2. *Potentials of the Simultaneous Equations* $(\Gamma^2 - aD)u_i = b\dfrac{\partial\theta}{\partial x_i}$.*

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In the fields of mathematical physics, i.e. elasticity, hydrodynamics, electricity, etc., there have frequently appeared simultaneous linear partial differential equations of the forms

$$(\Gamma^2 - aD)u_i = b\frac{\partial\theta}{\partial x_i} \qquad (i = 1, 2, 3), \tag{1}$$

where Γ^2 denotes Laplace's operator in three dimensions, D an operator such as 0, $\partial^2/\partial t^2$, $\partial/\partial t$, $\partial/\partial x + \partial/\partial y + \partial/\partial z$, etc., u_i a component of a vector aquntity, and θ a scalar quantity.

So far as the present stage of my work is concerned, potential equations of (1) have ever been obtained in the forms

$$(\Gamma^2 - aD)(\Gamma^2 - cD')\chi = 0 \tag{2}$$

and

$$(\Gamma^2 - aD)\varphi = 0 \quad \text{provided} \quad \theta = 0, \tag{3}$$

D' being some operator. The vector components, and the tensor components, are given by certain operations performed on χ and φ. It is noted that φ is in general independent of χ. From the mathematical point of view potential functions thus found may be termed 'solutions' of the original simultaneous equations (1).

The procedure of obtaining (2) will be sufficient by taking

$$u = \left\{\kappa\frac{\partial}{\partial x}\mathit{\Gamma}^2 + \kappa'\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\mathit{\Gamma}^2\right\}\chi + \mathcal{D}\chi, \quad \cdots\cdots, \tag{4}$$

$\mathcal{D}$ being an operator determinable according to aD in (1); κ, κ' being constants. The operator $\mathit{\Gamma}^2$ here proposed represents

$$\mathit{\Gamma}^2 = \frac{\partial^2}{\partial y\partial z} + \frac{\partial^2}{\partial z\partial x} + \frac{\partial^2}{\partial x\partial y}, \tag{5}$$

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which has frequently been used in the series of my work. The principal part of the right-hand side of (4) has been first obtained in the case of statical elasticity (cf. I), by commencing with the most generalized form of the so-called Maxwell's stress-functions and the Morera's.[1] The φ-part is due, for instance, to the substitution $u=0$, $v \neq 0$, $w \neq 0$; for χ and u will satisfy the same differential equation, so that the correspondence between them is complete; and then we may have

$$u = 0\,, \qquad v = \frac{\partial\varphi}{\partial z}\,, \qquad w = -\frac{\partial\varphi}{\partial y}\,; \tag{6}$$

and the cyclical interchanges of u, v, w and x, y, z are also solutions of (1). Again, the sum of these three sets of solutions is a solution of (1), which is the form for φ adopted in this paper.

Several examples of potentials which have been obtained hitherto will be shown in the following:

I. Statical Elasticity ruled by Hooke's Law.

In this case fundamental equations when no body forces act can be expressed in the forms

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\Delta + (1-2\sigma)\nabla^2(u, v, w) = 0\,, \tag{7}$$

Δ being cubical dilatation, and σ Poisson's ratio. Potential equations of (7) have been found to be

$$\nabla^4\chi = 0 \qquad \text{and} \qquad \nabla^2\varphi = 0\,. \tag{8}$$

In terms of these potentials the displacements are given by

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1) The potential χ of the statical elasticity was obtained by putting

$$\begin{aligned} Y_z = \Big[a\nabla^2 + b\frac{\partial^2}{\partial x^2} + c\frac{\partial^2}{\partial y\partial z} + d\frac{\partial}{\partial x}\Big(\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\Big)\Big]\chi_1 \\ + e\Big(\frac{\partial^2\chi_2}{\partial y^2}+\frac{\partial^2\chi_3}{\partial z^2}\Big) + f\Big(\frac{\partial^2\chi_2}{\partial z^2}+\frac{\partial^2\chi_3}{\partial y^2}\Big) + g\frac{\partial^2}{\partial x^2}(\chi_2+\chi_3) \\ + h\frac{\partial^2}{\partial y\partial z}(\chi_2+\chi_3) + i\frac{\partial}{\partial x}\Big(\frac{\partial\chi_2}{\partial y}+\frac{\partial\chi_3}{\partial z}\Big) + j\frac{\partial}{\partial x}\Big(\frac{\partial\chi_2}{\partial z}+\frac{\partial\chi_3}{\partial y}\Big), \;\cdots\cdots, \end{aligned}$$

and further by putting

$$\chi_1 = \Big[\alpha\nabla^2 + \beta\frac{\partial^2}{\partial x^2} + \gamma\frac{\partial^2}{\partial y\partial z} + \delta\frac{\partial}{\partial x}\Big(\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\Big)\Big]\chi, \;\cdots\cdots,$$

which is of homogeneous quadratic form, the linear form of the substitution having resulted in failure.

