

22. *Relation between the Gravity Anomalies and the Corresponding Subterranean Mass Distribution.* (V)

Isostatic Anomalies and the Undulation of the Isostatic Geoid in the United States of America.

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In the writer's previous paper bearing this title¹, the geographical distribution of the Bouguer gravity anomalies observed in the United States of America was discussed with particular reference to isostatic conditions of the earth's crust in this part of the world as indicated by that distribution. In that study, the double Fourier series method which the writer² had proposed was resorted to, and it was found, as we had expected, to greatly reduce the labour of numerical computation.

It was also found that the isostatic earth's crust with zero elevation (in Airy's sense) has here a thickness of about 56 km and that loads of topographies smaller than 145 km across are supported by some mechanism other than isostasy. On the other hand, it is easy to prove³ that if the thickness of the earth's crust for the latter case is taken as d , the gravity anomalies computed after Pratt's isostasy with depth of compensation $2d$ are in the first approximation identical with those computed after Airy's isostasy. The value of 56 km derived here on the basis of Airy's isostasy therefore agrees almost exactly with the value of 113.7 km, the well-known depth of compensation in the United States. Although neither of the conclusions drawn in the previous paper is novel, what is important is that they could be reached directly with the aid of this new method by comparing the Fourier coefficients for the topographical elevations in the area in question with those for the observed Bouguer gravity anomalies in this area without

1) C. TSUBOI, T. KANEKO, S. MIYAMURA and T. YABASI, *Bull. Earthq. Res. Inst.*, **17** (1939), 385.

2) C. TSUBOI and T. FUCHIDA, *Bull. Earthq. Res. Inst.*, **15** (1937), 636; **16** (1938), 273. C. TSUBOI, *ibid.*, **15** (1937), 650; **17** (1939), 351; *Proc. Imp. Acad. Tokyo*, **14** (1938), 170.

3) H. JEFFREYS, *Gerl. Beitr. z. Geophys.*, **15** (1926), 167. C. TSUBOI, *Disin*, **10** (1938), 109.

resorting to the customary trial and error method which, notwithstanding its complexities, had to be used in studies of this kind.

In the present paper, this double Fourier series method will further be used in computing the geographical distribution of the isostatic anomalies in the United States and next, with that as basis, the undulation of the isostatic geoid. It is hoped that by doing so it will be possible to obtain a connection between the two quantities, gravity anomalies and deflections of the vertical which, notwithstanding the close relation that exists between the two, have so far been measured and discussed independently. It has been shown by C. G. Stokes that if the gravity anomalies were known for the whole earth, the undulation of the geoid, and consequently the deflections of the vertical could be derived from them. Unfortunately the gravimetric points in the world that have been occupied so far are too scanty for actual application of Stokes' method, and the writer is aware of no successful attempt having been made to connect gravity anomalies with deflections of the vertical observed in the same area.

Let the topographical elevation $H(xy)$ and the Bouguer gravity anomalies $\Delta g''(xy)$ in a rectangular area to be studied be expressed by double Fourier series, such as

$$\left. \begin{aligned} H(xy) &= \sum_m \sum_n H_{mn} \frac{\cos mx}{\sin} \frac{\cos ny}{\sin}, \\ \Delta g''(xy) &= \sum_m \sum_n B_{mn} \frac{\cos mx}{\sin} \frac{\cos ny}{\sin}. \end{aligned} \right\} \quad (1)$$

If ρ is the density of the material of which the earth's crust is composed and d is the thickness of that crust, Airy's isostasy implies that there is at depth d below the earth's surface a distribution of mass which is expressed by

$$M(xy) = -\rho \sum_m \sum_n H_{mn} \frac{\cos mx}{\sin} \frac{\cos ny}{\sin}. \quad (2)$$

Here d is not the thickness of the earth's crust with zero elevation, but of an actual crust having an average elevation within the area, of say H_{co} . These thicknesses differ by amount

$$\frac{\rho}{\rho' - \rho} H_{co},$$

where ρ' is the density of the material that is supposed to underlie the earth's crust.

The attraction due to the mass (2) is given by

$$\Delta g''(xy) = -2\pi k^2 \rho \sum_m \sum_n H_{mn} \exp(-\sqrt{m^2 + n^2}d) \frac{\cos mx}{\sin} \frac{\cos ny}{\sin}, \quad (3)$$

and this will be seen to be the Bouguer anomalies on the earth's surface, provided that isostasy is perfect.

Since the isostatic anomaly is nothing else but the difference between the Bouguer anomalies that are computed according to the expression (3) and those that are actually observed and expressed by (1), it is given by

$$\Delta g_{is} = \sum_m \sum_n \{B_{mn} + 2\pi k^2 \rho H_{mn} \exp(-\sqrt{m^2 + n^2}d)\} \frac{\cos mx}{\sin} \frac{\cos ny}{\sin}, \quad (4)$$

which is easy to evaluate if B_{mn} and H_{mn} are known.

