32. *On the Propagation of Rayleigh-waves in Dispersive Elastic Media.*

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Although attempts have been made to solve a number of problems relating to dispersive Rayleigh-waves, their dispersive nature as shown by the results of calculation is because stratified layers of elastic media were assumed and not because the media themselves are dispersive. In this paper we shall consider the case in which the waves are dispersed even though the layer through which the waves are transmitted is not stratified.

If we assume that the vibration of a point in the media is resisted by a force proportional to the displacement of the same point, in addition to the elastic force, it is possible for the equations of motion for longitudinal waves and transverse waves respectively to have the forms

$$\rho\frac{\partial^2 u}{\partial t^2}=(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}-au, \tag{1}$$

$$\rho\frac{\partial^2 u'}{\partial t^2}=\mu\frac{\partial^2 u'}{\partial x^2}-a'u'. \tag{2}$$

The solutions for the velocities of both these waves are of the types

$$V=\sqrt{\frac{\lambda+2\mu}{\rho}+\frac{a}{\rho f^2}},\quad V'=\sqrt{\frac{\mu}{\rho}+\frac{a'}{\rho f^2}}, \tag{3}$$

respectively, where $2\pi/f$ is wave length. It thus holds that if there is vibrational resistance proportional to the displacement, bodily waves are also dispersive.

We shall next consider the case of a semi-infinite body, the equations of motion of which are then expressed by

$$\left.\begin{aligned}\rho\frac{\partial^2 u}{\partial t^2}&=(\lambda+\mu)\frac{\partial \Delta}{\partial x}+\mu\nabla^2 u-au,\\ \rho\frac{\partial^2 v}{\partial t^2}&=(\lambda+\mu)\frac{\partial \Delta}{\partial y}+\mu\nabla^2 v-av.\end{aligned}\right\} \tag{4}$$

It is remembered that

$$\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}, \quad 2\varpi=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}. \tag{5}$$

From (4), (5), we get

$$\rho\frac{\partial^2\Delta}{\partial t^2}=(\lambda+2\mu)\nabla^2\Delta-a\Delta, \quad \rho\frac{\partial^2\varpi}{\partial t^2}=\mu\nabla^2\varpi-a\varpi, \tag{6}$$

the solutions of which are

$$\Delta=Ae^{i(pt-fx)+ry}, \quad 2\varpi=Be^{i(pt-fx)+sy}, \tag{7}$$

where

$$r^2=f^2-h^2, \quad s^2=f^2-k^2, \quad h^2=\frac{\rho p^2-a}{\lambda+2\mu}, \quad k^2=\frac{\rho p^2-a}{\mu}. \tag{8}$$

The displacements corresponding to (7) are

$$\left.\begin{aligned} u_1&=\frac{if}{h^2}Ae^{i(pt-fx)+ry}, \quad v_1=-\frac{r}{h^2}Ae^{i(pt-fx)+ry}, \\ u_2&=\frac{s}{k^2}Be^{i(pt-fx)+sy}, \quad v_2=\frac{if}{k^2}Be^{i(pt-fx)+sy}. \end{aligned}\right\} \tag{9}$$

Substituting (7), (9) in the boundary conditions

$$\lambda\Delta+2\mu\frac{\partial}{\partial y}(v_1+v_2)=0, \quad \frac{\partial(v_1+v_2)}{\partial x}+\frac{\partial(u_1+u_2)}{\partial y}=0 \tag{10}$$

at $y=0$, we get

$$\left(2-\frac{k^2}{f^2}\right)^4=16\left(1-\frac{h^2}{f^2}\right)\left(1-\frac{k^2}{f^2}\right), \tag{11}$$

the form of which is the same as the characteristic equation for the usual Rayleigh-waves. Relation (11) shows that if $\lambda=\mu$,

$$V=\sqrt{0{\cdot}8453\mu/\rho+a/\rho f^2}, \tag{12}$$

and that if $\lambda/\mu=\infty$,

$$V=\sqrt{0{\cdot}9126\mu/\rho+a/\rho f^2}, \tag{13}$$

whence it follows that Rayleigh-waves in a semi-inifinite body would be dispersive, provided there is an additional resistance that is proportional to the displacement.