$$u = \left\{ -\frac{\partial}{\partial x}\nabla^2 + (1-\sigma)\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\nabla^2 \right\}\chi + \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right)\varphi, \cdots\cdots, \quad (9)$$

v, w being given by cyclical interchange of x, y, z. χ may be divided in two parts, one being harmonic, χ_1, say, and the other biharmonic proper. The former χ_1 is at once seen to be the same as the known displacement-potential, if we write $\nabla^2\chi_1 = \phi$. It can easily be verified that $x\phi$, $y\phi$, $z\phi$ and $r^2\phi$, where $r^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$, are all biharmonic, provided ϕ is harmonic.

An application of the present potentials has been made to one of Boussinesq's problems of simple type; and its solution is in general compatible with the shearing forces as well as the normal pressure.[2)]

II. Dynamical Elasticity ruled by Hooke's Law.

In this case fundamental equations can take the forms

$$\frac{\partial \Delta}{\partial x} + (1-2\sigma)\nabla^2 u = (1-2\sigma)\frac{\rho}{\mu}\frac{\partial^2 u}{\partial t^2}, \cdots\cdots. \quad (10)$$

Potential equations of these have been found to be

$$\Box_1\Box_2\chi = 0 \qquad \text{and} \qquad \Box_2\varphi = 0, \quad (11)$$

where

$$\Box_1 = \nabla^2 - \frac{\rho}{\lambda+2\mu}\frac{\partial^2}{\partial t^2}, \qquad \Box_2 = \nabla^2 - \frac{\rho}{\mu}\frac{\partial^2}{\partial t^2}. \quad (12)$$

In terms of these potentials the displacements are given by

$$u = \left\{ -\frac{\partial}{\partial x}\nabla^2 + (1-\sigma)\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\Box_1 \right\}\chi + \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right)\varphi, \cdots\cdots. \quad (13)$$

Two kinds of functions χ_1 and χ_2 which satisfy the equations $\Box_1\chi_1 = 0$ and $\Box_2\chi_2 = 0$ respectively are both included in χ; and it will easily be seen that χ_1 corresponds to the irrotational wave, and that χ_2 and φ correspond to the two kinds of equivoluminal waves.

III. Dynamical Visco-Elasticity ruled by Voigt's Law.

In this case fundamental equations take the forms

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2) The known Boussinesq's potential method is not compatible with the shearing forces, since it gives the expressions $Y_z = z\frac{\partial^2 V}{\partial y \partial z}$ and $Z_x = z\frac{\partial^2 V}{\partial z \partial x}$ (cf. Love's Elasticity, p. 193), both of which reduce to zero at the plane surface of a semi-infinite solid, where $z = 0$.

$$(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u+(\lambda'+\mu')\frac{\partial^2\Delta}{\partial x\partial t}+\mu'\nabla^2\frac{\partial u}{\partial t}=\rho\frac{\partial^2 u}{\partial t^2}\,,\quad\cdots\cdots. \tag{14}$$

Potential equations of these have been found to be

$$\Box_1'\Box_2'\chi=0 \quad\text{and}\quad \Box_2'\varphi=0\,, \tag{15}$$

where $$\Box_1'=\Box_1+\frac{\lambda'+2\mu'}{\lambda+2\mu}\frac{\partial}{\partial t}\nabla^2\,,\qquad \Box_2'=\Box_2+\frac{\mu'}{\mu}\frac{\partial}{\partial t}\nabla^2\,. \tag{16}$$

In terms of these potentials the displacements are given by

$$u=\left\{-\frac{\partial}{\partial x}\left(1+\frac{\lambda'+\mu'}{\lambda+\mu}\frac{\partial}{\partial t}\right)\nabla^2+(1-\sigma)\left(\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\Box_1'\right\}\chi+\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)\varphi,\ \cdots \tag{17}$$

