12. *On the Initial Movement of a Seismograph subjected to an Arbitrary Earthquake Motion, Solved with Operational Calculus. I.*

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1. *Introduction.*

The problem of the initial motion of a seismograph subjected to an arbitrary disturbance has been solved by a number of authors, mainly by means of Fourier's double integrals. On the other hand, Jeffreys[1] used Heaviside's operational calculus in solving the problem. We have recently found that the operational method, based on Mellin's inversion theorem, is more suitable in dealing with the problem.

If, in Mellin's theorem,

$$\phi(p) = p\int_0^\infty e^{-pt} f(t)\,dt, \tag{1}$$

then $f(t)$ is expressed by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{zt}\phi(z)}{z}\,dz, \tag{2}$$

where $\int_{c-i\infty}^{c+i\infty}$ constitutes the Bromwich-Wagner's integral in the complex z-plane. Thus, it is possible to say that $\phi(p)$ is an operational form of $f(t)$, and $f(t)$ an interpretation of $\phi(p)$.

2. *Expression of seismograph movements resulting from an arbitrary earthquake motion.*

Let ξ be the ground movement and x the displacement of the pendulum relative to the moving ground. Then, the equation of motion of the pendulum is expressed by

$$\ddot{x} + 2k\dot{x} + n^2 x = -\ddot{\xi}, \tag{3}$$

1) H. JEFFREYS, *Operational Method* (Cambridge Math. Tract, 1927).

Multiplying both sides of this equation by $pe^{-pt}dx$, and integrating, we get

$$p\int_0^\infty e^{-pt}dx' + 2kp\int_0^\infty e^{-pt}dx + n^2p\int_0^\infty e^{-pt}xdt = -p\int_0^\infty e^{-pt}d\xi', \qquad (4)$$

where x', ξ' mean dx/dt, $d\xi/dt$. Integrating (4) by parts, it becomes

$$(p^3+2kp^2+n^2p)\int_0^\infty e^{-pt}xdt = -p^3\int_0^\infty e^{-pt}\xi dt$$
$$+p^2(x_0+\xi_0)+p(x_0'+\xi_0'+2kx_0), \qquad (5)$$

where x_0, ξ_0, x_0', ξ_0' are the initial values of x, ξ, x', ξ'.

From (5) the operational form of x becomes

$$x \supset p\int_0^\infty e^{-pt}xdt = \frac{-p^3\int_0^\infty e^{-pt}\xi dt}{p^2+2kp+n^2} + \frac{p^2(x_0+\xi_0)+p(x_0'+\xi_0'+2kx_0)}{p^2+2kp+n^2}, \qquad (6)$$

from which the interpretation of the right-hand side of (6) becomes

$$x = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{-z\phi(z)e^{zt}dz}{z^2+2kz+n^2} + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\left\{z(x_0+\xi_0)+(x_0'+\xi_0'+2kx_0)\right\}e^{zt}dz}{z^2+2kz+n^2}. \qquad (7)$$

The first term on the right-hand side of (7) represents the movement of the pendulum for zero displacements and zero velocities of the same pendulum and of the ground at $t=0$, while the second term gives the effect of their initial displacements and the initial velocities.

If, assuming temporarily that the second term does not exist, we calculate the first term, we have $x_0+\xi_0=0$, $x_0'+\xi_0'+2kx_0=0$, from which condition the second term in (7) should vanish. If the initial displacement and the initial velocity of the ground, namely, ξ_0, ξ_0', were zero, the initial displacement and the initial velocity of the pendulum, namely, x_0, x_0', would be zero. Although $x_0+\xi_0$ indicates zero displacement of the pendulum in space, $x_0'+\xi_0'+2kx_0$ is somewhat complex. However, if the damping of the seismograph were zero, we get the relations $x_0+\xi_0=0$, $x_0'+\xi_0'=0$, indicating that the displacement and the velocity of the pendulum relative to space are both zero.

3. *The special case, $\xi=1$ for $0<t<h$ and $\xi=0$ for $t<0$, $t>h$.*

Since, in this case,

$$\xi \supset 1 \quad \text{for } 0<t<h, \quad \xi \supset 1-e^{-ph} \quad \text{for } t>h, \tag{8}$$

we get

$$\left.\begin{aligned} x \supset p\int_0^\infty e^{-pt}x\,dt &= \frac{-p^2}{p^2+2kp+n^2}, \quad (0<t<h) \\ &= \frac{-p^2(1-e^{-ph})}{p^2+2kp+n^2}, \qquad (t>h) \end{aligned}\right\} \tag{9}$$

from which

$$\left.\begin{aligned} x &= \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{zt}z\,dz}{z^2+2kz+n^2}, \quad (0<t<h) \\ x &= \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{zt}(1-e^{-zh})z\,dz}{z^2+2kz+n^2}. \quad (t>h) \end{aligned}\right\}. \tag{10}$$

