25. *A Problem of Weighted Mean.*

By Chuji TSUBOI,

Earthquake Research Institute.

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In a number of natural as well as social phenomena, two quantities $\Psi(\xi)$ and $\psi(\xi)$ are connected in such a way that one of them, $\Psi(\xi)$, is a weighted overlapping mean of the other, $\psi(\xi)$, taken over a certain range, say, from a to b, of the independent variable ξ, which may be either time or a co-ordinate. Mathematically, the relation between two such quantities may generally be written

$$\Psi(\xi)=\int_a^b \psi(\xi+\alpha)p(\alpha)d\alpha, \tag{1}$$

where $p(\alpha)$ is the weight to be assigned to $\psi(\xi+\alpha)$ according to α. As examples of pairs of quantities that are connected in this way, we have the amount of water flowing in a river as Ψ and the precipitation in its basin as ψ; ground temperature and solar radiation; the reading on a hair hygrometer and the actual humidity; supply and demand in economical problems; and so on. In connection with these problems, it is often required to find that law by which $\Psi(\xi)$ is affected by $\psi(\xi)$, in other words, to determine the functional form of $p(\xi)$, starting from $\Psi(\xi)$ and $\psi(\xi)$ which are both known. The method hitherto resorted to for solving such a problem has mostly been that of trial and error, which method consists first in assuming a certain reasonable functional form for $p(\xi)$ and secondly in adjusting the numerical values of the constants contained in it until such a good agreement is obtained between the given $\Psi(\xi)$ and the calculated $\int_a^b \psi(\xi+\alpha)p(\alpha)d\alpha$ that the $p(\xi)$ assumed may well be regarded as a satisfactorily accurate solution. Owing to the considerable time required for doing so, especially when $p(\xi)$ contains two or more constants to be adjusted, it naturally behooves us to devise some other method that would enable the problem to be solved more simply.

In the method which the writer proposes in this paper, by using Fourier series, it is possible directly to find $p(\xi)$, starting from the given $\Psi(\xi)$ and $\psi(\xi)$. From the standpoint of mathematical rigorousness, it is admitted that there are some weak points in the method that are not very satisfactory, but there can be no objection to applying the proposed method to actual problems.

Let $\Psi(\xi)$ and $\psi(\xi)$ be two functions that satisfy Dirichlet's conditions, hence expansible into Fourier series within the interval $a \leqq \xi \leqq b$. By means of transformation,

$$x = 2\pi \frac{\xi - a}{b - a},$$

the above assumption may be reduced to another, that $F(x)$ and $f(x)$ are both expansible into Fourier series within the interval $0 \leqq x \leqq 2\pi$, where

$$F(x) = \Psi\left(\frac{b-a}{2\pi}x + a\right),$$

and

$$f(x) = \psi\left(\frac{b-a}{2\pi}x + a\right).$$

Under this assumption, $F(x)$ and $f(x)$ may be written

$$F(x) = \sum_m A_m \cos mx + \sum_m B_m \sin mx \qquad (2)$$

$$f(x) = \sum_m a_m \cos mx + \sum_m b_m \sin mx \qquad (3)$$

where

$$A_m = \frac{1}{\varepsilon_m \pi} \int_0^{2\pi} F(\lambda) \cos m\lambda \, d\lambda$$

$$B_m = \frac{1}{\pi} \int_0^{2\pi} F(\lambda) \sin m\lambda \, d\lambda$$

$$a_m = \frac{1}{\varepsilon_m \pi} \int_0^{2\pi} f(\lambda) \cos m\lambda \, d\lambda$$

$$b_m = \frac{1}{\pi} \int_0^{2\pi} f(\lambda) \sin m\lambda \, d\lambda$$

and

$$\varepsilon_m = 2 \quad \text{for} \quad m = 0$$

$$\varepsilon_m = 1 \quad \text{for} \quad m \geqq 1.$$

Now if

$$F(x)=\int_0^{2\pi} f(x+\alpha)\varphi(\alpha)d\alpha \qquad (4)$$

is the fundamental equation from which $\varphi(x)$ is to be determined, this is nothing but a Fredholm integral equation of the first kind which, in the present case, is solvable provided $\varphi(x)$ is also expansible into Fourier series. If now we assume this, and this is true in almost all practical problems, then $\varphi(x)$ may be written

$$\varphi(x)=\sum_m \alpha_m \cos mx+\sum_m \beta_m \sin mx \qquad (5)$$

with

$$\alpha_m=\frac{1}{\varepsilon_m \pi}\int_0^{2\pi} \varphi(\lambda)\cos m\lambda d\lambda$$

and

$$\beta_m=\frac{1}{\pi}\int_0^{2\pi} \varphi(\lambda)\sin m\lambda d\lambda.$$

The equation (4) reduces then to

$$F(x)=\int_0^{2\pi} f(x+\alpha)\varphi(\alpha)d\alpha$$

$$=\int_0^{2\pi}\sum_m\Big\{a_m\cos m(x+\alpha)+b_m\sin m(x+\alpha)\Big\}\varphi(\alpha)\,d\alpha$$

