14. *Elastic Waves from a Point in an Isotropic Heterogeneous Sphere. Part 3.*

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1. As will become apparent by the following paragraph, in studying the propagation of waves through a medium, the heterogeneity of which in respect to its density and elastic constants is small, the problem eventually revolves itself into a mathematical treatment of a differential equation of the type

$$\frac{\partial^2\phi}{\partial T^2}=v^2\nabla^2\phi-\alpha^2\phi, \tag{117}$$

in which v stands for $\sqrt{\frac{\lambda+2\mu}{\rho}}$ or $\sqrt{\frac{\mu}{\rho}}$, and α^2 is some function of derivatives of ρ, λ, and μ. When the medium is homogeneous, these derivatives vanish, and the equation (117) reduces to the so-called "wave equation" in a homogeneous medium. In Parts 1 and 2,[1] the writer, dealing with a special heterogeneous medium, worked out equations of wave motion exactly similar to (117).

It is shown in this paper that, even through a more general heterogeneous medium, waves of a certain character are propagated in accordance with an equation of type (117), and it will be easily seen from the example in Part 2 that, in some cases, $\alpha^2=0$ or v is a constant, although the medium may be heterogeneous with respect to each of its elastic constants and density.

In certain branches of atomic physics, equations of the type (117) are studied, and some approximate solutions obtained. However, since the problems met with in seismology differ from those in atomic physics, it is not likely that those approximate solutions can be satisfactorily used for the former. Here, solutions of equation (117) are discussed, using the results obtained in Parts 1 and 2 of this paper.

2. After the notations in Part 1, equation (117) is written in the form

1) R. YOSIYAMA, *Bull. Earthq. Res. Inst.*, **11** (1933), 1~13., **18** (1940), 41~56.

$$\frac{\partial^2\phi}{\partial T^2}=\frac{h_2h_3}{h_1}\left\{\frac{\partial}{\partial t}\left(\frac{h_1}{h_2h_3}\frac{\partial\phi}{\partial t}\right)+\frac{\partial}{\partial\beta}\left(\frac{h_2}{h_3h_1}\frac{\partial\phi}{\partial\beta}\right)+\frac{\partial}{\partial\varphi}\left(\frac{h_3}{h_1h_2}\frac{\partial\phi}{\partial\varphi}\right)\right\}-a^2\phi. \tag{118}$$

Putting

$$\phi=\sqrt{\left|\frac{h_2h_3}{h_1}f(\beta)\right|}\quad\phi_1\begin{matrix}\sin m\varphi\\ \cos m\varphi\end{matrix}e^{\pm ipT}, \tag{119}$$

the equation transforms to

$$\frac{1}{\phi_1}\frac{\partial^2\phi_1}{\partial t^2}+p^2-a^2-a'^2+\frac{h_2^2}{h_1^2}\left\{\frac{1}{\phi_1}\frac{\partial^2\phi_1}{\partial\beta^2}\right.$$

$$\left.+\frac{1}{\phi_1}\frac{\partial\phi_1}{\partial\beta}\frac{\partial}{\partial\beta}\log\frac{h_2^2}{h_1^2}f(\beta)-m^2\frac{h_3^2}{h_2^2}\right\}=0, \tag{120}$$

in which

$$-a'^2=v^2\sqrt{\left|\frac{h_1}{h_2h_3f(\beta)}\right|}\quad\nabla^2\sqrt{\left|\frac{h_2h_3}{h_1}f(\beta)\right|}\,. \tag{121}$$

The expression of $f(\beta)$, which is arbitrary as yet, had best be determined in such a way that ϕ_1 in equation (120) may be solved in the most convenient form possible to discuss the problem.

We see that $\frac{d\beta}{h_2}$ is equal to the length of a line-element, perpendicular to two wave rays in a plane $\varphi=$const., making an angle $d\beta$ with each other at the origin, and that, consequently, $\frac{1}{h_2}$ increases with t if, as in most conceivable cases, the wave-front spreads out as the wave motions are propagated, with the result that $\frac{h_1}{h_2}$, h_1 being equal to $\frac{1}{v}$, also tends generally to increase with t. The same conclusion also holds true with respect to $\frac{h_1}{h_3}$. Consequently, if m is small, we may put for sufficiently large values of t and p, neglecting the variations of a^2 and a'^2,

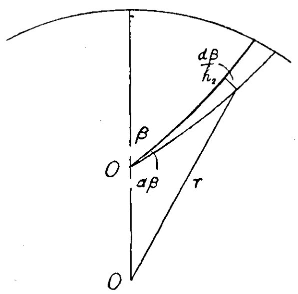

Fig. 1.

$$\phi_1=e^{\pm i\sqrt{p^2-a^2-a'^2}\,t} \tag{122}$$

in which $f(\beta)$ is required to be such that a'^2 is practically constant.

If we can get $f(\beta)$ for which $\alpha^2+\alpha'^2$ is a constant, then $\sqrt{\left|\frac{h_2h_3}{h_1}f(\beta)\right|}$ $\cdot e^{\pm i\sqrt{p^2-\alpha^2-\alpha'^2}\,t}$ becomes one of the exact particular solutions of (117). This solution corresponds to that for a single source in a homogeneous medium, and from this solution it is possible to find a solution for the wave from a multiple source in a heterogeneous medium by the same method of procedure as that in the case of a homogeneous medium. Although the solution thus obtained must be equivalent to that obtained directly from (117) or (118), determination of such $f(\beta)$ is obviously equivalent to obtaining a particular solution of the original equation (117), which is none the less difficult. However, if we can get an approximate solution of $f(\beta)$, an approximate solution of (117) naturally follows. When v^2 and a^2 are both constant, we have $f(\beta)=\sin\beta$ and $\alpha'^2=0$, whence it may be assumed that generally $f(\beta)\doteqdot\sin\beta$. And if p is sufficiently large, $p^2-\alpha^2-\alpha'^2$ being considered practically constant, we can use (122) as a particular solution of ϕ_1.

In the case of a homogeneous medium, or such as that considered in Parts 1, 2, then ϕ_1 is expressible by a product of two functions, one of which is a function of t only, and the other a function of β only. Put, for example, X for the former and Y for the latter. In all the cases just given, $f(\beta)=\sin\beta$ and $Y=P_n^m(\cos\beta)$; $X=\sqrt{t}\,H_{n+\frac{1}{2}}(pt)$ in the case of a homogeneous medium, and $X=\left(\frac{\sinh 2\sqrt{ab}\,t}{2\sqrt{ab}}\right)^{\frac{1}{2}}U_n$ in the case considered in Parts 1, 2.

Now it is a debatable point whether or not ϕ_1 is in all cases expressed by a product of two functions. The answer to this question must be relevant to one of the theoretical bases of the method of procedure, by means of which we may discuss the mechanism of earthquake occurrence at the origin from observations of the amplitudes and the directions of initial motions on the earth surface.