In the case of $k \neq n$ $(\alpha_2 \neq 0)$, the integral (10) transforms to

$$\left.\begin{aligned} x &= \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{zt}z\,dz}{\{z+(k-i\alpha_2)\}\{z+(k+i\alpha_2)\}}, \quad (0<t<h) \\ x &= \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{zt}z(1-e^{-zh})\,dz}{\{z+(k-i\alpha_2)\}\{z+(k+i\alpha_2)\}}, \qquad (t>h) \end{aligned}\right\} \tag{11}$$

where $\alpha_2=\sqrt{n^2-k^2}$, from which we get

$$\left.\begin{aligned} x &= \frac{n}{\sqrt{n^2-k^2}}\left[e^{-kt}\sin\left(t\alpha_2-\tan^{-1}\frac{\alpha_2}{k}\right)\right], \quad (0<t<h) \\ x &= \frac{n}{\sqrt{n^2-k^2}}\left[e^{-kt}\sin\left(t\alpha_2-\tan^{-1}\frac{\alpha_2}{k}\right)\right. \\ &\qquad \left.-e^{-k(t-h)}\sin\left\{\alpha_2(t-h)-\tan^{-1}\frac{\alpha_2}{k}\right\}\right]. \quad (t>h) \end{aligned}\right\} \tag{12}$$

In the case of $k=n$, the integral (10) becomes of the types

$$x=\frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{zt}z\,dz}{(z+k)^2}, \quad x=\frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{zt}z(1-e^{-zh})\,dz}{(z\pm k)^2}, \tag{13}$$

so that

$$\left.\begin{aligned} x &= -(1-kt)e^{-kt}, \qquad (0<t<h) \\ x &= -(1-kt)e^{-kt}+\{1-k(t-h)\}e^{-k(t-h)}. \quad (t>h) \end{aligned}\right\} \tag{14}$$

From the results in (12), (14), it will be seen that in the present type of earth movement, there is apparently no term in the equations corresponding to forced vibration. The results of calculation for the two cases (i) $k/n=0$, $k/n=0{\cdot}6$, both for the same condition $nh=1$, are shown in Figs. 1, 2. The full lines represent the seismographic displacement

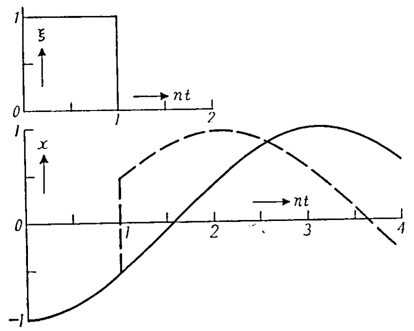

Fig. 1. $\xi=1$ for $0<t<h$, $\xi=0$ for $t<0$ and $t>h$. $nh=1$, $k/n=0$.

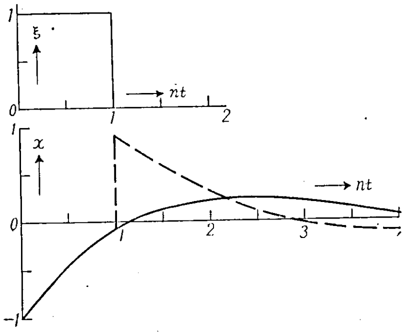

Fig. 2. $\xi=1$ for $0<t<h$, $\xi=0$ for $t<0$ and $t>h$. $nh=1$, $k/n=0{\cdot}6$.

for $0<t$ arising from the earth's displacement $\xi=1$ for $t>0$ and the broken lines that for $t>h$ arising from the displacement $\xi=1$ for $0<t<h$ and $\xi=0$ for $t>h$.

It will be seen that, as in common sense reasoning, the steepness of the damping curve for the seismographic movement increases with increase in logarithmic decrement, but the beginning of each damping curve corresponds to the forced vibration. The amount of initial movement of the pendulum is the same as that of the earth's displacement, but in the sense that is opposite to it for such a sudden movement. At all events, it is possible to say that the sense of the initial motion of the pendulum is opposite to that of the ground movement.

4. *The special case, $\xi=e^{-at}$ for $t>0$ and $\xi=0$ for $t<0$.*

In this case,

$$\xi \supset p\int_0^\infty e^{-pt}\xi dt=\frac{p}{p+a}, \tag{15}$$

so that

$$x \supset p\int_0^\infty e^{-pt}x dt=\frac{-p^3}{(p+a)(p^2+2kp+n^2)}, \tag{16}$$

from which

$$x=\frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{zt}z^2dz}{(z+a)(z^2+2kz+n^2)}=\frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{zt}z^2dz}{(z+a)(z+\alpha)(z+\beta)}, \tag{17}$$

where $\qquad \alpha=k-i\sqrt{n^2-k^2}, \qquad \beta=k+i\sqrt{n^2-k^2}.$ (18)