$$=\sum_m a_m\cos mx\int_0^{2\pi}\varphi(\alpha)\cos m\alpha\,d\alpha$$

$$-\sum_m a_m\sin mx\int_0^{2\pi}\varphi(\alpha)\sin m\alpha\,d\alpha$$

$$+\sum_m b_m\sin mx\int_0^{2\pi}\varphi(\alpha)\cos m\alpha\,d\alpha$$

$$+\sum_m b_m\cos mx\int_0^{2\pi}\varphi(\alpha)\sin m\alpha\,d\alpha$$

$$=\pi\left\{\sum_m \varepsilon_m a_m \alpha_m \cos mx-\sum_m a_m\beta_m \sin mx\right.$$

$$\left.+\sum_m \varepsilon_m b_m \alpha_m \sin mx+\sum_m b_m\beta_m \cos mx\right\}. \tag{6}$$

Comparing (6) with (2), we get

$$A_m=\pi(\varepsilon_m a_m \alpha_m+b_m\beta_m)$$

$$B_m=\pi(-a_m\beta_m+\varepsilon_m b_m \alpha_m)$$

and consequently

$$\left.\begin{aligned}\alpha_m&=\frac{1}{\varepsilon_m\pi}\,\frac{a_mA_m+b_mB_m}{a_m^2+b_m^2}\\ \beta_m&=\frac{1}{\pi}\,\frac{b_mA_m-a_mB_m}{a_m^2+b_m^2}\end{aligned}\right\}. \tag{7}$$

In other words, the Fourier coefficients of the m-th order of the weight function $\varphi(x)$ are simple algebraic functions of those of $F(x)$ and $f(x)$ of the corresponding order, so that it is easy to determine the form of $\varphi(x)$, provided both $F(x)$ and $f(x)$ are known.

The point that is most unsatisfactory from the mathematical point of view in the theory developed above is that we have restricted the range of x to be between 0 and 2π. This makes it hardly possible to apply this method, should $\varphi(x)$ actually extend beyond this range. The range of 2π, in which $F(x)$ and $f(x)$ are to be developed in Fourier series, should be taken sufficiently wider than the range of x, in which $\varphi(x)$ is supposed to have sensible values.

In order to see at the outset what accurate results will practically be obtained by means of this method, the following test was made. $n(x)$ in Table I is a series of integers taken at random, $N(x)$ being constructed according to

$$N(x)=n(x-4)+2n(x-3)+3n(x-2)+4n(x-1)+5n(x)$$

$$+4n(x+1)+3n(x+2)+2n(x+3)+n(x+4).$$

The curves for $n(x)$ and $N(x)$ are both shown in Fig. 1. In this case, we know that the weight assigned to $n(x)$ has been linear with respect to x, so that the form of $\varphi(x)$ is

$$\varphi(x)=5\mp x,$$

the double sign before x being taken according as whether x is positive or negative. What we shall do here is to determine $\varphi(x)$ anew, starting

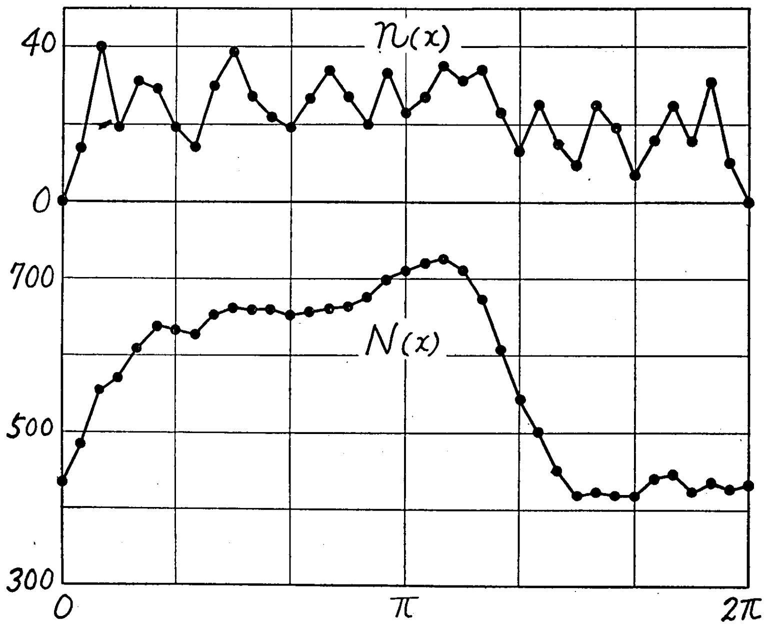

Fig. 1. $n(x)$.........series of random numbers
$N(x)$.........overlapping sum

from the given $n(x)$ and $N(x)$, acting as if the value of $\varphi(x)$ were not known beforehand. According to the method developed in the foregoing paragraphs, the Fourier coefficients for $n(x)$ and $N(x)$ were first calculated by means of ordinary harmonic analysis, with the results given in

Table I.