IV. Oseen's Equations for Viscous Fluid.

Fundamental equations of motion take the forms

$$U\frac{\partial}{\partial x}(u,v,w)=-\frac{1}{\rho}\left(\frac{\partial}{\partial x}\,,\frac{\partial}{\partial y}\,,\frac{\partial}{\partial z}\right)p+\nu\nabla^2(u,v,w)\,. \tag{18}$$

(Cf. Lamb's Hydrodynamics, p. 610.) Potential equations of these have been found to be

$$\nabla^2\left(\nabla^2-2k\frac{\partial}{\partial x}\right)\chi=0 \quad\text{and}\quad \left(\nabla^2-2k\frac{\partial}{\partial x}\right)\varphi=0\,. \tag{19}$$

In terms of these potentials the velocities and pressure are given by

$$\left.\begin{aligned} u&=\left(\frac{\partial^2}{\partial x^2}-\nabla^2\right)\chi+\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)\varphi, & v&=\frac{\partial^2\chi}{\partial x\partial y}+\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial x}\right)\varphi\,,\\ w&=\frac{\partial^2\chi}{\partial x\partial z}+\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)\varphi\,, & p&=\mu\frac{\partial}{\partial x}\left(\nabla^2-2k\frac{\partial}{\partial x}\right)\chi\,. \end{aligned}\right\} \tag{20}$$

If the uniform flow U is directed to the diagonal line whose direction cosines are all $1/\sqrt{3}$, the fundamental equations of motion will take the forms

$$\frac{U}{\sqrt{3}}\left(\frac{\partial}{\partial x'}+\frac{\partial}{\partial y'}+\frac{\partial}{\partial z'}\right)u'=-\frac{1}{\rho}\frac{\partial p}{\partial x'}+\nu\nabla'^2 u',\quad\cdots\cdots, \tag{21}$$

which reduce to the general formulation (1).[3]

3) The derivation of (19) and (20) has been done by beginning with (21). It will be found that the transformation between (21) and (18) is effected by the following scheme:

	x'	y'	z'
x	α	α	α
y	$-\alpha$	β	γ
z	$-\alpha$	γ	β

$$\alpha=\frac{1}{\sqrt{3}},\quad \beta=\frac{1}{2}\left(1+\frac{1}{\sqrt{3}}\right),\quad \gamma=-\frac{1}{2}\left(1-\frac{1}{\sqrt{3}}\right);$$

and in this case the operator ∇^2 transforms to

$$\nabla'^2=\frac{\partial^2}{\partial y'\partial z'}+\frac{\partial^2}{\partial z'\partial x'}+\frac{\partial^2}{\partial x'\partial y'}=\frac{1}{2}\left(-\nabla^2+3\frac{\partial^2}{\partial x^2}\right).$$

V. UNSTEADY OSEEN'S EQUATIONS.

When the uniform flow, U, in x-direction varies with time t, fundamental equations will be written

$$\frac{\partial}{\partial t}(U+u,\, v,\, w)+U\frac{\partial}{\partial x}(u,\, v,\, w) = -\frac{1}{\rho}\left(\frac{\partial}{\partial x},\, \frac{\partial}{\partial y},\, \frac{\partial}{\partial z}\right)p + \nu\nabla^2(u,\, v,\, w). \quad (22)$$

Potential equations of these have been found to be

$$\nabla^2\left(\nabla^2 - 2k\frac{\partial}{\partial x} - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\chi = 0 \quad \text{and} \quad \left(\nabla^2 - 2k\frac{\partial}{\partial x} - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\varphi = 0. \quad (23)$$

In terms of these potentials the velocities and pressure are given by

$$\left.\begin{aligned} u &= \left(\frac{\partial^2}{\partial x^2} - \nabla^2\right)\chi - U + \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right)\varphi, \quad & v &= \frac{\partial^2\chi}{\partial x\partial y} + \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial x}\right)\varphi, \\ w &= \frac{\partial^2\chi}{\partial x\partial z} + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\varphi, & p &= \mu\frac{\partial}{\partial x}\left(\nabla^2 - 2k\frac{\partial}{\partial x} - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\chi. \end{aligned}\right\} \quad (24)$$