(i) If $\alpha \fallingdotseq \beta \fallingdotseq a$ $(k \fallingdotseq n)$,

$$\left.\begin{aligned}
x&=\frac{-a^2e^{-at}}{(a-\alpha)(a-\beta)}-\frac{\alpha^2e^{-\alpha t}}{(\alpha-a)(\alpha-\beta)}-\frac{\beta^2e^{-\beta t}}{(\beta-a)(\beta-\alpha)}\\
&=\frac{-e^{-at}}{1-\dfrac{2k}{a}+\dfrac{n^2}{a^2}}+\frac{\dfrac{n}{a}e^{-kt}}{\sqrt{\left(1-\dfrac{k^2}{n^2}\right)\left(1-\dfrac{2k^2}{n^2}+\dfrac{n^2}{a^2}\right)}}\\
&\quad\cdot\sin\left(nt\sqrt{1-\frac{k^2}{n^2}}\quad \tan^{-1}\frac{2\dfrac{k}{n}\sqrt{1-\dfrac{k^2}{n^2}}}{2\dfrac{k^2}{n^2}-1}+\tan^{-1}\frac{\dfrac{n}{a}\sqrt{1-\dfrac{k^2}{n^2}}}{-1+\dfrac{k}{a}}\right),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\alpha \fallingdotseq \beta \fallingdotseq a,\ t>0)\\
x&=0. \qquad\qquad\qquad\qquad\qquad\qquad\qquad (\alpha \fallingdotseq \beta \fallingdotseq a,\ t<0)
\end{aligned}\right\} \quad (19)$$

(ii) If $n=k$, $a \fallingdotseq n$, then $\alpha=\beta$, $a \fallingdotseq \alpha$, so that

$$\left.\begin{aligned}
x&=\frac{-1}{2\pi i}\left[\int_{c-i\infty}^{c+i\infty}\frac{1}{(a+\alpha)^2}\frac{z^2e^{zt}}{(z-a)}dz-\int_{c-i\infty}^{c+i\infty}\frac{1}{(a+\alpha)^2}\frac{z^2e^{zt}\left\{z+(a+2\alpha)\right\}}{(z+\alpha)^2}dz\right],\\
&=\frac{-a^2e^{-at}}{(a-n)^2}-\frac{n\left\{nt(a-n)-(2a-n)\right\}e^{-nt}}{(a-n)^2},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\alpha=\beta,\ a \fallingdotseq \alpha,\ t>0)\\
x&=0. \qquad\qquad\qquad\qquad\qquad\qquad\qquad (\alpha=\beta,\ a \fallingdotseq \alpha,\ t<0)
\end{aligned}\right\} \quad (20)$$

(iii) If $a=n=k$ $(a=\alpha=\beta)$,

$$\left.\begin{aligned}
x&=\frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{z^2e^{zt}dz}{(z+a)^3}=-\frac{n^2t^2}{2(1+nt)}-(1-2nt)e^{-nt}, \qquad (t>0)\\
x&=0. \qquad\qquad\qquad\qquad\qquad\qquad\qquad (t<0)
\end{aligned}\right\} \quad (21)$$

The respective first terms in (19), (20), (21) represent forced oscillation and the respective second terms the free oscillation. The results of calculation for the two cases of k/n, a/n being unity, are shown in Figs. 3, 4. The thick lines correspond to the resultant displacements of the pendulum. With increase in damping coefficient the vibrations of the pendulum decays with increasing rapidity.

The results of calculation for different ratios of a/n and the various values of k/n are given in Figs. 5~7. These figures show that although the damping of the pendulum vibration increases with increasing steep-

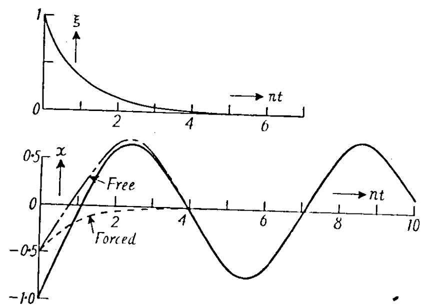

Fig. 3. $\xi = e^{-at}$ for $t > 0$, $\xi = 0$ for $t < 0$. $a/n = 1$, $k/n = 0$.

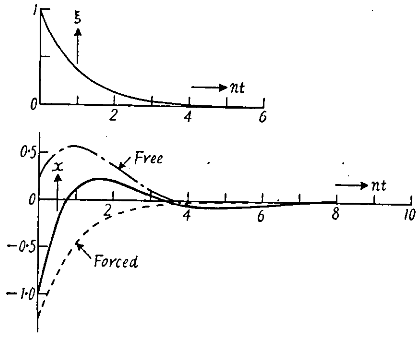

Fig. 4. $\xi = e^{-at}$ for $t > 0$, $\xi = 0$ for $t < 0$. $a/n = 1$, $k/n = 1$.

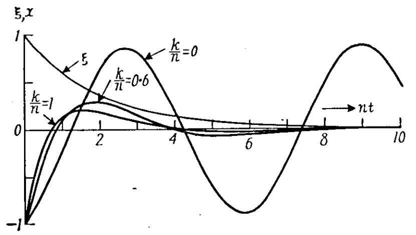

Fig. 5. $\xi = e^{-at}$ for $t > 0$, $\xi = 0$ for $t < 0$. $a/n = 0{\cdot}5$, $k/n = 0$, $0{\cdot}6$, 1.