Although a mathematical treatment of this problem without assuming the functional forms of v^2 and a^2 seems very difficult, there is no doubt, at any rate, that $Y=P_n^m(\cos\beta)$ holds true for only a few special cases. Even when $v=a-br^2$, a^2 being a function of r, the assumption $\phi_1=X\cdot Y$ is clearly inadmissible, even if Y is not restricted to $P_n^m(\cos\beta)$.

At all events, unsatisfactory conditions caused by putting

$$\phi_1=e^{\pm i\sqrt{p^2-\alpha^2-\alpha'^2}\,t}\;P_n^m(\cos\beta) \tag{123}$$

decrease as $\frac{h_2^2}{h_1^2}$, and $\frac{h_3^2}{h_1^2}$ decreases compared with p^2. This means that, when the wave front duly spreads out, waves of sufficiently short period are propagated independently along each path, as determined by Fermat's principle, the wave form observed on each path not being affected by the phase difference between waves along contiguous paths.

The foregoing remarks refer to a stationary wave. Solitary waves, for example, the P wave and the S wave observed in seismograms, may be considered, from a knowledge of Fourier's series, as superpositions of stationary waves of various periods. Consequently it will be accepted that, at the beginning of the P phase, the S phase and other solitary phases, whose time of incidence are clearly identified in seismograms, waves of fairly short period predominate, with the result that, (123) being accepted, the effect of heterogeneity on the beginning of such phases, especially on the direction, push and pull, of the incidence, will generally be negligible. Moreover, by applying the results of the calculations in Part 2, the velocity of propagation of incidence of disturbances will be equal to v. The effect of heterogeneity of the medium becomes very marked in later observations, the most marked being a certain reduction in the maximum amplitude of the disturbance, besides the appearance of damped oscillation of a characteristic frequency, which we shall call the "tail." These conclusions above stated can be deduced from the results obtained in Part 2. Obviously, the oscillation just mentioned has a physical significance only when $\alpha^2+\alpha'^2>0$, as will be seen from (92) in Part 2. When $\alpha^2+\alpha'^2<0$, the effect increases exponentially with time, in which case we must take into consideration that, in building up the wave equation (117), we have neglected, as stated in Part 2, the terms of the product of any two of ϕ, $\frac{\partial \phi}{\partial t}$, and so on, and also that the stress-strain relation in the medium, in which there must be a very close connection between every molecule, will naturally be altered when the deformation increases. Therefore, when $\alpha^2+\alpha'^2<0$, the effect above stated loses its fitness for being interpreted physically, particularly in the case of an ordinary elastic medium. However, in a fluid medium, whose molecules are to a certain extent free and not closely bound to one another, such a state as $\alpha^2+\alpha'^2<0$ may be regarded as being unstable.

At any rate, referring here again to the results from (91) to (93), described in Part 2, we find that the terms expressing these effects are proportional to certain integral forms, such as $\int_0^{t_0}\psi_1(\omega)d\omega$ or $\int_{t-T}^{t_0}\psi_1(\omega)d\omega$. This clearly means that these effects, namely, the reduction of maximum amplitude or the amplitude of the "tail," decrease, so to speak, as the apparent period of the disturbance decreases.

Although the solution of (117) will be expressed in various forms, we have attempted to find a solution that contains explicitly a factor of a harmonic function with regard to t, because if the results of observations are considered, such a solution would find wide application in the study of seismology.

From the foregoing discussions, we may use the expression $\sqrt{\left|\frac{h_2 h_3}{h_1}\sin\beta\right|}$ as the attenuation factor due to the distance of the station from the origin in obtaining from the observations a more theoretically suitable magnitude of ϕ at the origin, while the area of the wave front, enclosed by wave rays, and spreading out from the origin in a certain solid angle, is equal to $\frac{d\beta d\varphi}{h_2 h_3}$, from which last expression we may build up another formula for the attenuation factor.

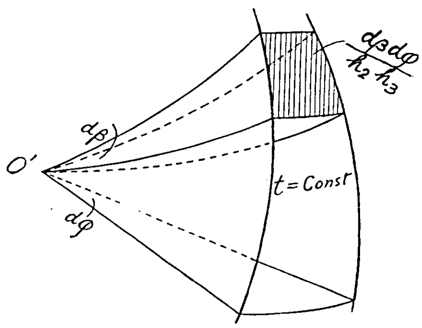

Fig. 2.

It will be seen that, in some cases, $\frac{1}{h_2}$ decreases as t increases, or vanishes at a certain value of t for a certain range of value of β, and that $\frac{1}{h_2}$ changes its sign, from positive to negative, for a smaller or a larger value than that of t, in which case, in the neighbourhood of that region where $\frac{1}{h_2}$ vanishes, $e^{\pm i\sqrt{p^2\alpha^2-\alpha'^2}\,t}$ is not admissible as an approximate solution even though t be large. Considering the significance of $\frac{d\beta d\varphi}{h_2 h_3}$ above stated, $\frac{1}{h_2}=0$ means intersection of the wave rays. From (13) in Part 1, $\frac{1}{h_2}=0$ gives

$$\int^{r} \frac{g^2(r)\,dr}{r\left\{g^2(r)-\kappa^2\right\}^{\frac{3}{2}}}=0, \tag{124}$$

which is equivalent to the equation of the envelope of the wave rays. Although the solution in such a case may be applicable in explaining, say, the mirage, in optics, other mathematical technique will be necessary in order to find a solution suitable for use in the neighbourhood of that region where $\frac{1}{h_2}=0$. This the writer hopes to treat on a future occasion.

In the following paragraph, we shall treat the problem more concretely and make clear the physical significances of ϕ and α^2 in connection with the elastic constants of the medium.

3. From the θ- and the φ-components of equation of motion (2), we get the following equation with respect to the r-component of rot $\boldsymbol{D}$, ϖ_r,

$$\rho\frac{\partial^2}{\partial T^2}\sqrt{\mu}\,r\varpi_r=\mu\nabla^2\sqrt{\mu}\,r\varpi_r-\left(\sqrt{\mu}\,\nabla^2\sqrt{\mu}+\frac{1}{r}\frac{d\mu}{dr}\right)\sqrt{\mu}\,r\varpi_r. \tag{125}$$

By performing divergence operations on both members of the equation (2), we get

$$\frac{d\rho}{dr}\frac{\partial^2 u_r}{\partial T^2}+\rho\frac{\partial^2\Delta}{\partial T^2}=\nabla^2\left\{(\lambda+2\mu)\Delta\right\}-\frac{2}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{d\mu}{dr}\Delta\right)$$
$$+\frac{2}{r}\frac{d\mu}{dr}\nabla^2 ru_r+2r\frac{d}{dr}\left(\frac{1}{r}\frac{d\mu}{dr}\right)\frac{\partial u_r}{\partial r}. \tag{126}$$

