}{x}-a\right)^{-\frac{1}{2}} + F'\left(\frac{z}{x}\right)\right]\sin z\cdot\frac{1}{x}dz$$

$$= x^{-1}[F^0(b)\sin bx - F^0(a)\sin ax] - x^{-\frac{3}{2}}F_1(a)\cdot\frac{1}{2}S - x^{-2}\int_{ax}^{bx}F^1\left(\frac{z}{x}\right)\sin z\,dz \tag{3·4}$$

Introducing (3·2) and (3·4) into (3·3), we have

$$I_c(x) = x^{-\frac{1}{2}}F_{-1}(a)\sqrt{\pi}\cdot\cos(ax+\pi/4) + x^{-1}[F(b)\sin bx - F_0(a)\sin ax]$$
$$-x^{-\frac{3}{2}}F_1(a)\frac{1}{2}\sqrt{\pi}\cdot\sin(\quad ,, \quad) + x^{-2}[\cdots\cdots\cdots\cdots\cdots\cdots] + \cdots \tag{3·5}$$

Similarly

$$I_s(x) = x^{-\frac{1}{2}}F_{-1}(a)\sqrt{\pi}\cdot\sin(ax+\pi/4) - x^{-1}[F(b)\cos bx - F_0(a)\cos ax]$$
$$+x^{-\frac{1}{2}}F_1(a)\frac{1}{2}\sqrt{\pi}\cdot\cos(\quad ,, \quad) + x^{-2}[\cdots\cdots\cdots\cdots\cdots\cdots] + \cdots \tag{3·6}$$

When $F(\xi)$ contains $\sqrt{\xi^2-b^2}$, we may have similar results also. Using proper consideration with respect to the argument of $\sqrt{\xi^2-b^2}$, we obtain the following results

$$I_c(x) = -ix^{-\frac{1}{2}}F_{-1}(b)\sqrt{\pi}\cdot\sin(bx+\pi/4) + x^{-1}[F_0(b)\sin bx - F(a)\sin ax]$$
$$-ix^{-\frac{3}{2}}F_1(b)\frac{1}{2}\sqrt{\pi}\cdot\cos(\quad ,, \quad) + x^{-2}[\cdots\cdots\cdots\cdots\cdots\cdots] + \cdots \tag{3·7}$$

$$I_s(x) = ix^{-\frac{1}{2}}F_{-1}(b)\sqrt{\pi}\cdot\cos(\quad ,, \quad) - x^{-1}[F_0(b)\cos bx - F(a)\cos ax]$$
$$-ix^{-\frac{3}{2}}F_1(b)\frac{1}{2}\sqrt{\pi}\cdot\sin(\quad ,, \quad) + x^{-2}[\cdots\cdots\cdots\cdots\cdots\cdots] + \cdots \tag{3·8}$$

Where $\quad F(\xi) = F_{-1}(b)(\xi-b)^{-\frac{1}{2}} + F_0(b) + F_1(b)(\xi-b)^{\frac{1}{2}} + \cdots$ ·········· (3·9)

$$\equiv \exp(-i\pi/2)F_{-1}(b)(b-\xi)^{-\frac{1}{2}} + F^0(\xi)$$

§ 4 When $F(\xi)$ contains $\sqrt{\xi^2-c^2}$, where $a<c<b$.

Using the results of the above article, we can easily obtain the formulae which are corresponding to this case. Dividing the integral $\int_a^b$ to two parts $\int_{c}^{c}$ and $\int_{c}^{b}$, and applying (3·5), (3·6), (3·7) and (3·8), we have at once,

$$\left\{\begin{aligned} I_c(x) &= x^{-\frac{1}{2}}F_{-1}(c)\sqrt{\pi}\cdot\exp\{-i(cx+\pi/4)\}+x^{-1}[F(b)\sin bx-F(a)\sin ax] \\ &\quad -x^{-\frac{3}{2}}F_1(c)\frac{1}{2}\sqrt{\pi}\cdot\exp\{-i(cx-\pi/4)\}+x^{-2}[\cdots\cdots] \\ I_s(x) &= x^{-\frac{1}{2}}F_{-1}(c)\sqrt{\pi}\cdot\exp\{-i(cx-\pi/4)\}+x^{-1}[F(b)\cos bx-F(a)\cos ax] \\ &\quad +x^{-\frac{3}{2}}F_1(c)\frac{1}{2}\sqrt{\pi}\cdot\exp\{-i(cx+\pi/4)\}+x^{-2}[\cdots\cdots] \end{aligned}\right. \tag{4·1}$$

From the above expressions we notice at a glance, that the asymptotic expansions are determined by the coefficients at the two limits of integration and the points where the derivative becomes infinity. This is what we expected at the beginning of this paper. Here, we will add that the coefficients at a and b are involved as the same forms with §2.

§ 5. Further development of formulae.

For the convenience of practical use, we shall here set forth another formula. Generally, if $F(\xi)$ is expressed as

$$F(\xi) = A_{-1}(\xi)(\xi^2-c^2)^{-\frac{1}{2}}+A_0(\xi)+A_1(\xi)(\xi^2-c^2)^{\frac{1}{2}}+\cdots \tag{5·1}$$

where $A_{-1}(\xi)$, $A_0(\xi)$, $\cdots$ are functions that are identically zero, or not zero and differantiable at $\xi = c$.

In this case, $F(\xi)$ may be developed in the following scheme; the proof is self-evident.

$$\begin{aligned} F(\xi) &= A_{-1}(c)\cdot(2c)^{-\frac{1}{2}}(\xi-c)^{-\frac{1}{2}}+A_0(c) \\ &\quad +\{-A_{-1}(c)/(4c)(+A_{-1}'(c)+A_1(c)\cdot 2c\}(2c)^{-\frac{1}{2}}(\xi-c)^{\frac{1}{2}}+\cdots \end{aligned} \tag{5·2}$$

Combining (4·1) with the above expression, we arrive at formulae which are convenient for practical use.

We may test the validity of new formulae by applying them to integrals whose asymptotic expansion is already known; e.g. Bessel function $J_0(x) = \frac{2}{\pi}\int_1^\infty \frac{\sin \xi x}{\sqrt{\xi^2-1}}d\xi$,[5] or the problem of generation of elastic waves treated by H. Lamb. $u_0 = \frac{Q}{\pi\mu}\int_0^\infty \frac{\xi(2\xi^2-k^2-\alpha\beta)}{(2\xi^2-k^2)^2-4\xi^2\alpha\beta}\sin \xi x\, d\xi$,[6] etc. ($\alpha = \sqrt{\xi^2-k^2}$, $\beta = \sqrt{\xi^2-k^2}$).

(5) WATSON, *Bessel Functions* p. 194.
(6) Ibid. (1), Expression (52).