x	$n(x)$	$N(x)$	x	$n(x)$	$N(x)$	x	$n(x)$	$N(x)$	x	$n(x)$	$N(x)$
10°	14	485	100°	27	659	190°	27	720	280°	25	423
20	40	553	110	22	658	200	35	725	290	19	418
30	19	570	120	19	652	210	31	712	300	7	419
40	31	610	130	26	656	220	34	672	310	16	441
50	29	637	140	34	661	230	23	607	320	25	446
60	19	633	150	27	664	240	13	544	330	15	424
70	14	627	160	20	675	250	25	501	340	31	437
80	30	652	170	34	699	260	15	450	350	10	426
90	39	662	180	23	708	270	9	416	360	0	433

Table II. It may be pointed out here that the Fourier coefficients P_m and Q_m for $N(x)$ do not exactly correspond to the coefficients A_m and

Table II. Fourier Cofficients for $n(x)$, $N(x)$ and $\varphi(x)$.

m	$n(x)$		$N(x)$		$\varphi(x)$	
	a_m	b_m	P_m	Q_m	α_m	β_m
0	22·972	——	574·310	——	0·695	——
1	− 4·582	3·890	−107·743	92·463	1·330	0·007
2	1·176	2·090	22·889	41·736	1·103	− 0·012
3	− 1·442	− 1·540	− 20,083	−21·064	0·766	0·007
4	− 1·871	1·605	− 15·530	13·146	0·459	− 0·003
5	− 1·276	0·728	− 4·794	2·539	0·205	− 0·006
6	− 5·722	− 0·144	− 5·722	− 0·192	0·056	0·000
7	− 2·187	− 1·577	− 0·051	− 0·037	0·001	0·000
8	0·056	0·724	− 0·010	0·215	0·015	− 0·008
9	− 1·778	3·556	− 1·778	3·611	0·056	0·000
10	− 1·454	− 1·103	− 2·187	− 1·605	0·083	0·001
11	− 3·653	− 1·578	− 5·403	− 2·349	0·082	0·000
12	− 4·556	− 1·203	− 4·556	− 1·251	0·056	− 0·001
13	0·438	2·130	0·175	0.832	0·022	0·000
14	− 1·056	− 1·172	− 0·036	− 0·041	0·002	0·000
15	0·386	0·096	0·027	0·009	0·004	− 0·011
16	1·145	− 0·592	0·463	− 0·229	0·022	0·000
17	2·593	0·097	2·146	0·116	0·046	− 0·001
18	0·806	——	0·806	——	0·056	——

B_m of the function $F(x)$, because the values of $n(x)$ are given only at finite intervals of $\frac{2\pi}{360}\times 10$. Since

$$A_m = P_m \times \frac{2\pi}{360} \times 10$$

$$B_m = Q_m \times \frac{2\pi}{360} \times 10,$$

we have

$$\begin{aligned}\alpha_m &= \frac{1}{\varepsilon_m \pi}\,\frac{a_m A_m + b_m B_m}{a_m^2 + b_m^2} \\ &= \frac{1}{\varepsilon_m \pi}\,\frac{a_m P_m + b_m Q_m}{a_m^2 + b_m^2}\,\frac{2\pi}{36} \\ &= \frac{1}{18\varepsilon_m}\,\frac{a_m P_m + b_m Q_m}{a_m^2 + b_m^2}\end{aligned}$$

and similarly,

$$\left.\beta_m=\frac{1}{18}\frac{b_mP_m-a_mQ_m}{a_m^2+b_m^2}\right\} . \qquad (8)$$

The values of α_m and β_m calculated from (8) are also given in Table II. Since $\varphi(x)$ is an even function with respect to x, the Fourier coefficients β_m of all sine terms of $\varphi(x)$ ought to have vanished, but in Table II, we see that there are slight departures from this, which is the result of accumulations of errors in the actual numerical computations.

The function $\varphi(x)$ is then evaluated by summing up the series

$$\varphi(x)=\sum_m \alpha_m \cos mx+\sum_m \beta_m \sin mx,$$

with the results given in Table III. The values of $\varphi(x)$ calculated in this way agree well with what were first assumed, showing the feasibility of the proposed method.

Table III.

x	$\varphi(x)$	x	$\varphi(x)$	x	$\varphi(x)$	x	$\varphi(x)$
0°	5·08	100°	0·00	190°	− 0·04	280°	0·02
10	4·02	110	− 0·04	200	0·01	290	− 0·04
20	3·07	120	0·01	210	− 0·04	300	0·07
30	2·00	130	− 0·04	220	0·02	310	0·00
40	1·06	140	0·02	230	− 0·04	320	1·06
50	0·00	150	− 0·04	240	0·01	330	2·00
60	0·07	160	0·01	250	− 0·04	340	3·07
70	− 0·04	170	− 0·04	260	0·00	350	4·02
80	0·02	180	0·00	270	-- 0·04	360	5·08
90	− 0·04						

Now in order to apply this method to a few actual problems, we shall consider first the relation between the amount of discharge from

Table IV.