The r-component of equation (2) is modified to

$$\rho\frac{\partial^2 u_r}{\partial T^2}=\frac{\mu}{r}\nabla^2 ru_r+2\frac{d\mu}{dr}\frac{\partial u_r}{\partial r}+\frac{\partial}{\partial r}(\lambda\Delta)+r^2\mu\frac{\partial}{\partial r}\left(\frac{\Delta}{r^2}\right). \tag{127}$$

Since no one of the three equations, (125), (126), (127), is derived from the remaining two by mere mathematical operation, we can use these three equations instead of equation (2), consequently we can use the three quantities ϖ_r, Δ, and u_r instead of the three components of displacement, with the magnitudes of which three it is possible to discuss the deformation in the elastic medium. We find that ϖ_r appears only in equation (125), and also that equation (125) contains ϖ_r only. Consequently, the problem can be treated by considering that, in such a medium, the deformed state, generally, is a superposition of two kinds of waves, the one satisfying equation (125) and $u_r=\Delta=0$, and the other equations (126), (127), and $\varpi_r=0$.

We know that two kinds of waves are propagated through a homogeneous medium. Although, in a heterogeneous medium, the propriety of assuming that it is possible to classify the waves is questionable, such assumption is clearly very convenient in solving the equations. And since, as a matter of fact, there are two distinct phases, such as the P phase and the S phase, as observed in seismograms, solutions of problems on such assumption will also be of some use in seismological investigations.

Prof. K. Sezawa, for convenience in dealing with earthquake mechanism, classified the waves through a homogeneous medium into three kinds, namely, the irrotational dilatational wave and equivoluminal rotational waves of the first kind and of the second kind. In this paper, it is shown that it is convenient, in the case of a heterogeneous medium, to assort the waves according to their characteristic mode of propagation. Since waves of magnitude of ϖ_r, such that $u_r = \Delta = 0$, are propagated as a wave independent of other disturbance as already stated, we shall call it the "*transverse wave of type B.*" It will be easily understood that, generally speaking, if the elastic constants and density of the medium are some functions of one of the coordinates of a certain orthogonal curvilinear coordinate system, that component of rot $\boldsymbol{D}$ is propagated as a "*transverse wave of type B*" independently of other waves. In the case of a homogeneous medium, there are two other types of waves, one of which is the "*longitudinal wave,*" and the other the transverse wave, such that $\varpi_r = \Delta = 0$.

Neglecting the gradient terms of ρ, λ, and μ in (126) and (127), we get the following, as approximate equations in a heterogeneous medium,

$$\rho\frac{\partial^2 \Delta}{\partial T^2} = \nabla^2\left\{(\lambda+2\mu)\Delta\right\}, \tag{128}$$

$$\rho\frac{\partial^2 u_r}{\partial T^2} = \frac{\mu}{r}\nabla^2 r u_r + \frac{\partial}{\partial r}(\lambda\Delta) + r^2\mu\frac{\partial}{\partial r}\left(\frac{\Delta}{r^2}\right), \tag{129}$$

from which it will be seen that the magnitude of div $\boldsymbol{D}$ is propagated as a wave of velocity $\sqrt{\frac{\lambda+2\mu}{\rho}}$, which we shall call the "*longitudinal wave.*" The solution of u_r consists of two parts, one of which is a particular solution corresponding to the magnitude of Δ, and the other a general solution obtained by putting $\Delta = 0$. The former is nothing but the magnitude of u_r due to the longitudinal wave, and is necessarily propagated with velocity $\sqrt{\frac{\lambda+2\mu}{\rho}}$. The latter, however, represents another wave that is propagated with velocity $\sqrt{\frac{\mu}{\rho}}$, and in which $\Delta = 0$. To sum up, the wave that is propagated with velocity $\sqrt{\frac{\mu}{\rho}}$ consists of two types of waves, one of which we have called the "*transverse* *wave of type B*" and the other which we shall call the "*transverse wave of type A.*" If we can construct an equation that contains Δ only, and no u_r, such as (128), div $\boldsymbol{D}$ will always vanish in the case of a transverse wave. However, the same is not the case in a heterogeneous medium, as may be seen from (126) and, according to the heterogeneity of the medium, there are some changes of volume necessarily associated with the "*transverse wave of type A.*"

4. *Transverse wave of type B.* For the transverse wave of type B,

$$u_r = \Delta = 0, \tag{130}$$

whence we may put

$$u_\theta = \frac{1}{\sin\theta}\frac{\partial F}{\partial \varphi}, \quad u_\varphi = -\frac{\partial F}{\partial \theta}, \tag{131}$$

where u_θ and u_φ are the θ- and φ-components of displacement $\boldsymbol{D}$. Substituting (131) into the original equation (2) we get, as would have been naturally expected, an equation of the same type as (125), namely,

$$\rho\frac{\partial^2}{\partial T^2}\sqrt{\mu}F = \mu\nabla^2\sqrt{\mu}F - \left(\sqrt{\mu}\nabla^2\sqrt{\mu} + \frac{1}{r}\frac{d\mu}{dr}\right)\sqrt{\mu}F. \tag{132}$$

Solving (132) or (125) in t, β, φ-coordinates, it is now possible to discuss the wave of type B having its source at any point. Putting $\sqrt{\mu}F = \phi$, $\dfrac{\mu}{\rho} = v^2$, and $\dfrac{1}{\rho}\left(\sqrt{\mu}\,\nabla^2\sqrt{\mu} + \dfrac{1}{r}\dfrac{d\mu}{dr}\right) = a^2$, (132) transforms to an equation of exactly the same type as (117), so that the results of the discussion of ϕ apply equally well to that of $\sqrt{\mu}\,F$, and naturally, also, to that of $\sqrt{\mu}\,r\varpi_r$. Dispersion of the wave increases with a^2, and also with its wave length.

From the θ-, φ-components of displacement given above, the t-, β- and φ-components of displacement can be calculated geometrically as follows,

$$u_t = u_\theta \sin i, \quad u_\beta = u_\theta \cos i,$$

where $g(r)\sin i = g(h)\sin\beta = \kappa$, and $g(r) = \dfrac{r}{v}$. The magnitude of the transverse wave of the first kind and of the second kind, according to Sezawa's definition, were also calculated by similar formulae, and we have, as the relation between ϖ_r, ϖ_θ and ϖ_φ in the transverse wave of type B,

$$2\varpi_r = -\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\varphi^2}\right\},$$

$$\left.\begin{aligned} 2\varpi_\theta &= \frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial F}{\partial \theta}\right\}, \\ 2\varpi_\varphi &= \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left\{r\frac{\partial F}{\partial \varphi}\right\}. \end{aligned}\right\} \tag{133}$$

By putting

$$\frac{\mu}{\rho} = (a - br^2)^2,$$

$$\frac{1}{\rho}\left(\sqrt{\mu}\,\nabla^2\sqrt{\mu} + \frac{1}{r}\frac{d\mu}{dr}\right) = \alpha^2 (\text{const}),$$

the wave will be discussed more concretely as done in Part 2.