If attention is confined to the two potentials of the second degree in χ, (24) can be written

$$\left.\begin{aligned} u &= -\frac{\partial\Phi}{\partial x} + \frac{1}{2k}\frac{\partial\Psi}{\partial x} - \Psi - \frac{1}{U}\frac{\partial}{\partial t}\int\Psi dx - U, \quad & v &= -\frac{\partial\Phi}{\partial y} + \frac{1}{2k}\frac{\partial\Psi}{\partial y}, \\ w &= -\frac{\partial\Phi}{\partial z} + \frac{1}{2k}\frac{\partial\Psi}{\partial z}, & p &= \rho U\frac{\partial\Phi}{\partial x} + \rho\frac{\partial\Phi}{\partial t}, \end{aligned}\right\} \quad (25)$$

where $\quad \nabla^2\Phi = 0 \quad$ and $\quad \left(\nabla^2 - 2k\frac{\partial}{\partial x} - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\Psi = 0. \quad (26)$

By means of this system of equations, I have found an approximation to the resistance experienced by a sphere moving unsteadily through a viscous fluid, the result of which is $6\pi\mu aU/(1-3\nu U'/U^3)$, U' being the given acceleration of the sphere.

Appendix. In the case of symmetrical strain of a solid of revolution, stress-equations are in general written

$$\frac{\partial\widehat{rr}}{\partial r} + \frac{\partial\widehat{rz}}{\partial z} + \frac{\widehat{rr}-\widehat{\theta\theta}}{r} - \frac{\partial U}{\partial r} = 0, \quad \frac{\partial\widehat{rz}}{\partial r} + \frac{\partial\widehat{zz}}{\partial z} + \frac{\widehat{rz}}{r} - \frac{\partial W}{\partial z} = 0, \quad (27)$$

U, W being some functions of r, z and t. If Hooke's law is referred to, we can obtain, after some amount of calculation,[4]

4) Cf. Love's Elasticity, pp. 274-276; and further introduce the integral $\iint Urdrdz$.

$$\left.\begin{aligned} &(1-\sigma)\Gamma^4\chi = -(1-2\sigma)\Gamma^2\int Udz - \frac{\partial U}{\partial z} + \frac{\partial W}{\partial z}\,, \\ &u = -\frac{1}{2\mu}\frac{\partial^2\chi}{\partial r\partial z}\,, \quad w = \frac{1}{2\mu}\left\{2(1-\sigma)\Gamma^2 - \frac{\partial^2}{\partial z^2}\right\}\chi + \frac{1-2\sigma}{\mu}\int Udz\,, \end{aligned}\right\} \qquad (28)$$

where and in what follows Γ^2 denotes $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$.

In cases of a uniform gravitation (ρg) in the positive z-direction and of a centrifugal force of constant angular velocity (ω) rotating about the z-axis, it is sufficient to take $U = W$, and $U = -\rho gz$ and $U = -\frac{1}{2}\rho\omega^2 r^2$ respectively. In the case of elastic vibration, we can take $\frac{\partial U}{\partial r} = \rho\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial W}{\partial z} = \rho\frac{\partial^2 w}{\partial t^2}$; and then we obtain

$$\Box_1\Box_2\chi = 0\,, \quad u = -\frac{1}{2\mu}\frac{\partial^2\chi}{\partial r\partial z}\,, \quad w = \frac{1}{2\mu}\left\{2(1-\sigma)\Box_1 - \frac{\partial^2}{\partial z^2}\right\}\chi\,. \qquad (29)$$

Further if Voigt's law is referred to, we can obtain

$$\left.\begin{aligned} &\Box_1'\Box_2'\chi = 0\,, \\ &u = -\frac{1}{2\mu}\left(1 + \frac{\lambda'+\mu'}{\lambda+\mu}\frac{\partial}{\partial t}\right)\frac{\partial^2\chi}{\partial r\partial z}\,, \quad w = \frac{1}{2\mu}\left\{2(1-\sigma)\Box_1' - \left(1 + \frac{\lambda'+\mu'}{\lambda+\mu}\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial z^2}\right\}\chi\,. \end{aligned}\right\} \qquad (30)$$

In the case of thermo-elastic equations ruled by Duhamel's law, similar calculation will afford

$$\left.\begin{aligned} &\Gamma^4\chi = -\frac{Ec}{1-\sigma}\Gamma^2\int Tdz\,, \\ &u = -\frac{1}{2\mu}\frac{\partial^2\chi}{\partial r\partial z}\,, \quad w = \frac{1}{2\mu}\left\{2(1-\sigma)\Gamma^2 - \frac{\partial^2}{\partial z^2}\right\}\chi + 2(1+\sigma)c\int Tdz\,, \end{aligned}\right\} \qquad (31)$$

T being a distribution of temperature, E Young's modulus, and c the linear coefficient of thermal expansion of the solid.

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