	F (Discharge)	f (Prec.)	F'	F-F'		F	f	F'	F-F'
April	10·70 l/m	103·1 mm	3·34	7·36	Oct.	12·29	58·5	4·87	7.42
May	10·80	154·8	3·65	7·15	Nov.	11·86	69·5	4·68	7·18
June	11·57	220·5	3·98	7·59	Dec.	11·42	53·5	4·11	7·31
July	11·93	209·8	4·65	7·28	Jan.	11·01	43·9	3·93	7·08
Aug.	11·87	206·8	4·29	7·58	Feb.	10·76	78·0	3·44	7·32
Sept.	12·67	360·5	5·44	7·23	March	10·73	116·3	3·65	7·08

the Beppu hot springs and the precipitations in nearby regions, as reported by T. Nomitsu and others[1]. According to these investigators, the mean monthly discharge from the springs (per orifice) F and the mean monthly precipitation in nearby regions f are as given in Table IV. $F(x)$ and $f(x)$ are graphically shown in Fig. 2.

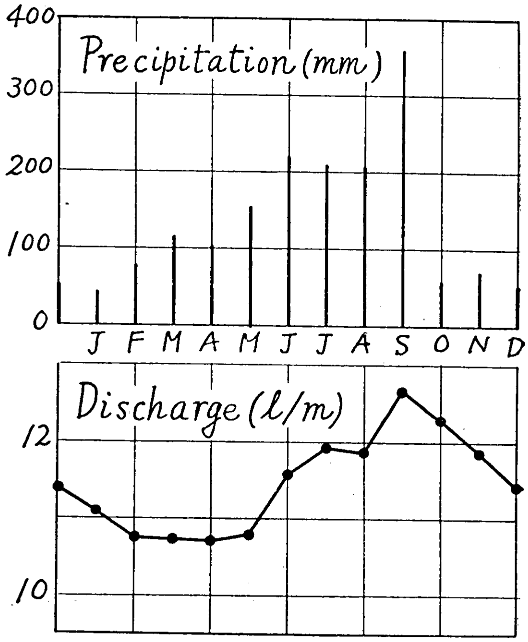

Fig. 2. Precipitation near Beppu and the discharge of the Beppu Hot Springs (per orifice).

Both these quantities were analysed into Fourier series with the results shown in Table V, in which are also given the Fourier coefficients for the function $\varphi(x)$, computed from those for the discharge and the precipitation, according to (7). With these coefficients, the Fourier series for $\varphi(x)$ was synthesised, with the results given in Table VI and shown graphically in Fig. 3. It is clear from Fig. 3 that the weight function $\varphi(x)$ is almost exactly linear with respect to time, that is to say, the discharge of the hot-spring water in a certain month is most strongly

Table V.

m	F (Discharge)		f (Prec.)		φ (Weight)	
	cos	sin	cos	sin	cos	sin
0	11·475	——	139·6	——	0·00685	——
1	− 0·563	− 0·380	− 59·6	78·5	64	−0·00115
2	0·095	0·052	31·2	− 22·0	21	− 43
3	− 0·008	− 0·068	− 27·3	1·1	3	− 43
4	0·112	0·128	48·1	− 13·5	25	− 51
5	− 0·120	− 0·023	− 35·3	6·0	53	− 20
6	0·077	——	19·5	——	66	——

1) T. Nomitsu and others, *Tikyubuturi*, 2 (1938), 97; *Mem. Coll. Sc., A*, 23 (1940), 41.

affected by the precipitation during that same month, and less and less by those in the preceding months.

Table VI.

x	$\varphi(x)$	x	$\varphi(x)$	x	$\varphi(x)$
0°	0·00917	150°	0·00579	270°	0·00715
30	435	180	677	300	857
60	707	210	637	330	817
90	531	240	761	360	917
120	581				

One point in this problem needs some amplifying remarks. It is generally believed that a considerable portion, say, J, out of the total discharge F of the hot-spring water comes from a deep origin, and that this portion is scarcely affected by precipitation and other meteorological elements. It might therefore have been better to assume that it is the difference $(F-J)$, not F itself, that is to be correlated with the precipitation, and to write the starting equation in the form

$$F(x)-J=\int f(x+a)\varphi(a)da.$$

Since J is taken as a constant, although unknown, the constant term in the Fourier expansion of $(F-J)$ differs from that in the expansion of F by that constant amount. The term a_0 in the series for $\varphi(x)$ is therefore indeterminate by another unknown constant amount. Now if $\varphi(x)$ is put zero at $x=2\pi$, that is, if it is assumed that after a year precipitation ceases to have any effect on the discharge of the hot-water, then the weight function $\varphi'(x)$ ought to be

$$\varphi'(x)=\varphi(x)-0{\cdot}00435.$$

Using these $\varphi'(x)$, the monthly discharges F', which were also given in Table IV, are calculated according to

$$F'(x)=\sum_{n=-12}^{0} f\left(x+\frac{2\pi}{12}n\right)\varphi'\left(\frac{2\pi}{12}n\right). \qquad (9)$$

Between the observed monthly discharges and those calculated according to (9), we find almost constant differences of about $7\,l/m$, which may be identified with what we believed came from a deep origin.