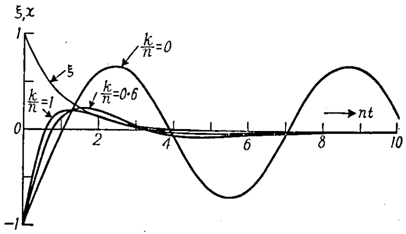

Fig. 6. $\xi = e^{-at}$ for $t > 0$, $\xi = 0$ for $t < 0$. $a/n = 1$, $k/n = 0$, $0{\cdot}6$, 1.

ness of the damping curve of the forced vibration, the amplitude of the first motion of the same pendulum is the same as that of the initial motion of the ground, but in the sense opposite to it—a feature closely resembling the case of the preceding section. This feature results from the condition that the initial motion of the ground is quite instantaneous.

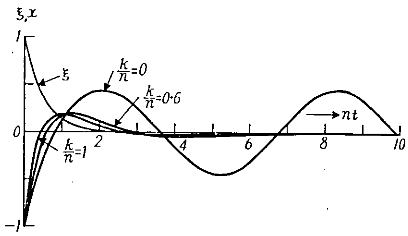

Fig. 7. $\xi = e^{-at}$ for $t > 0$, $\xi = 0$ for $t < 0$. $a/n = 2$, $k/n = 0$, $0{\cdot}6$, 1.

5. *The special case,* $\xi = \sin bt$ *for* $t > 0$, $\xi = 0$ *for* $t < 0$.[2]

In this case,

2) This case was solved by S. T. NAKAMURA and also by C. TSUBOI by simple elementary methods. *Proc. Imp. Acad.* **3** (1927), 32; *Bull. Earthq. Res. Inst.*, **12** (1934), 426.

$$\left.\begin{aligned} \xi \supset p\int_0^\infty e^{-pt}\xi dt &= \frac{pb}{p^2+b^2}, \quad (t>0) \\ &= 0, \qquad\qquad (t<0) \end{aligned}\right\} \tag{22}$$

so that

$$\left.\begin{aligned} x \supset p\int_0^\infty e^{-pt}x dt &= \frac{pb}{p^2+b^2}\,\frac{-p^2}{p^2+2kp+n^2} \\ &= \frac{-p^3 b}{(p+ib)(p-ib)(p+\alpha)(p+\beta)}, \quad (t>0) \\ &= 0, \qquad\qquad (t<0) \end{aligned}\right\} \tag{23}$$

where $p^2+2kp+n^2=(p+\alpha)(p+\beta)$. Thus,

$$x = \frac{-b}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{z^2 e^{zt}dz}{(z+ib)(z-ib)(z+\alpha)(z+\beta)}. \tag{24}$$

(i) If $\alpha \neq \beta$, from (24)

$$\left.\begin{aligned} x &= \frac{\sin\left(bt-\tan^{-1}\dfrac{\frac{2k}{b}}{\frac{n^2}{b^2}-1}\right)}{\sqrt{\left(\frac{n^2}{a^2}-1\right)^2+\frac{4k^2}{a^2}}} \\ &\quad - \frac{\frac{n}{b}e^{-kt}\sin\left\{nt\sqrt{1-\frac{k^2}{n^2}}-\tan^{-1}\dfrac{2\frac{k}{n}\sqrt{1-\frac{k^2}{n^2}}}{2\frac{k^2}{n^2}-1}+\tan^{-1}\dfrac{\frac{2k}{n}\sqrt{1-\frac{k^2}{n^2}}}{2\frac{k^2}{n^2}-1+\frac{b^2}{n^2}}\right\}}{\sqrt{\left(1-\frac{k^2}{n^2}\right)\left\{\left(\frac{n^2}{b^2}-1\right)^2+4\frac{k^2}{b^2}\right\}}} \qquad (t>0) \\ x &= 0. \qquad\qquad (t<0) \end{aligned}\right\} \tag{25}$$

(ii) If $\alpha=\beta$ $(n=k)$, then

$$x = \frac{-b}{(k^2+b^2)^2 2\pi i}\left[\int_{c-i\infty}^{c+i\infty}\frac{z^2\left\{-2kz+(k^2-b^2)\right\}e^{zt}dz}{(z+ib)(z-ib)} + \int_{c-i\infty}^{c+i\infty}\frac{z^2 e^{zt}\left\{2kz+(3k^2+b^2)\right\}dz}{(z+k)^2}\right]$$

$$\left.\begin{aligned}&=\frac{1}{\left(\frac{n^2}{b^2}+1\right)}\left[\sin\left(bt-\tan^{-1}\frac{2\frac{n}{b}}{\frac{n^2}{b^2}-1}\right)+\frac{n}{b}\left\{\frac{2}{\left(\frac{n^2}{b^2}+1\right)}-nt\right\}e^{-nt}\right],\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(n=k,\ t>0)\\&x=0.\qquad\qquad\qquad\qquad\qquad\qquad\qquad(n=k,\ t<0)\end{aligned}\right\}\quad(26)$$

(iii) If $k=0$, $n=b$, then

$$\left.\begin{aligned}x&=\frac{-b}{2\pi i}\left[\int_{c-i\infty}^{c+i\infty}\frac{z^2e^{zt}(iz-2b)\,dz}{4b^3(z+ib)^2}-\int_{c-i\infty}^{c+i\infty}\frac{z^2e^{zt}(iz+2b)\,dz}{4b^3(z-ib)^2}\right]\\&=-\tfrac{1}{2}(\sin bt+bt\cos bt),\qquad(k=0,\ n=b,\ t>0)\\x&=0.\qquad\qquad\qquad\qquad\qquad(k=0,\ n=b,\ t<0)\end{aligned}\right\}\quad(27)$$