When elastic media composing stratified layers have a resistance that is proportional to the displacement, the dispersive nature of the surface waves alters considerably. The equations of motion in this case are expressed by

$$\left.\begin{aligned}\rho'\frac{\partial^2 u'}{\partial t^2}&=(\lambda'+\mu')\frac{\partial\Delta'}{\partial x}+\mu'\nabla^2 u'-a'u',\\ \rho'\frac{\partial^2 v'}{\partial t^2}&=(\lambda'+\mu')\frac{\partial\Delta'}{\partial y}+\mu'\nabla^2 v'-a'v',\end{aligned}\right\}\qquad(14)$$

$$\left.\begin{aligned}\rho\frac{\partial^2 u}{\partial t^2}&=(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u-au,\\ \rho\frac{\partial^2 v}{\partial t^2}&=(\lambda+\mu)\frac{\partial\Delta}{\partial y}+\mu\nabla^2 v-av,\end{aligned}\right\}\qquad(15)$$

for the upper layer and the lower layer, respectively. The characteristic equation has the form

$$\begin{aligned}&\frac{4r's'}{f^2}\left(2-\frac{k'^2}{f^2}\right)\eta-\frac{r's'}{f^2}\left\{4\vartheta+\left(2-\frac{k'^2}{f^2}\right)^2\zeta\right\}\cosh r'H\cos s'H\\ &+\frac{r'}{f}\varphi\left\{\frac{4rs'^2}{f^3}+\frac{s}{f}\left(2-\frac{k'^2}{f^2}\right)^2\right\}\cosh r'H\sin s'H\\ &+\frac{r'}{f}\varphi\left\{-\frac{4sr'^2}{f^3}+\frac{r}{f}\left(2-\frac{k'^2}{f^2}\right)^2\right\}\sinh r'H\cos s'H\\ &+\left\{-\frac{4r'^2s'^2}{f^4}\zeta+\left(2-\frac{k'^2}{f^2}\right)^2\vartheta\right\}\sinh r'H\sin s'H=0,\end{aligned}\qquad(16)$$

where H is the thickness of the surface layer and

$$\left.\begin{aligned}&\Delta=\frac{\mu'k^2k'^2}{\mu f^4},\quad \zeta=\frac{4rs}{f^2}\left(\frac{\mu'}{\mu}-1\right)^2-\alpha^2,\quad \eta=\frac{2rs}{f^2}\left(\frac{\mu'}{\mu}-1\right)\beta-\alpha\gamma,\\ &\vartheta=\frac{rs}{f^2}\beta^2-\gamma^2,\quad \alpha=\frac{2\mu'}{\mu}-\left(2-\frac{k^2}{f^2}\right),\quad \beta=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-2,\\ &\gamma=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-\left(2-\frac{k^2}{f^2}\right),\\ &r^2=f^2-h^2,\quad s^2=f^2-k^2,\quad r'^2=f^2-h'^2,\quad s'^2=k'^2-f^2,\\ &h^2=\frac{\rho p^2-a}{\lambda+2\mu},\quad k^2=\frac{\rho p^2-a}{\mu},\quad h'^2=\frac{\rho' p^2-a'}{\lambda'+2\mu'},\quad k'^2=\frac{\rho' p^2-a'}{\mu'}.\end{aligned}\right\}\qquad(17)$$

Using (16), (17), it is possible to calculate the relation between k'/f and fH. Since $k'^2/f^2=(\rho'p^2-a')/\mu'f^2$, by assuming the value of $a'/\mu'f^2$, we get the velocity of transmission of Rayleigh-waves for different values of fH and $a'/\mu' f^2$. Since, however, the ratio of $a'/\mu' f^2$ is not a known quantity, the parameters will be so changed that the velocity of transmission of the waves is represented as functions of fH and aH^2/μ'. The results of calculation for the three cases, (i) $\lambda=\mu$, $\lambda'=\mu'$, $\mu/\mu'=5$, $\rho'/\rho=1$, $a'/a=1$, (ii) $\lambda=\mu$, $\lambda'=\mu'$, $\mu/\mu'=20$, $\rho'/\rho=1$, $a'/a=1$, (iii) $\lambda=\mu=\lambda'=\mu'$, $\rho'/\rho=1/2$, $a'/a=1/2$, are shown in Figs. 1, 2, 3. The numerical data for the present calculation are those that we obtained previously.[1),2),3)]