It is interesting to compare the foregoing results with those of Nomitsu and others that were obtained in a quite different way.

They assumed from the outset that the effect of precipitation on the discharge of hot-water diminishes exponentially with respect to time, and finally obtained

$$\varphi(x) = 0{\cdot}00390 \exp\left(-0{\cdot}11 \times \frac{12}{360}x\right) \tag{10}$$

as the weight function. For comparison, the values of $\varphi(x)$ according to (10) were also plotted in Fig. 3. According to Nomitsu's expression, precipitation affects sensibly even after the lapse of a few years, but according to the results obtained in the present paper, it is not necessary to assume that the effect of precipitation lasts so long.

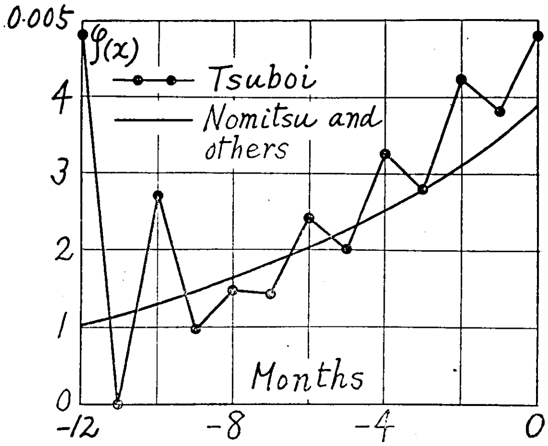

Fig. 3. Effect of precipitation on the discharge of the Ceppu Hot Spring.

As to the value of J, Nomitsu and others found 6·3 l/m, while our value is 7·3 l/m.

The second example relates to the dependence of the daily height of the hot-water table in a certain hot-spring, at Rendaizi, on the daily precipitations in nearby regions. The daily readings of both these quantities from Sept. 6 to Oct. 11,

Table VII.

Date	Water Head	Prec.	Date	Water Head	Prec.	Date	Water Head	Prec.
Sept. 6	97 cm	0·2mm	Sept. 18	72 cm	15·7mm	Sept. 30	140	—mm
7	98	0·1	19	67	0·1	Oct. 1	136	—
8	98	—	20	64	—	2	132	37·7
9	97	—	21	63	15·8	3	130	8·3
10	96	11·5	22	66	38·8	4	128	26·8
11	93	11·6	23	70	42·7	5	126	—
12	84	—	24	75	2·8	6	125	—
13	83	—	25	84	105·2	7	125	0·4
14	82	1·2	26	116	10·2	8	118	—
15	78	8·5	27	128	—	9	95	—
16	74	0·1	28	135	—	10	113	—
17	73	—	29	137	—	11	112	6·1

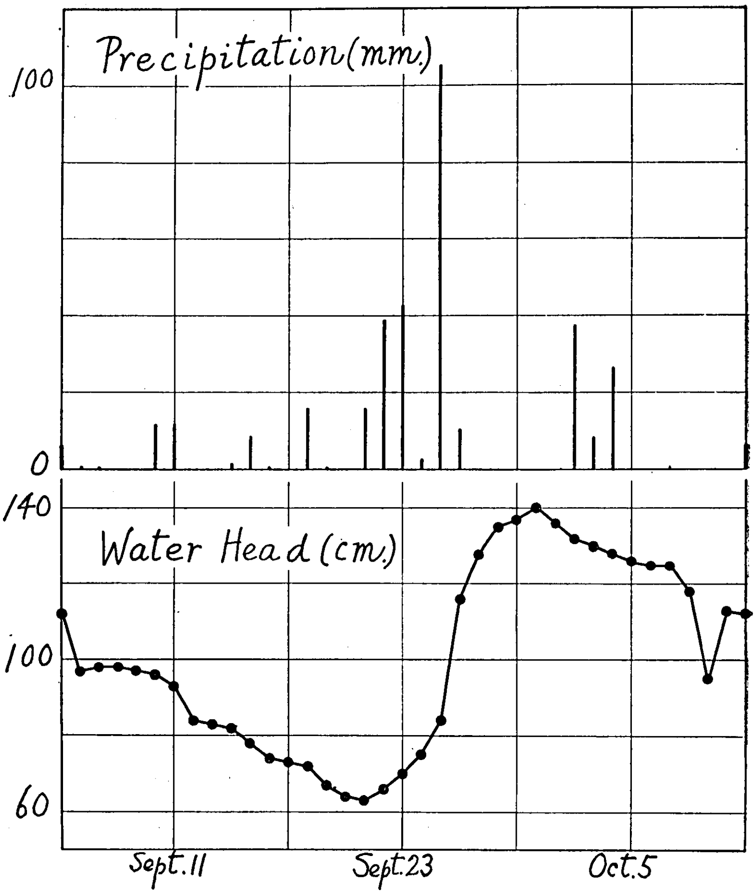

Fig. 4. Precipitation near Rendaizi and the head of the hot water table.