The respective first terms in (25), (26) represent forced oscil-

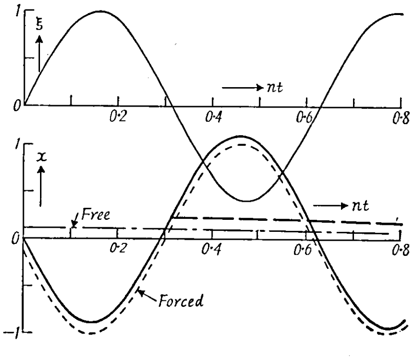

Fig. 8. $\xi=\sin bt$ for $t>0$, $\xi=0$ for $t<0$. $n/b=0{\cdot}1$, $k/n=0{\cdot}6$.

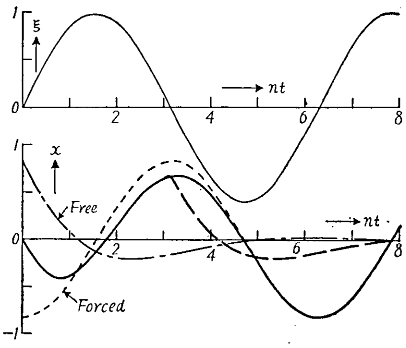

Fig. 9. $\xi=\sin bt$ for $t>0$, $\xi=0$ for $t<0$. $n/b=1$, $k/n=0{\cdot}6$.

lation and the respective second terms free oscillation. The results of calculation for the three cases of n/b, namely $n/b=0{\cdot}1$, 1, 10 for $k/n=0{\cdot}6$, are shown in Figs. 8, 9, 10. The full lines give the resultant vibrations for $t>0$ arising from $\xi=\sin bt$ for $t>0$ and $\xi=0$ for $t<0$, and the thick broken lines those for $t>\pi/2b$ arising from $\xi=\sin bt$ for $0<t<\pi/2b$ and $\xi=0$ for $t>\pi/2b$. The condition $n/b\ll 1$ corresponds to the state of the dis-

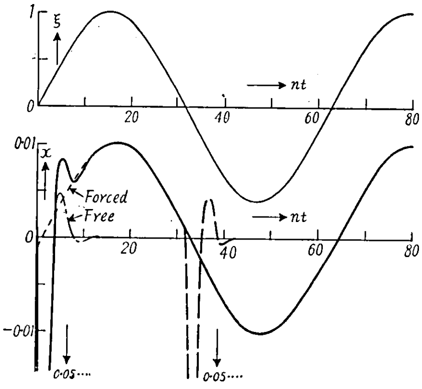

Fig. 10. $\xi=\sin bt$ for $t>0$, $\xi=0$ for $t<0$. $n/b=10$, $k/n=0{\cdot}6$.

placement seismometer, and that of $n/b \gg 1$ to that of the accelerometer. Comparing the results in Figs. 8~10, it will be seen that, for sinusoidal motion of the ground occurring at $t=0$, although the displacement seismometer can record the movement of the ground rather accurately, at least for the initial motion, it is not possible to do so with the acceleration seismometer, not at any rate, for the same initial motion. In the case of the acceleration seismometer, the duration of the initial motion of the pendulum is fairly small.

For ascertaining the effect of damping of the seismometer, we calculated two cases with respect to k/n for $n/b=1$, with results as shown in Figs. 11~12. Since the condition $n/b=1$ represents resonance, the vibration amplitude of the pendulum without damping increases without limit with time increase, whereas the vibration of the pendulum with critical damping fairly resembles the motion of the ground.

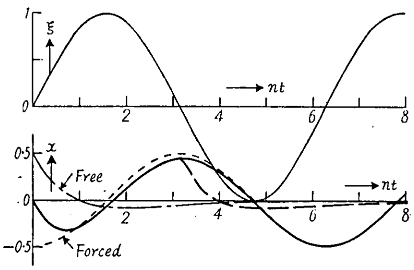

Fig. 11. $\xi=\sin bt$ for $t>0$, $\xi=0$ for $t<0$. $n/b=1$, $k/n=1$.

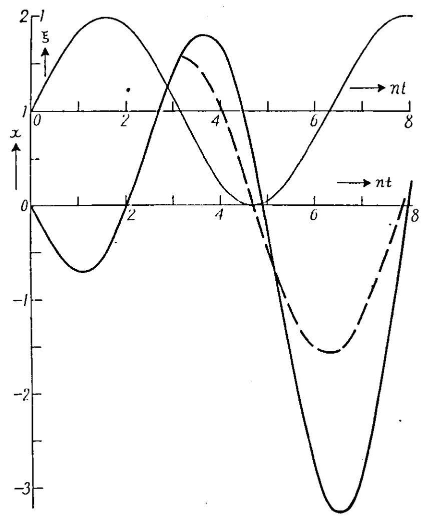

Fig. 12. $\xi=\sin bt$ for $t>0$, $\xi=0$ for $t<0$. $n/b=1$, $k/n=0$.