If, for example, $\mu = \left(c + \dfrac{d}{r^3}\right)^2$, then

$$F = \frac{1}{\sqrt{\mu}\sqrt{a - br^2}}\sqrt{\frac{2\sqrt{ab}}{\sinh 2\sqrt{ab}\,t}}\,U_n^{(i)}(\sqrt{p^2 - ab}\,t)\,P_n^m(\cos\beta)\,\frac{\sin m\varphi}{\cos m\varphi}\,e^{\pm ipT} \tag{134}$$

5. A special case of the wave problem mentioned above is the Love wave that is propagated over a spherical surface, which has already been discussed by the writer.[2]

6. *Transverse wave of type A and the longitudinal wave.* From the results obtained in Part 2, which, however, concerned a special medium, there is no doubt that so long as either its density or its rigidity is not constant, there are no longitudinal waves propagated through a heterogeneous medium in accordance with the usual conception, that is rot $\boldsymbol{D}=0$, which is established when the medium is homogeneous. A similar conclusion may be drawn for the transverse wave of type A, because div $\boldsymbol{D}$ does not vanish. We should conclude, as already suggested, that, through a heterogeneous medium, there is a wave such that $u_r=0$ and div $\boldsymbol{D}=0$, whose characteristics will be studied by equation (131) and (132), and which has been classified above as the transverse wave of type B, and that there is another type of waves, the law of propagation of which is given by equations (126), (127) and $\varpi_r=0$. When the medium is not very heterogeneous, the latter will be observed as a wave composed of two waves, one of which is that called the longitudinal wave, and the other that called in this paper the transverse wave of type A.

The two equation (126) and (127) suggest a coupled motion of two

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2) R. YOSIYAMA, *Disin*, **10** (1938), 272~276.

systems in dynamics. The derivatives of ρ, λ, and μ may correspond to the coupling factor. When the two systems are tightly coupled, the characteristic mode of motion of each system will not be clearly observed. A similar argument could be used in the present problem, and we should expect theoretically that when the medium is heterogeneous to a certain degree, the phase of a longitudinal wave and that of a transverse wave of type A can be distinguished only with difficulty. Accordingly the observed fact that onsets of these waves, or time of incidence of these waves, in seismograms, are easily identified should be interpreted as suggesting that the inner part of the earth is only slightly heterogeneous.

In this paper, we are solving the equations approximately, assuming first the existence of waves of equivoluminal rotation and that of irrotational dilatation as in the case of a homogeneous medium, attempting to work out the magnitude of dilatation or that of rotation attached to each wave as a small correction.

7. *Transverse wave of type A, and change in volume.* As a first approximation, the transverse wave of type A is discussed with the aid of the following equation, obtained from (127). Putting $\Delta=0$ we get

$$\rho\frac{\partial^2}{\partial T^2}ru_r=\mu\nabla^2 ru_r+2r\frac{d\mu}{dr}\frac{\partial u_r}{\partial r}, \tag{135}$$

with $\Delta=\varpi_r=0$. Putting $\mu ru_r=\phi$, $\frac{\mu}{\rho}=v^2$ and $\frac{1}{\rho}\left(\nabla^2\mu+\frac{2}{r}\frac{d\mu}{dr}\right)=\alpha^2$, the equation transforms to (117), the results thus obtained being applicable to our case. The beginning of this wave is propagated with velocity $\sqrt{\frac{\mu}{\rho}}$. This wave, together with the transverse wave of type B, may form the S waves that are observed in seismograms. The amplitudes of these two waves which must be projected from the origin at the same time, are determined by the particular mechanism of earthquake occurrence. It is of some interest to know that the quantities which stand for α^2 may very likely differ in each case, so that a phase difference should be expected between these two waves, although the effect due to such phase difference, such for example, as the possibility of two peaks appearing in the S phase, or of a prolongation of the apparent period of the S phase, will be too feeble to be observed in practice because of the fact that the earth is only slightly heterogeneous.

We shall now try to find the change in volume in the S wave. Assuming that the period of the wave is $\frac{2\pi}{p}$, the time factor consequently being e^{ipT} in common, and neglecting the second term in the right hand members in order to solve to a first approximation, we get from (126),

$$-\frac{d\rho}{dr}p^2u_r-\rho p^2\Delta=\nabla^2\Big\{(\lambda+2\mu)\Delta\Big\}+\frac{2}{r}\frac{d\mu}{dr}\nabla^2ru_r+2r\frac{d}{dr}\Big(\frac{1}{r}\frac{d\mu}{dr}\Big)\frac{\partial u_r}{\partial r}, \tag{136}$$

and from (135),

$$-\rho p^2ru_r=\mu\nabla^2ru_r. \tag{137}$$

Putting

$$(\lambda+2\mu)\Delta=\varepsilon ru_r+\delta r\frac{\partial}{\partial r}ru_r, \tag{138}$$

in which ε and δ are necessarily some functions of r. However, these, being assumed small, are treated as constant in the following calculation. Substituting (138) in (136), and using formula

$$\frac{\partial}{\partial r}r^2\nabla^2\phi=r\nabla^2r\frac{\partial\phi}{\partial r} \tag{139}$$

we get

$$-\frac{d\rho}{dr}p^2u_r-\frac{\rho p^2}{\lambda+2\mu}\Big\{\varepsilon ru_r+\delta r\frac{\partial}{\partial r}ru_r\Big\}=-\varepsilon\frac{\rho p^2}{\mu}ru_r+\frac{\delta}{r}\frac{\partial}{\partial r}r^2\nabla^2ru_r$$

$$-2\frac{d\mu}{dr}\frac{\rho p^2}{\mu}u_r+2r\frac{d}{dr}\Big(\frac{1}{r}\frac{d\mu}{dr}\Big)\frac{\partial u_r}{\partial r}. \tag{140}$$