1935, as reported by T. Fukutomi[2], are given in Table VII, and shown graphically in Fig. 4.

The Fourier coefficients for the height of the water head and the precipitation are given in Table VIII, together with those for the weight functions that were calculated from them. Since the water head was measured from an arbitrary origin, no attention was paid to the constant term of the Fourier series for $\varphi(x)$. The values of $\varphi(x)$ computed from these coefficients are given in Table IX, and graphically shown in Fig. 5. In this case, the summation of the series for $\varphi(x)$ was carried out to various orders of harmonics. The rather large fluctuations of $\varphi(x)$ that result when the summation is made to highter orders, are due to slow convergence on the part of the Fourier coefficients for the preci-

2) T. FUKUTOMI, *Zisin*, **12** (1940), 195.

pitation. At any rate, it is clear from Fig. 5. that the water head on a certain day is most strongly affected, not by the precipitation on that identical day, but by the precipitation 4 or 5 days prior to it.

Table VIII.

m	Water Head		Precipitation		Weight function	
	cos	sin	cos	sin	cos	sin
0	——	——	9·558	——	——	——
1	12·192	−28·607	−10·384	− 3·333	−0·0146	−0·1577
2	−10·412	8·424	5·567	2·025	−0·0648	−0·1077
3	5·135	− 1·512	− 9·124	− 0·362	− 308	−0·0104
4	− 4·516	− 0·464	6·741	5·613	− 238	− 161
5	1·046	1·628	0·008	− 7·259	− 124	− 80
6	0·640	− 0·831	− 0·557	0·710	− 646	− 6
7	1·350	1·233	1·978	0·142	402	− 317
8	2·558	− 0·984	− 1·701	1·108	− 812	156
9	0·723	0·833	4·048	3·555	113	− 16
10	2·042	− 1·318	− 5·769	− 5·290	− 43	− 167
11	0·530	− 1·861	4·932	5·438	− 77	124
12	1·474	− 1·023	− 0·427	− 6·350	81	− 134
13	0·348	− 1·632	− 1·116	3·479	− 252	− 25
14	0·284	− 0·574	1·735	− 4·079	80	− 4
15	− 0·355	− 1·320	− 2·964	8·288	− 86	− 49
16	− 0·788	− 0·304	7·779	− 4·781	− 31	41
17	0·244	− 0·713	− 5·584	− 1·043	− 11	− 73
18	− 0·557	——	1·496	——	− 207	——

Table IX.

x	Up to 3rd Harmonies.	Up to 6th Harmonies.	Up to 12th Harmonies.	Up to 18th Harmonies.
0°	−0·1102	−0·2111	−0·2448	−0·2954
10	− 1714	− 2470	− 2804	− 2427
20	− 2109	− 2049	− 1623	− 1780
30	− 2276	− 1131	− 959	− 972
40	− 2234	− 1593	− 2009	− 1983
50	− 2034	− 1955	− 1985	− 1877
60	− 1742	− 2119	− 1664	− 1939
70	− 1409	− 1729	− 1599	− 1262
80	− 1003	− 1002	− 1434	− 1673
90	− 825	− 498	− 1009	− 962

(to be continued.)

Table IX. (*Continued.*)

x	Up to 3 rd Harmonies.	Up to 6 th Harmonies.	Up to 12 th Harmonies.	Up to 18 th Harmonies.
100°	− 615	− 529	− 173	− 76
110	− 460	− 841	132	53
120	− 346	− 879	− 1082	− 1131
130	− 261	− 444	− 1582	− 1251
140	− 200	306	200	− 239
150	− 159	579	1579	1921
160	− 143	202	816	722
170	− 157	− 534	− 684	− 806
180	− 193	− 953	− 1362	− 1171
190	− 239	− 712	− 706	− 705
200	− 266	− 92	1544	1223
210	− 236	323	1099	1701
220	− 111	242	− 1383	− 2062
230	140	− 16	− 1025	− 496
240	523	129	1249	983
250	1018	915	1923	1986
260	1575	1960	1610	1583
270	2121	2608	1742	1887
280	2568	2567	2236	1939
290	2836	2182	2801	3154
300	2857	2060	2721	2442
310	2607	2415	2685	2266
320	2094	2781	2143	2086
330	1374	2877	2806	2926
340	0534	1079	1619	1323
350	− 326	− 741	− 427	0039
360	− 1102	− 2111	− 2448	− 2952

Finally, the foregoing method may, with some modifications, be readily applied to many other problems that can be reduced to solving an equation of the form

$$F(x)=\int_{-s}^{+s} f(x+a)da,$$

where $F(x)$ is given and $f(x)$ is to be sought. Such a problem is encountered, for instance, as Mr. Z. Koana has noticed, when it is required to estimate the intensities of individual spectral lines that were taken on a photographic plate from the microphotometric curve of the plate obtained by a microphotometer, equipped with a slit of finite width. If the deflection of the photometer, when the photographic plate is run