It may be remarked here that in the present problem, since $x_0=0$, $\xi_0=0$, it is possible from (7) for the initial conditions $x'_0=-\xi'_0$ to exist. At all events, the initial motion of the pendulum is always in the sense opposite to that of the ground motion.

6. *The special case, $\xi=e^{-at}\sin bt$ for $t>0$, $\xi=0$ for $t<0$.*

Here

$$\left.\begin{aligned} \xi \supset p\int_0^\infty e^{-pt}\xi dt &= \frac{pb}{(p+a)^2+b^2}, \quad (t>0) \\ &=0, \qquad\qquad\quad (t<0) \end{aligned}\right\} \tag{28}$$

from which

$$\left.\begin{aligned} x \supset p\int_0^\infty e^{-pt}x\,dt &= \frac{-p^3 b}{\left\{(p+a)^2+b^2\right\}(p^2+2kp+n^2)}, \quad (t>0) \\ &= 0. \qquad\qquad\qquad\qquad\qquad (t<0) \end{aligned}\right\} \tag{29}$$

Thus,

$$\left.\begin{aligned} x &= \frac{-b}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{z^2e^{zt}dz}{\left\{z+(a-ib)\right\}\left\{z+(a+ib)\right\}\left\{z+(k-i\alpha_2)\right\}\left\{z+(k+i\alpha_2)\right\}}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t>0) \\ x &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad (t<0) \end{aligned}\right\} \tag{30}$$

where $\alpha_2=\sqrt{n^2-k^2}$.

(i) If $a\neq k$, $b\neq\alpha_2$,

$$x=\frac{-\left(1+\dfrac{a^2}{b^2}\right)e^{-at}\sin\left\{bt-\tan^{-1}\dfrac{\dfrac{2a}{b}}{\dfrac{a^2}{b^2}-1}+\tan^{-1}\dfrac{2\left(\dfrac{a}{b}-\dfrac{k}{b}\right)}{\left(\dfrac{a}{b}-\dfrac{k}{b}\right)^2+\dfrac{n^2}{b^2}-\dfrac{k^2}{b^2}-1}\right.}{\sqrt{\left\{\left(\dfrac{a}{b}-\dfrac{k}{b}\right)^2+\dfrac{n^2}{b^2}-\dfrac{k^2}{b^2}-1\right\}^2+4\left(\dfrac{a}{b}-\dfrac{k}{b}\right)^2}}$$

$$-\frac{\dfrac{n^2}{b^2}e^{-kt}\sin\left\{nt\sqrt{1-\dfrac{k^2}{n^2}}-\tan^{-1}\dfrac{2\dfrac{k}{n}\sqrt{1-\dfrac{k^2}{n^2}}}{2\dfrac{k^2}{n^2}-1}-\tan^{-1}\dfrac{\dfrac{2n}{b}\sqrt{1-\dfrac{k^2}{n^2}}\left(\dfrac{a}{b}-\dfrac{k}{b}\right)}{\left(\dfrac{a}{b}-\dfrac{k}{b}\right)^2-\dfrac{n^2}{b^2}+\dfrac{k^2}{b^2}-1}\right\}}{\sqrt{\left(\dfrac{n^2}{b^2}-\dfrac{k^2}{b^2}\right)\left[\left\{\left(\dfrac{a}{b}-\dfrac{k}{b}\right)^2-\dfrac{n^2}{b^2}+\dfrac{k^2}{b^2}+1\right\}^2+4\left(\dfrac{n^2}{b^2}-\dfrac{k^2}{b^2}\right)\left(\dfrac{a}{b}-\dfrac{k}{b}\right)^2\right]}},$$

$$(t>0)$$

$$x=0, \qquad (t<0) \tag{31}$$

(ii) If $a=k$, $b=\alpha_2$,

$$\left.\begin{aligned} x &= \frac{-b}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{z^2e^{zt}dz}{\left\{z+(k-i\alpha_2)\right\}^2\left\{z+(k+i\alpha_2)\right\}^2} \\ &= \frac{\sqrt{1-\dfrac{k^2}{n^2}}\,e^{-kt}\sqrt{P^2+Q^2}\,\sin nt\left(\sqrt{1-\dfrac{k^2}{n^2}}-\tan^{-1}\dfrac{Q}{P}\right)}{8\left(1-\dfrac{k^2}{n^2}\right)^{\frac{3}{2}}}, \quad (t>0) \\ x &= 0, \qquad\qquad\qquad\qquad\qquad\qquad (t<0) \end{aligned}\right\} \tag{32}$$

where $P=kt\left(7-8\frac{k^2}{n^2}\right)+\left(6\frac{k^2}{n^2}-5\right),\qquad Q=\sqrt{1-\frac{k^2}{n^2}}\left\{nt\left(3-8\frac{k^2}{n^2}\right)+6\frac{k}{n}\right\}.$