Substituting (137) in the second term of the right hand members of the above equation, we get an equation which contains u_r and $\frac{\partial}{\partial r}ru_r$ only. Since each of the factors should then vanish, we have

$$-\frac{d\rho}{dr}p^2-\frac{\rho p^2}{\lambda+2\mu}\varepsilon r=-\varepsilon r\frac{\rho p^2}{\mu}-\delta p^2\frac{d}{dr}\frac{r^2\rho}{\mu}-2\frac{d\mu}{dr}\frac{\rho p^2}{\mu}-2\frac{d}{dr}\Big(\frac{1}{r}\frac{d\mu}{dr}\Big),$$

$$-\frac{\rho p^2}{\lambda+2\mu}\delta=-\frac{\rho p^2}{\mu}\delta+\frac{2}{r}\frac{d}{dr}\Big(\frac{1}{r}\frac{d\mu}{dr}\Big),$$

from which ε and δ are otained as follows:

$$\delta=\frac{2}{r}\frac{(\lambda+2\mu)\mu}{(\lambda+\mu)\rho p^2}\frac{d}{dr}\Big(\frac{1}{r}\frac{d\mu}{dr}\Big), \tag{141}$$

$$\frac{\lambda+\mu}{\lambda+2\mu}\frac{\rho p^2}{\mu}\varepsilon r=\rho p^2\Big\{\frac{1}{\rho}\frac{d\rho}{dr}-\frac{2}{\mu}\frac{d\mu}{dr}\Big\}-\frac{2(3\lambda+5\mu)}{\lambda+\mu}\frac{d}{dr}\Big(\frac{1}{r}\frac{d\mu}{dr}\Big)$$

$$-2r\frac{\lambda+2\mu}{\lambda+\mu}\frac{\mu}{\rho}\frac{d}{dr}\Big(\frac{1}{r}\frac{d\mu}{dr}\Big)\frac{d}{dr}\frac{\rho}{\mu}. \tag{142}$$

Substituting ε and δ thus obtained in (138), we get the expression for div $\boldsymbol{D}$, and substituting it into equation (127), we get the following equation for u_r to a second approximation,

$$-\rho p^2 \mu r u_r = \mu \nabla^2 \mu r u_r - a \mu r u_r - b \frac{\partial}{\partial r} \mu r u_r - c \frac{\partial^2}{\partial r^2} \mu r \varpi_r,$$

where a, b, and c are some functions of p^2, ε, δ, and the derivatives of ρ, λ, μ. It will be seen from (141) that, when p tends to infinity, δ vanishes, and consequently c vanishes. Therefore, notwithstanding the presence of the term $\frac{\partial^2}{\partial r^2} \mu r u_r$ on the right hand side of the above equation, it will be seen that the velocity of propagation of a wave of unlimited short period, or, in consequence, the beginning of a disturbance (the S phase in seismograms), is not affected by the heterogeneity of the medium, being equal to $\sqrt{\frac{\mu}{\rho}}$. Since ε does not vanish even when $p \to \infty$, there is a change in volume at the beginning of the S phase.

Although we have put $\frac{1}{\rho}\left(\nabla^2 \mu + \frac{2}{r}\frac{d\mu}{dr}\right) = \alpha^2$ by way of estimating the order of magnitude of dispersion, it is clear from the results above obtained that $\frac{\partial}{\partial r}(\lambda \Delta)$ and $r^2 \mu \frac{\partial}{\partial r}\left(\frac{\Delta}{r^2}\right)$ are of the same order of magnitude as $\left(\nabla^2 \mu + \frac{2}{r}\frac{d\mu}{dr}\right) u_r$. Now, $\frac{1}{|ru_r|}\left|\frac{\partial (ru_r)}{\partial r}\right|$ is not greater than $\sqrt{\frac{\rho}{\mu}} p$, so that, for a large value of p, $\delta r \frac{\partial}{\partial r}(ru_r)$ is negligible compared with $\varepsilon r u_r$, and putting $(\lambda + 2\mu)\Delta = \varepsilon r u_r$, $\varepsilon r = \frac{\lambda + 2\mu}{\lambda + \mu} \mu \left\{\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{\mu}\frac{d\mu}{dr}\right\}$, we get from (127),

$$\rho \frac{\partial^2}{\partial T^2} r u_r = \mu \nabla^2 r u_r + \frac{\mu}{\rho}\frac{d\rho}{dr}\frac{\partial}{\partial r} r u_r - \alpha'^2 r u_r,$$

$$\alpha'^2 = \frac{2}{r}\frac{d\mu}{dr} - r\frac{d}{dr}\frac{\lambda\varepsilon}{\lambda + 2\mu} - r^3 \mu \frac{d}{dr}\frac{\varepsilon}{r^2(\lambda + 2\mu)},$$

which is easily transformed to the type,

$$\rho \frac{\partial^2}{\partial T^2} \sqrt{\rho}\, r u_r = \mu \nabla^2 \sqrt{\rho}\, r u_r - \alpha^2 \sqrt{\rho}\, r u_r, \tag{143}$$

$$\alpha^2 = \alpha'^2 + \frac{\mu}{\sqrt{\rho}} \nabla^2 \sqrt{\rho}.$$

8. *Longitudinal waves.* It is easier to discuss the longitudinal wave by the following method of mathematical treatment than to trouble with equations (126) and (127).

In the longitudinal wave, rot $\boldsymbol{D}=0$ to a first approximation, and strictly $(\text{rot } \boldsymbol{D})_r=0$, so that we may put

$$u_r=\frac{\partial P}{\partial r}+Q, \quad u_\theta=\frac{1}{r}\frac{\partial P}{\partial \theta}, \quad u_\varphi=\frac{1}{r\sin\theta}\frac{\partial P}{\partial \varphi}, \tag{144}$$

when Q is treated as a small correction. The equation with respect to P and Q are

$$\rho\frac{\partial^2 P}{\partial T^2}=(\lambda+2\mu)\nabla^2 P+2r\frac{d\mu}{dr}\frac{\partial}{\partial r}\frac{P}{r}$$

$$+(\lambda+2\mu)\frac{1}{r^2}\frac{\partial}{\partial r}r^2 Q-\mu\frac{\partial Q}{\partial r}+\frac{d\mu}{dr}Q, \tag{145}$$

$$-\frac{d\rho}{dr}\frac{\partial^2}{\partial T^2}\frac{P}{r}+\rho\frac{\partial^2}{\partial T^2}\frac{Q}{r}=\mu\nabla^2\frac{Q}{r}-\left(\frac{d^2\mu}{dr^2}+\frac{4}{r}\frac{d\mu}{dr}\right)\frac{Q}{r}$$

$$-\frac{2}{r}\frac{d\mu}{dr}\left\{\nabla^2 P-\frac{\partial}{\partial r}\frac{P}{r}\right\}-2\frac{d^2\mu}{dr^2}\frac{\partial}{\partial r}\frac{P}{r} \tag{146}$$

Putting $Q=0$ in (145), and by a slight modification, we get the following equation, from which an approximate solution for the longitudinal wave can be obtained, namely,

$$\rho\frac{\partial^2}{\partial T^2}gP=(\lambda+2\mu)\nabla^2 gP-\left(\frac{2}{r}\frac{d\mu}{dr}+\frac{\lambda+2\mu}{g}\nabla^2 g\right)gP, \tag{147}$$

in which $\dfrac{1}{g}\dfrac{dg}{dr}=\dfrac{1}{\lambda+2\mu}\dfrac{d\mu}{dr}$. If $\lambda=n\mu$, n being constant, $g=\mu^{\frac{1}{2+n}}$, and in this case the wave path, determined with the aid of Fermat's principle, being common with that of transverse waves, the same t, β, φ-coordinates are available.