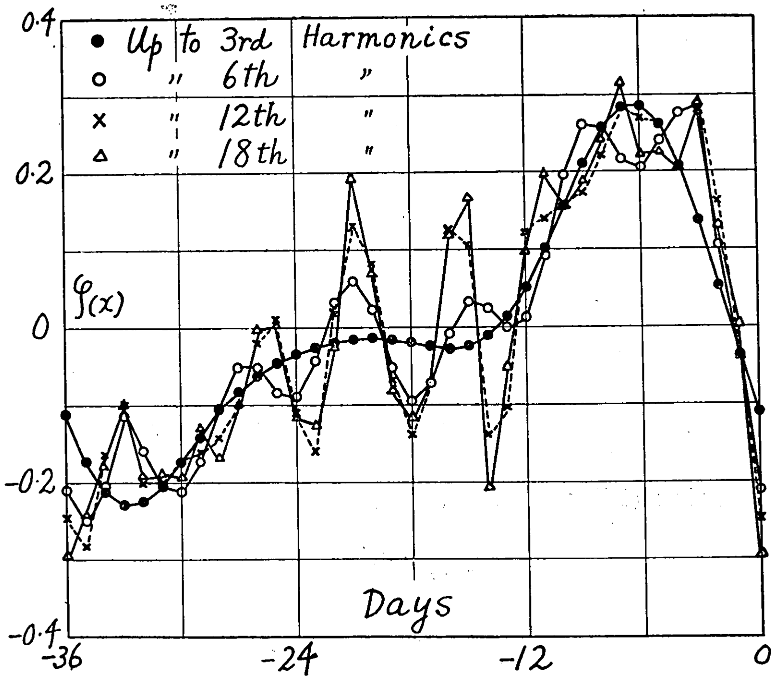

Fig. 5. Effect of precipitation on the head of the hot water table at Rendaizi.

under it, is $F(x)$, and the intensity distribution along the plate is $f(x)$, then the relation between $F(x)$ and $f(x)$ is exactly

$$F(x)=\int_{-s}^{+s} f(x+\alpha)\,d\alpha, \tag{11}$$

where s is half the width of the slit of the photometer. If, as before, we put

$$\left.\begin{aligned} F(x)&=\sum_m A_m\cos mx+\sum_m B_m\sin mx\\ f(x)&=\sum_m a_m\cos mx+\sum_m b_m\sin mx, \end{aligned}\right\}, \tag{12}$$

(11) reduces to

$$\begin{aligned} F(x)&=\int_{-s}^{+s} f(x+\alpha)\,d\alpha\\ &=\int_{-s}^{+s}\left\{\sum_m a_m\cos m(x+\alpha)+\sum_m b_m\sin m(x+\alpha)\right\}d\alpha \end{aligned}$$

$$=\int_{-s}^{+s}\Big\{\sum_m a_m\cos mx\cos ma-\sum_m a_m\sin mx\sin ma$$

$$+\sum_m b_m\sin mx\cos ma+\sum_m b_m\cos mx\sin ma\Big\}da$$

$$=\Big[\sum_m\frac{a_m}{m}\cos mx\sin ma+\sum_m\frac{a_m}{m}\sin mx\cos ma$$

$$+\sum_m\frac{b_m}{m}\sin mx\sin ma-\sum_m\frac{b_m}{m}\cos mx\cos ma\Big]_{-s}^{+s}$$

$$=\sum_m\frac{2a_m}{m}\cos mx\sin ms+\sum_m\frac{2b_m}{m}\sin mx\sin ms. \qquad (13)$$

Comparing (13) with (12), we get

$$\left.\begin{aligned} a_m&=\frac{mA_m}{2\sin ms} \\ \text{and} \qquad b_m&=\frac{mB_m}{2\sin ms}\end{aligned}\right\}. \qquad (14)$$

From these relations, the intensity distribution on a photographic plate can be evaluated from the microphotometric curve taken with a slit that may even be wider than the individual spectral lines to be measured.

In view of the fact that $\sin ms$ comes into the denominators, it is necessary to choose s so that $\sin ms$ for any m shall not vanish. In other words, s should be so chosen that it will not be an integral measure of π. For the case $m=0$,

$$a_0=\frac{d}{dm}(mA_m)\Big/\frac{d}{dm}(2\sin ms)$$

$$=\frac{A_m}{2s}.$$

In order to demonstrate the usefulness of this procedure, it was applied to the following example. In Fig. 6, $f(x)$ represents two "spectral lines" that are two probability curves separated by $\frac{2\pi}{360}\times 80$ from peak to peak, while $F(x)$ in the same figure is the "microphotometric curve" taken with a slit of half the width, namely, $\frac{2\pi}{360}\times 35$. The values of $F(x)$ and $f(x)$ are given in Table X. What is required here is to find $f(x)$ anew, knowing the form of $F(x)$, and that it was taken with a slit, half the width of which is $\frac{2\pi}{360}\times 35$. Using the Fourier coefficients for $F(x)$, those for $f(x)$ were calculated according to (14), the results being given in Table XI. The values of $f(x)$, which were obtained by synthesising the series

$$f(x)=\sum_m \frac{mA_m}{2\sin ms}\cos mx+\sum_m \frac{mB_m}{2\sin ms}\sin mx\ ,$$

and which are also given in Table X, agree well with the given $f(x)$.