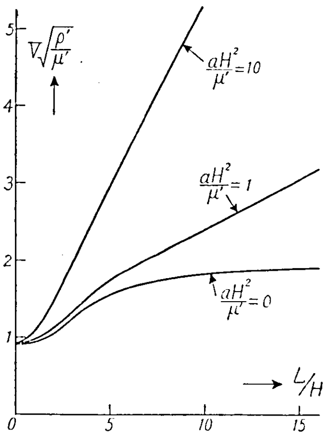

Fig. 1. $\lambda=\mu$, $\lambda'=\mu'$, $\mu/\mu'=5$, $\rho/\rho'=1$, $a/a'=1$.

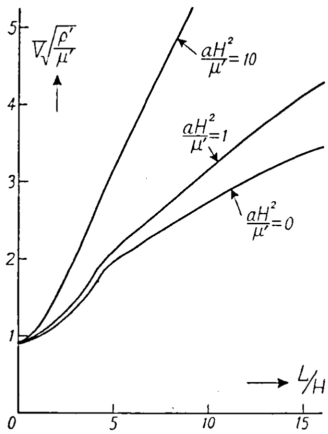

Fig. 2. $\lambda=\mu$, $\lambda'=\mu'$, $\mu/\mu'=20$, $\rho/\rho'=1$, $a/a'=1$.

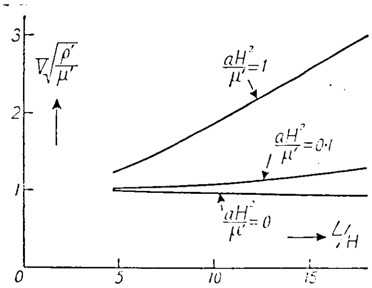

Fig. 3. $\lambda=\mu$, $\lambda'=\mu'$, $\rho'/\rho=1/2$, $a'/a=1/2$, $\mu'/\mu=1$.

From these figures it will be seen that the greater the value of a, the more the increase in the velocity of transmission of Rayleigh-waves. This feature is more pronounced with increase in the length of the waves. Although, in case (iii), the velocity of the waves in non-dispersive media diminishes with wave length, should the value of a exceed a certain limit, the velocity of the same waves would tend to increase with wave length.

1) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **3** (1927), 1.

2) K. SEZAWA and K. KANAI, "Discontinuity in the Dispersion Curves of Rayleigh-waves," *Bull. Earthq. Res. Inst.*, **13** (1935), 237~244.

3) K. SEZAWA and K. KANAI, "Anomalous dispersion of Elastic Surface Waves," *Bull. Earthq. Res. Inst.*, **16** (1938), 225~237.

Finally, there remains to be seen whether or not it is possible for a resistance that is proportional to the displacement of the waves to exist in the actual crust. Although no elastic medium with such a property exists, if there were earth blocks having statistically the nature of the so-called oscillators, resistance such as that just mentioned would be conceivable.

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32. 固體自體が分散性なる場合に於けるレーレー波の傳播

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今まで分散性のレーレー波について數多くの研究を試みたが，何れの場合にも地層の存在を假定する事によつてその分散性が現れ得たのである．玆に固體自體が分散性である場合を探して見たが，固體に彈性力の外に振動の變位に比例する抵抗が存在すると地層がなくても分散性がある事が確められた．勿論この分散性は表面波に限らず固體波にも存在する譯である．何れの波にしても上述のやうな分散性があると，波の傳播速度が波長とともに增加する．その速度の方則は本文中に示してある．地表層があるために既に分散性を有するレーレー波は只今のやうな固體の性質によつて更に分散性が增加する．地表層によつて異常分散をなす場合例へば波長とともに速度が減ずる場合には只今の性質によつて速度が補はれ，波長とともに速度が增加する事もあり得るのである．實際問題として變位に比例する振動抵抗が地殼にあり得るかどうかといふ疑問が當然起るけれども，之は統計的に見て振動子と考へられるやうな地塊が存在すれば可能になるものと思はれる．