To get Q as a small correction, we approximate $-\rho p^2 P=(\lambda+2\mu)\nabla^2 P$, the time factor being assumed to be e^{ipT} as in the case of the transverse wave. Neglecting the second term on the right hand side of (146), and substituting the above approximate relation, we get

$$\frac{d\rho}{dr}p^2\frac{P}{r}-\rho p^2\frac{Q}{r}=\mu\nabla^2\frac{Q}{r}+\frac{2}{r}\frac{d\mu}{dr}\frac{\rho p^2}{\lambda+2\mu}P-2r\frac{d}{dr}\left(\frac{1}{r}\frac{d\mu}{dr}\right)\frac{\partial}{\partial r}\frac{P}{r}. \tag{148}$$

Putting,

$$\frac{Q}{r}=\varepsilon' P+\delta' r\frac{\partial P}{\partial r}, \tag{149}$$

into equation (148), we get an equation containing only P and $\frac{\partial P}{\partial r}$, from which, putting each factor zero, we have

$$\delta'=\frac{2}{\rho p^2}\frac{\lambda+2\mu}{\lambda+\mu}\frac{1}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\mu}{dr}\right), \tag{150}$$

$$\frac{\lambda+\mu}{\lambda+2\mu}\rho p^2\varepsilon' r=\rho p^2\left\{\frac{1}{\rho}\frac{d\rho}{dr}-\frac{2}{\lambda+2\mu}\frac{d\mu}{dr}\right\}-\frac{2(\lambda-\mu)}{\lambda+\mu}\frac{d}{dr}\left(\frac{1}{r}\frac{d\mu}{dr}\right)$$

$$+\frac{2(\lambda+2\mu)\mu}{(\lambda+\mu)\rho}r\frac{d}{dr}\left(\frac{1}{r}\frac{d\mu}{dr}\right)\frac{d}{dr}\frac{\rho}{\lambda+2\mu}. \tag{151}$$

When p tends to infinity, δ' vanishes, so that the velocity of propagation of the beginning of the longitudinal wave, or the P phase of seismology, is, as would have been expected, $\sqrt{\frac{\lambda+2\mu}{\rho}}$.

We obtain the following equation for the longitudinal wave to a second approximation after a method of procedure similar to that used in getting (143),

$$\rho\frac{\partial^2}{\partial T^2}\sqrt{\rho}P=(\lambda+2\mu)\nabla^2\sqrt{\rho}P-\alpha^2\sqrt{\rho}P, \tag{152}$$

$$\alpha^2=\frac{2}{r}\frac{d\mu}{dr}-\varepsilon' r\frac{d\mu}{dr}+\mu\frac{d(\varepsilon' r)}{dr}-\frac{\lambda+2\mu}{r^2}\frac{d}{dr}(\varepsilon' r^3)+\frac{\lambda+2\mu}{\sqrt{\rho}}\nabla^2\sqrt{\rho}.$$

We get the r-, θ- and the φ-component of rot $\boldsymbol{D}$, associated with the longitudinal wave, and consequently propagated with velocity $\sqrt{\frac{\lambda+2\mu}{\rho}}$, as follows,

$$\left.\begin{aligned}
2\varpi_r&=0\\
2\varpi_\theta&=\frac{1}{r\sin\theta}\frac{\partial Q}{\partial\varphi}=\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\left\{\varepsilon' P+\delta' r\frac{\partial P}{\partial r}\right\},\\
2\varpi_\varphi&=-\frac{1}{r}\frac{\partial Q}{\partial\theta}=-\frac{\partial}{\partial\theta}\left\{\varepsilon' P+\delta' r\frac{\partial P}{\partial r}\right\},
\end{aligned}\right\} \tag{153}$$

and we may also put, approximately,

$$\Delta \doteqdot \nabla^2 P \doteqdot -\frac{\rho p^2}{\lambda+2\mu} P. \tag{154}$$

These seem to be the essential points of the longitudinal wave through a heterogeneous medium.

If we put $\mu=0$,

$$\alpha^2 = \frac{\lambda}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} - \frac{\lambda}{r^2} \frac{d}{dr}\left(\frac{r^2}{\rho} \frac{d\rho}{dr}\right) = \lambda \sqrt{\rho}\, \nabla^2 \frac{1}{\sqrt{\rho}},$$

equation (152) then becomes

$$\rho \frac{\partial^2}{\partial T^2} \sqrt{\rho}\, P = \lambda \nabla^2 \sqrt{\rho}\, P - \left(\lambda \sqrt{\rho}\, \nabla^2 \frac{1}{\sqrt{\rho}}\right) \sqrt{\rho}\, P$$

and

$$(D_2)_r = \frac{\partial P}{\partial r} + \frac{1}{\rho} \frac{d\rho}{dr} P = \frac{1}{\rho} \frac{\partial}{\partial r} \rho P,$$

$$(D_2)_\theta = \frac{1}{r} \frac{\partial P}{\partial \theta} = \frac{1}{\rho} \frac{1}{r} \frac{\partial}{\partial \theta} \rho P.$$

These are exactly the results obtained in Part 2.

9. *Transverse waves of type A.* Although this wave has been dealt with in the previous paragraph, it will be considered again. Because of complexity of procedure, we have not yet given the magnitude of rot $\boldsymbol{D}$. It is clear that substitution (144) can be used, although neither P nor Q in this case is a small correction, and both of these, the magnitudes of which are obtained by the following calculation, are propagated with velocity $\sqrt{\frac{\mu}{\rho}}$, while those in the last paragraph are propagated with velocity $\sqrt{\frac{\lambda+2\mu}{\rho}}$. For convenience in discriminating P, Q, Δ and other quantites in the transverse wave of type A from those in the longitudinal wave, we shall affix the notation "$'$" to those in the transverse wave of type A, for example, P', Q', Δ' and so on.