(iii) If $n=k$,

$$\left.\begin{aligned} x&=\frac{-b}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{z^2e^{zt}dz}{\left\{z+(a-ib)\right\}\left\{z+(a+ib)\right\}\left\{z+k\right\}^2}\\ &=\frac{-(a^2+b^2)}{(a-k)^2+b^2}e^{-at}\sin\left\{bt-\tan^{-1}\frac{2ab}{a^2-b^2}+\tan^{-1}\frac{2b(a-k)}{(a-k)^2-b^2}\right\}\\ &\quad-\frac{bke^{-kt}}{\left\{(a-k)^2+b^2\right\}^2}\left[\left\{(a-k)^2+b^2\right\}(kt-2)+2k(k-a)\right], \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t>0)\\ x&=0. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad (t<0) \end{aligned}\right\} \quad (33)$$

The respective first terms in (31), (33) represent forced oscillation and the respective second terms free oscillation. The results of calculation for the three cases of n/b (for $a/b=0{\cdot}1$ or 1) for $k/n=0{\cdot}6$ are shown in Figs. 13∼15. Although, in these cases, the damping is relatively high, if n/b were not small, that is to say, if the condition were in that of the accelerometer, the movement of the seismograph would not necessarily agree with that of the ground motion. In such a condition, the duration of the initial motion of

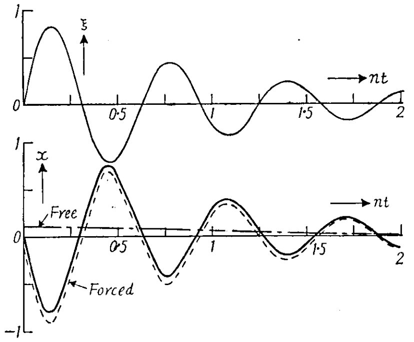

Fig. 13. $\xi=e^{-at}\sin bt$ for $t>0$, $\xi=0$ for $t<0$. $a/b=0{\cdot}1$, $n/b=0{\cdot}1$, $k/n=0{\cdot}6$.

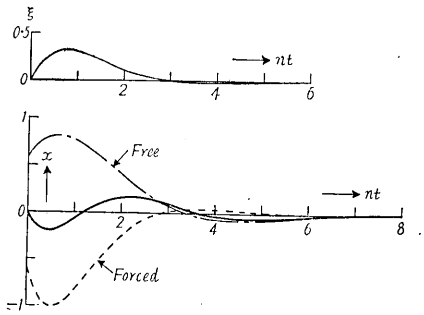

Fig. 14. $\xi=e^{-at}\sin bt$ for $t>0$, $\xi=0$ for $t<0$. $a/b=1$, $n/b=1$, $k/n=0{\cdot}6$.

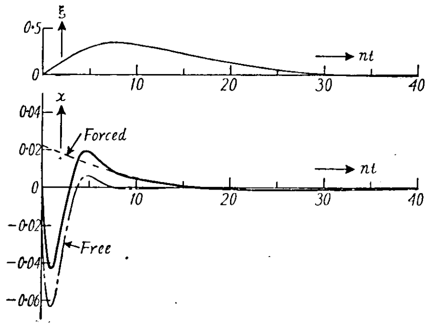

Fig. 15. $\xi=e^{-at}\sin bt$ for $t>0$, $\xi=0$ for $t<0$. $a/b=1$, $n/b=10$, $k/n=0{\cdot}6$.

the pendulum is unduly small and the amplitude of the same motion fairly large. It is however possible to say that the initial motion of the pendulum is in the sense opposite to that of the ground. Owing to the fact that $x_0=0$, $\xi_0=0$, it is also possible for condition $x'_0=-\xi'_0$ to exist.

7. *The special case of* $\xi=e^{-at}\,(2\sin bt-\sin 2bt)$ *for* $t>0$, $\xi=0$ *for* $t<0$.

In the examples shown in the foregoing sections, it was assumed that the ground motion begins suddenly with initial velocity. In such cases, the duration of the initial motion of the pendulum is unduly small and the amplitude of the same motion fairly large. If, on the other hand, the ground motion begins without initial velocity, the feature in the seismographic movements, particularly, that in the amplitude of the initial motion, is modified to a certain extent.

For the present purpose, we write

$$\left.\begin{aligned} &\xi=e^{-at}\,(2\sin bt-\sin 2bt), &&(t>0) \\ &\xi=0, &&(t<0) \end{aligned}\right\} \tag{34}$$

by means of which expressions it is possible for the initial displacement and velocity of the ground to be made zero.

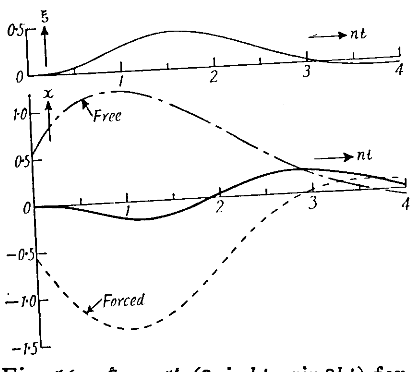

Fig. 16. $\xi=e^{-at}\,(2\sin bt-\sin 2bt)$ for $t>0$, $\xi=0$ for $t<0$. $a/b=1$, $n/b=1$, $k/b=0{\cdot}6$.