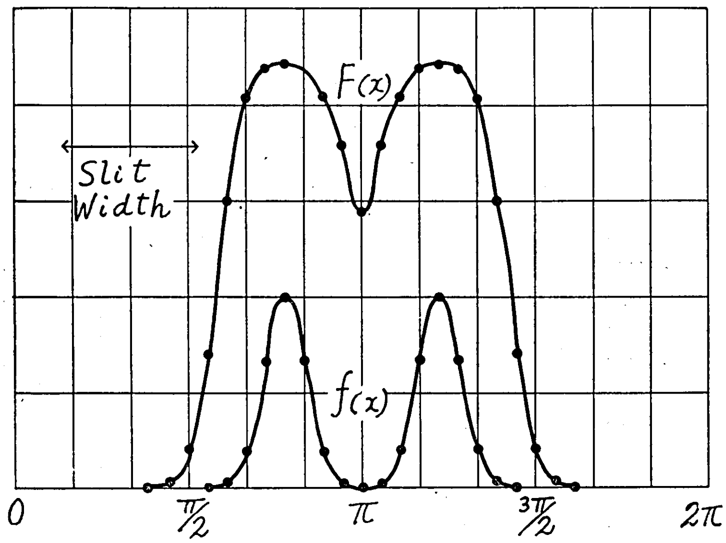

Fig. 6. $f(x)$........." Spectral Lines "
$F(x)$........." Photometric Curve " taken with a slit of finite width.

Table X.

x	f (assumed)	F	f (calc.)	x	f (assumed)	F	f (calc.)
0°	0·0	0·0	0·1	60°	0·0	0·0	− 0·9
10	0·0	0·0	− 0·2	70	0·0	17·0	0·8
20	0·0	0·0	− 0·7	80	0·0	263·0	− 1·1
30	0·0	0·0	0·9	90	0·0	1928·0	0·3
40	0·0	0·0	0·3	100	1·0	7205·0	1·5
50	0·0	0·0	− 0·3	110	25·0	14934·0	26·3

(*to be continued.*)

Table X. (*Continued.*)

x	f (assumed)	F	f (calc.)	x	f (assumed)	F	f (calc.)
120°	194·0	20215·0	192·7	250°	25·0	14934·0	26·3
130	664·0	21876·0	665·1	260	1·0	7205·0	1·5
140	1000·0	22105·0	1000·0	270	0·0	1928·0	0·3
150	664·0	21887·0	665·4	280	0·0	263·0	− 1·1
160	194·0	20473·0	192·1	290	0·0	17·0	0·8
170	25·0	16857·0	26·3	300	0·0	0·0	− 0·9
180	1·0	14404·0	2·3	310	0·0	0·0	− 0·3
190	25·0	16857·0	26·3	320	0·0	0·0	0·3
200	194·0	20473·0	192·1	330	0·0	0·0	0·9
210	644·0	21887·0	665·4	340	0·0	0·0	− 0·7
220	1000·0	22105·0	1000·0	350	0·0	0·0	− 0·2
230	664·0	21876·0	665·1	360	0·0	0·0	0·1
240	194·0	20215·0	192·7				

Table XI.

m	A_m	$a_m=mA_m/2\sin ms$	m	A_m	$a_m=mA_m/2\sin ms$
0	8609	15·4	10	− 59	3·7
1	−12156	−23·1	11	− 19	− 0·6
2	2133	5·0	12	− 70	− 1·6
3	3841	13·0	13	88	1·3
4	− 3163	−21·5	14	− 38	− 0·8
5	290	18·2	15	4	0·2
6	602	− 7·9	16	− 1	0·1
7	255	− 2·1	17	4	− 0·1
8	− 811	7·2	18	0	0
9	493	− 6·9			

It may be added, in conclusion, that the object of the present paper is merely to propose a method that will be useful in solving a certain class of problems related to weighted overlapping means.

The writer intends to use this method in a forthcoming paper discussing the mechanism of regional isostasy.

25. 重價平均に關する一問題

地震研究所 坪井忠二

$f(x)$ なる量と，それに $\varphi(x)$ なる重みを附して求めた平均

$$F(x)=\int_0^{2\pi}f(x+\alpha)\varphi(\alpha)d\alpha$$

とが與へられて居る時，$\varphi(x)$ を近似的に求める方法を述べた．即ち若し

$$f(x)=\sum a_m\cos mx+\sum b_m\sin mx$$

$$F(x)=\sum A_m\cos mx+\sum B_m\sin mx$$

であるならば

$$\varphi(x)=\sum\frac{1}{\varepsilon_m\pi}\,\frac{a_mA_m+b_mB_m}{a_m^2+b_m^2}\cos mx+\sum\frac{1}{\pi}\,\frac{b_mA_m-a_mB_m}{a_m^2+b_m^2}\sin mx$$

で與へられる．二三の實例に就いて其の應用例を示した．