To a first approximation, we can easily obtain from (145) and (146), neglecting the terms of the derivatives of ρ and μ, and putting

$$\Delta' = \nabla^2 P' + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 Q') = 0,$$

the equation

$$P' = \frac{\mu}{\rho p^2} \frac{\partial Q'}{\partial r}. \tag{155}$$

For the components of rot $\boldsymbol{D}$ we have

$$\left.\begin{aligned} 2\varpi_r &= 0, \\ 2\varpi_\theta &= \frac{1}{r\sin\theta} \frac{\partial Q'}{\partial \varphi}, \\ 2\varpi_\varphi &= -\frac{1}{r} \frac{\partial Q'}{\partial \theta}. \end{aligned}\right\} \tag{156}$$

10. *Recapitulation of the foregoing results, and remarks.* Through a heterogeneous medium, there are propagated two energy fronts of waves, one advancing with velocity $\sqrt{\frac{\lambda+2\mu}{\rho}}$ and the other with velocity $\sqrt{\frac{\mu}{\rho}}$, while we have three different phase fronts of waves. These three kinds of waves we call the "longitudinal wave," the "transverse wave of type A" and the "transverse wave of type B." The characteristics of dispersion of the transverse wave of type B are simple, while those of the other two are very complicated, with the result that all we got was a rough approximation. At any rate, owing to the characteristic dispersion of each of these waves, there are certain reductions of maximum amplitudes, the appearance of characteristic, damped oscillation, called the "tail"; and possibilities of the appearance of two peaks in the S phase in seismograms. If observations of these effects of dispersion are possible, they will be useful, with the aid of the results of investigations of earthquake mechanism, in studying the heterogeneity of ρ, λ, and μ respectively, whereas analyses of travel-time curves are of help only in the study of the heterogeneity of $\sqrt{\frac{\lambda+2\mu}{\rho}}$ or $\sqrt{\frac{\mu}{\rho}}$.

Displacement $\boldsymbol{D}$ due to the wave motion, generally speaking, may be expressed as a vector sum of those due to the three kinds of waves, namely,

$$\boldsymbol{D} = \boldsymbol{D}_1(\sqrt{p^2-\alpha_1^2}\, t) e^{ipT} + \boldsymbol{D}_2(\sqrt{p^2-\alpha_2^2}\, t) e^{ipT} + \boldsymbol{D}_3(\sqrt{p^2-\alpha_3^2}\, t') e^{ipT} \tag{157}$$

in which t is calculated by putting $v=\sqrt{\frac{\mu}{\rho}}$, while t' is calculated by putting $v=\sqrt{\frac{\lambda+2\mu}{\rho}}$, and

$\boldsymbol{D}_1$ is due to the transverse wave of type B,

$\boldsymbol{D}_2$ 〃 〃 of type A,

$\boldsymbol{D}_3$ 〃 the longitudinal wave.

Obviously, the r-component of $\boldsymbol{D}_1$ is nil, and the beginning of the former two are propagated with velocity $\sqrt{\frac{\mu}{\rho}}$, while that of the last is propagated with velocity $\sqrt{\frac{\lambda+2\mu}{\rho}}$. These are affected by some dispersion characteristic of each of the three waves just mentioned.

We shall now consider some illustrative problems with the hope of throwing some fresh light on the investigation of earthquake mechanism, and consider the following two cases,

Case 1. $\operatorname{div} \boldsymbol{D}=0$, $\operatorname{rot} \boldsymbol{D}\neq 0$

Case 2. $\operatorname{div} \boldsymbol{D}\neq 0$, $\operatorname{rot} \boldsymbol{D}=0$.

on the surface, $t=t_0$.

When dealing with a homogeneous medium, we know that in case 1, transverse waves spread out from the surface with velocity $\sqrt{\frac{\mu}{\rho}}$, there being no (longitudinal) waves that are propagated with velocity $\sqrt{\frac{\lambda+2\mu}{\rho}}$, which is quite in contrast with case 2, in which only longitudinal waves of velocity $\sqrt{\frac{\lambda+2\mu}{\rho}}$ spread out, there being no (transverse) waves of velocity $\sqrt{\frac{\mu}{\rho}}$.

Owing to the property of the transverse wave of type B, as given by (130), $\operatorname{div} \boldsymbol{D}_1=0$, and from (138) $\operatorname{div} \boldsymbol{D}_2=\frac{1}{\lambda+2\mu}\left\{\varepsilon+\delta r\frac{\partial}{\partial r}\right\}(r\boldsymbol{D}_2)_r$, whence $\operatorname{div} \boldsymbol{D}=0$ gives

$$\operatorname{div} \boldsymbol{D}_3=-\frac{1}{\lambda+2\mu}\left\{\varepsilon+\delta r\frac{\partial}{\partial r}\right\}(r\boldsymbol{D}_2)_r \tag{158}$$

on the surface $t=t_0$, where $(\boldsymbol{D}_2)_r$ represents the r-component of $\boldsymbol{D}_2$, ε and δ being given by (141) and (142). Since, from (144) and (154), $\boldsymbol{D}_3=\operatorname{grad} P$, approximately, where $P=-\frac{\lambda+2\mu}{\rho p^2}\operatorname{div} \boldsymbol{D}_3$,

$$\boldsymbol{D}_3=\operatorname{grad}\frac{1}{\rho p^2}\left\{\varepsilon+\delta r\frac{\partial}{\partial r}\right\}(r\boldsymbol{D}_2)_r \tag{159}$$

We shall suppose an earthquake mechanism such that a small rigid ball performs a rotational oscillation at the origin. When the axis of rotation coincides with the direction of $\boldsymbol{r}$, since there are no r-components of displacement and no change of volume, no waves other than transverse wave of type B, represented by $\boldsymbol{D}_1 e^{ipT}$ and such that $(\boldsymbol{D}_1)_r = \operatorname{div} \boldsymbol{D}_1 = 0$, will be generated and propagated. However, when the axis does not coincide with the direction of $\boldsymbol{r}$, the r-component of displacement necessarily exists, so that, although there are no changes in volume, there will be longitudinal waves although their magnitudes may be small, as given by (158) and (159). Then if the medium is heterogeneneous, but not otherwise, we shall observe waves of velocity $\sqrt{\dfrac{\lambda+2\mu}{\rho}}$ from a nondilatational origin.

Now, as to any function ϕ, representing a wave motion propagated with velocity v, approximately $\dfrac{1}{\phi}\operatorname{grad}\phi = \dfrac{p}{v}\cos\omega$ for a large value of p, in which ω is an angle between the normal to the wave front and the direction to which the gradient of ϕ is taken. Therefore, for a wave of short period, we can put, neglecting the term of δ compared with that of ε,

$$\boldsymbol{D}_3 = \operatorname{grad} \frac{(\lambda+2\mu)\mu}{(\lambda+\mu)\rho p^2}\left\{\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{\mu}\frac{d\mu}{dr}\right\}(\boldsymbol{D}_2)_r, \tag{160}$$

or, further, we see that $\dfrac{\boldsymbol{D}_3}{\boldsymbol{D}_2}$ decreases, approximately inversely to p, as p increases. Consequently, considering again the above illustrative example, the amplitude of the longitudinal wave decreases as the frequency of the rotational oscillation of the rigid ball increases.