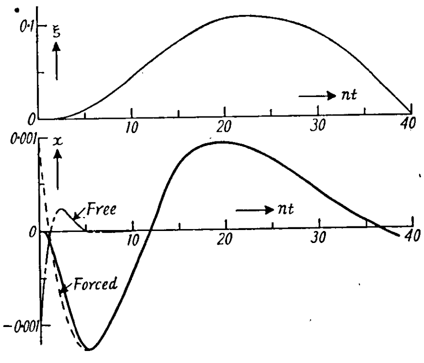

Fig. 17. $\xi=e^{-at}\,(2\sin bt-\sin 2bt)$ for $t>0$, $\xi=0$ for $t<0$. $a/b=2$, $n/b=20$, $k/n=0{\cdot}6$.

The results of calculation for two cases of n/b are shown in Figs. 16~17. It will be seen that although the duration of the initial motion of the pendulum becomes small for quick motion of the ground, the amplitude of the same initial motion is not abnormally large.

8. *Concluding remarks.*

Using the operational calculus, we calculated the type of movement of a seismograph subjected to arbitrary earthquake motion. Generally speaking, the motion of the pendulum consists of two parts, one of which corresponds to forced oscillation and the other to free oscillation. In any case, the initial motion of the pendulum is in the sense opposite to that of the ground motion. In the case of a displacement seismometer, the motion of the pendulum fairly agrees with that of the ground, whereas in the acceleration seismometer it does not. When the disturbance occurs suddenly, the amount of initial displacement of the seismometer is the same as that of the ground motion, but in opposite sense. If the ground motion of quick type begins with zero displacement and zero velocity, then, although the duration of the initial movement of the pendulum becomes very small, the amplitude of the same motion is not abnormally large.

In conclusion, we wish to express our thanks to Mr. Watanabe, who assisted us greatly in our calculations. We also wish to express our sincere thanks to the officials of the Division of Scientific Research, in the Ministry of Education, for financial aid (Funds for Scientific Research) granted us for a series of investigations, of which this study is a part.

Added notes—For getting, operationally, the earth's displacement from a seismographic record (without magnification), we write

$$\ddot{x}+2k\dot{x}+n^2x=-\ddot{\xi}. \tag{1}$$

Using the sympol p of the operator for d/dt, (1) transforms to

$$\xi=-x-\frac{(2kp+n^2)}{p^2}x\equiv -x+\xi_1. \tag{2}$$

Let $\phi(p)$ be the operational form of x. Then

$$\xi_1 \supset -\frac{(2kp+n^2)}{p^2}\phi(p), \tag{3}$$

from which

$$\xi_1 = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\frac{2kz+n^2}{z^3}\phi(z)\,dz. \tag{4}$$

If the pendulum movement were of the form $f(t)$, then, from Mellin's theorem, it would follow that

$$\phi(z) = z\int_0^\infty e^{-zt_1}f(t_1)\,dt_1. \tag{5}$$

Substituting (5) in (4), and changing the orders of the integrations,

$$\xi_1 = \frac{-1}{2\pi i}\int_0^\infty f(t_1)\,dt_1\int_{c-i\infty}^{c+i\infty} e^{z(t-t_1)}\frac{2kz+n^2}{z^2}\,dz. \tag{6}$$

Considering the nature of the contour of the integration,

$$\xi_1 = \frac{-1}{2\pi i}\int_0^t f(t_1)\,dt_1\int_{c-i\infty}^{c+i\infty} e^{z(t-t_1)}\frac{2kz+n^2}{z^2}\,dz = -\int_0^t f(t_1)\,dt_1\Big[2k+n^2(t-t_1)\Big]. \tag{7}$$

Integrating by parts,

$$\xi_1 = -2k\int_0^t f(t_1)\,dt_1 - n^2\int_0^t dt_2\int_0^{t_2} f(t_1)\,dt_1. \tag{8}$$

Putting this in (2), we naturally get

$$\xi = -x+\xi_1 = -f(t_1) - 2k\int_0^t f(t_1)\,dt_1 - n^2\int_0^t dt_2\int_0^{t_2} f(t_1)\,dt_1. \tag{9}$$

Although this formula can be obtained directly from (1), the foregoing calculation is specially to show the method of applying operational calculus to the present case. The case of Galitzin's pendulum can also be discussed in the same way.

12. 演算法によりて解いた地震計の初動問題について（第1報）

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金井清

演算法を用ひる事によつて任意の地震動を受ける地震計の動きを計算した．一般的にいふと，振子の動きの表式は強制振動の部分と自己振動の部分とから成立つ．如何なる場合にも振子の動きは地動と反對方向になる．變位地震計の場合には振子の動きは地動と可成り一致するけれども，加速度地震計は必しも然らず．地動が極めて急激に起るときは地震計の動きは地動の動きと同じであつて地動とは反對方向になるといふ事も出るのである．尚，地動が最初に變位も速度も零から始まるときには，地震計の初動の繼續時間は短いが振幅は必しも大きくならぬ．之に反して地動の最初の速度が零でないときには地震計の初動の繼續時間も短いし，その振幅も非常に大きくなるのである．