In order to study the case 2, we perform the calculation for the operation of rotation on both members of (157). We get easily, from (133), (153) and (156),

$$\left.\begin{aligned}
2\varpi_r &= -\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\varphi^2}\right\},\\
2\varpi_\theta &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial Q'}{\partial\varphi} + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\left\{\varepsilon' P + \delta' r\frac{\partial P}{\partial r}\right\},\\
2\varpi_\varphi &= \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial\varphi}\right) - \frac{1}{r}\frac{\partial Q'}{\partial\theta} - \frac{\partial}{\partial\theta}\left\{\varepsilon' P + \delta' r\frac{\partial P}{\partial r}\right\}.
\end{aligned}\right\} \tag{161}$$

Consequently rot $\boldsymbol{D}=0$ gives,

$$F=0,$$

$$\frac{Q'}{r}=-\left\{\varepsilon' P+\delta' r\frac{\partial P}{\partial r}\right\}, \tag{162}$$

in which, from (144) and (154), approximately,

$$\boldsymbol{D}_3=\text{grad } P, \quad P=-\frac{\lambda+2\mu}{\rho p^2}\text{div } \boldsymbol{D}_3 \tag{163}$$

From (155) and (162),

$$P'=\frac{\mu}{\rho p^2}\frac{\partial Q'}{\partial r}=-\frac{\mu}{\rho p^2}\frac{\partial}{\partial r}r\left\{\varepsilon' P+\delta' r\frac{\partial P}{\partial r}\right\} \tag{164}$$

$F=0$ means that there are no transverse wave of type B. The displacement owing to the transverse wave of type A can be calculated from P' and Q' by a formula that is exactly the same as (144) as already stated. Thus it has been shown that in a heterogeneous medium, even from a pure dilatational or condensational origin, we should expect not only longitudinal waves, but also transverse waves of type A, which are propagated at the beginning with velocity $\sqrt{\frac{\mu}{\rho}}$, according to the following equation (approximate):

$$-\rho p^2\frac{Q'}{r}=\mu\nabla^2\frac{Q'}{r}. \tag{165}$$

From (162) and (164), we get, for example, the vertical and horizontal components of the amplitude of displacement due to such transverse wave, for large values of p, as follows;

$$\left.\begin{aligned}
(\boldsymbol{D}_2)_r&=Q'+\frac{\mu}{\rho p^2}\frac{\partial^2 Q'}{\partial r^2}=-\left\{1+\frac{\mu}{\rho p^2}\frac{\partial^2}{\partial r^2}\right\}\\
&\qquad\cdot\frac{\lambda+2\mu}{\lambda+\mu}\left(\frac{1}{\rho}\frac{d\rho}{dr}-\frac{2}{\lambda+2\mu}\frac{d\mu}{dr}\right)P\\
(\boldsymbol{D}_2)_\theta&=(\text{grad } P')_\theta=-\frac{1}{r}\frac{\mu}{\rho p^2}\frac{\partial}{\partial r}\left\{\frac{\lambda+2\mu}{\lambda+\mu}\left(\frac{1}{\rho}\frac{d\rho}{dr}-\frac{2}{\lambda+2\mu}\frac{d\mu}{dr}\right)\frac{\partial P}{\partial\theta}\right\}\\
&=-\frac{1}{r}\frac{\mu}{\rho p^2}\frac{\partial}{\partial r}r\frac{\lambda+2\mu}{\lambda+\mu}\left(\frac{1}{\rho}\frac{d\rho}{dr}-\frac{2}{\lambda+2\mu}\frac{d\mu}{dr}\right)(\boldsymbol{D}_3)_\theta,
\end{aligned}\right\} \tag{166}$$

$$(D_2)_\varphi = (\text{grad } P')_\varphi = -\frac{1}{r}\frac{\mu}{\rho p^2}\frac{\partial}{\partial r}r\frac{\lambda+2\mu}{\lambda+\mu}\left(\frac{1}{\rho}\frac{d\rho}{dr}-\frac{2}{\lambda+2\mu}\frac{d\mu}{dr}\right)(D_3)_\varphi,$$

on the surface $t=t_0$. The operator of the "grad" is equivalent to the factor $\sqrt{\frac{\rho}{\mu}}p\cos\omega$, so that $\frac{D_2}{D_3}$ decreases inversely as p increases. When $\frac{\partial P}{\partial\theta}=\frac{\partial P}{\partial\varphi}=0$, and the wave is propagated in the direction of r, $-\rho p^2 Q' =\mu\frac{\partial^2 Q'}{\partial r^2}$ from (165), consequently no transverse wave is generated.

It was shown above that the amplitude of the longitudinal and the transverse wave, due to the heterogeneity of the medium, from the transverse and longitudinal wave respectively diminishes with shortening of the period. Needless to say, it should be remembered that so far as observations far from the surface $t=t_0$ are concerned, the attenuation factor of D_2 in (166), for example, differs from that of D_3.

The structure of the upper layer of the earth-crust is so complicated that application of the results of study outlined above to practical problems in seismology is clearly inadequate for their solution. However, in the case of a deep-seated earthquake, some application will be found in the interpretation of seismograms.

For several years past, the mechanism of earthquakes at the origin have been investigated by a number of workers from the amplitudes of the P phase or the S phase, and many reasonable conclusions have been drawn, especially for deepseated earthquakes. According to these investigations, if the earth is a homogeneous elastic medium, there ought to be "Nodal lines" of the P phase. There should also be "Nodal points" where only the P phase is observed, and no S phase. As a matter of fact, many seismograms have been recorded, clearly proving the existence of the "Nodal lines," and seismograms that virtually give the "Nodal points" are also recorded, although naturally less frequently.

However, upon closer investigations, owing to the heterogeneity of the earth, these conclusions regarding the necessity for Nodal lines and points, naturally seem to leave something to be desired.

From (159) and (152), we may calculate the amplitude of the P phase that will be observed on the "Nodal lines," and from (164), (165), and (166), we may calculate the amplitude of the S phase that is expected at the "Nodal points." From the foregoing discussion, the "Nodal lines" and the "Nodal points" should be clearly identifiable for a wave of short period, and, by close observations of waves of longer period, as also by observations of the effects of dispersion of the

P phase or the S phase, there are possibilities of estimating the order of heterogeneity of the earth at dephth with respect to ρ, λ and μ.

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14. 等方不均一球内の一點より起る彈性波（第3報）

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密度及び彈性常數の變り方が小さいとして彈性波の性質を調べて見た．波はその性質に從つて三種類に分類され，各々近似的に第2報で調べたと同樣な形の方程式を滿足する．三種類の波を假りに A型の横波，B型の横波，及び縱波と呼ぶことにする．不連續面に於ける反射屈折を調べる際横波を SH 波と SV 波に分類して考へると便利であるのと同じ事である．B型の横波は全く獨立に一つの方程式を滿足する．他の二つの波は互に或る關聯を有つてゐるので地震の發震機構の問題に於いて云ふ所の節線或は節點は媒質が均質である場合と樣子が異つて來る．然し波の週期が短ければ短い程不均質性の影響は小さくなる．