In the present case of the United States, that area approximately 4000 km \times 2000 km, bounded by two latitudes $\varphi=30^\circ\text{N}$ and 48°N and two longitudes $\lambda=77^\circ\text{W}$ and 125°W , was regarded as a plane rectangle, and the double Fourier coefficients for the topographical elevation H_{mn} and those for the Bouguer anomalies B_{mn} within this area were already computed in the paper just referred to. In determining B_{mn} , use was made of the Bouguer anomalies at 925 gravimetric points in the United States, as published by the U. S. Coast and Geodetic Survey⁴⁾. The coefficients, both for H and $\Delta g_o''$, which are down to the 18-18th order, amount to 1296 in number. With these H_{mn} 's and B_{mn} 's, and taking $\rho=2.7$ and $d=0.096$, the Fourier coefficients for the isostatic anomalies, namely,

$$B_{mn} + 2\pi k^2 \rho H_{mn} \exp(-\sqrt{m^2 + n^2}d)$$

were computed for every combination of m and n down to the 18-18th order. The value of d , which is 0.096, corresponds to 61 km in length, since

$$\frac{4000 \text{ km}}{2\pi} \times 0.096 = 61 \text{ km}.$$

The coefficients obtained are given in Table I.

With these coefficients, the series

$$\Delta g_{is} = \sum_m \sum_n \{B_{mn} + 2\pi k^2 \rho H_{mn} \exp(-\sqrt{m^2 + n^2}d)\} \frac{\cos mx}{\sin} \frac{\cos ny}{\sin} \quad (4)$$

were then evaluated for $36 \times 36 = 1296$ points at every 10° interval of x and y . On the basis of these values, the contour lines of the isostatic anomalies were drawn, as shown in Fig. 1.

As already mentioned, the coefficients B_{mn} used in these calculations are those that were determined by means of double Fourier analysis of the distribution of the Bouguer anomalies, in doing which it was necessary that the anomalies at $36 \times 36 = 1296$ points within the

4) U. S. Coast and Geodetic Survey, Principal Facts for Gravity Stations in the United States, Part 1-4, (1934-1938).

Table I D. Fourier Coefficients for the Isostatic Anomalies (mgal). sin-sin.

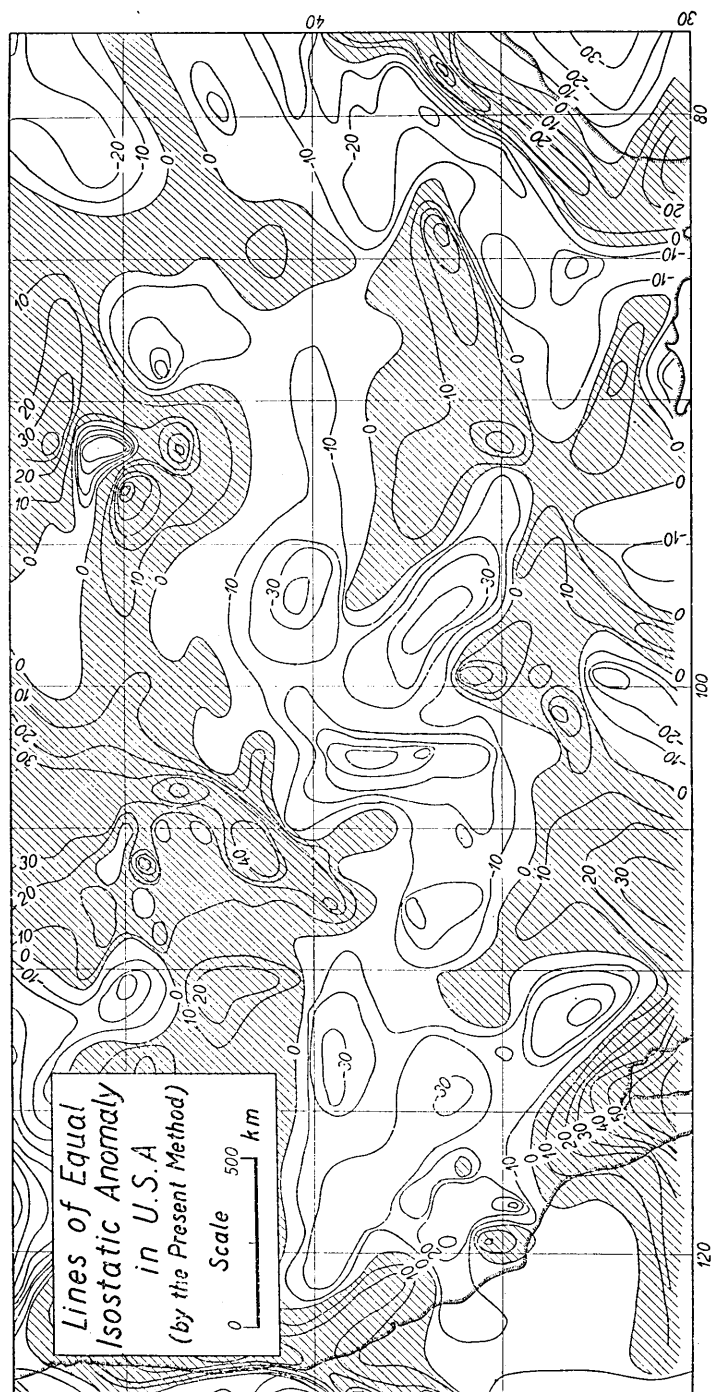


Fig. 1. Isostatic Anomalies in U.S.A (mgal) as calculated by the Present Method.

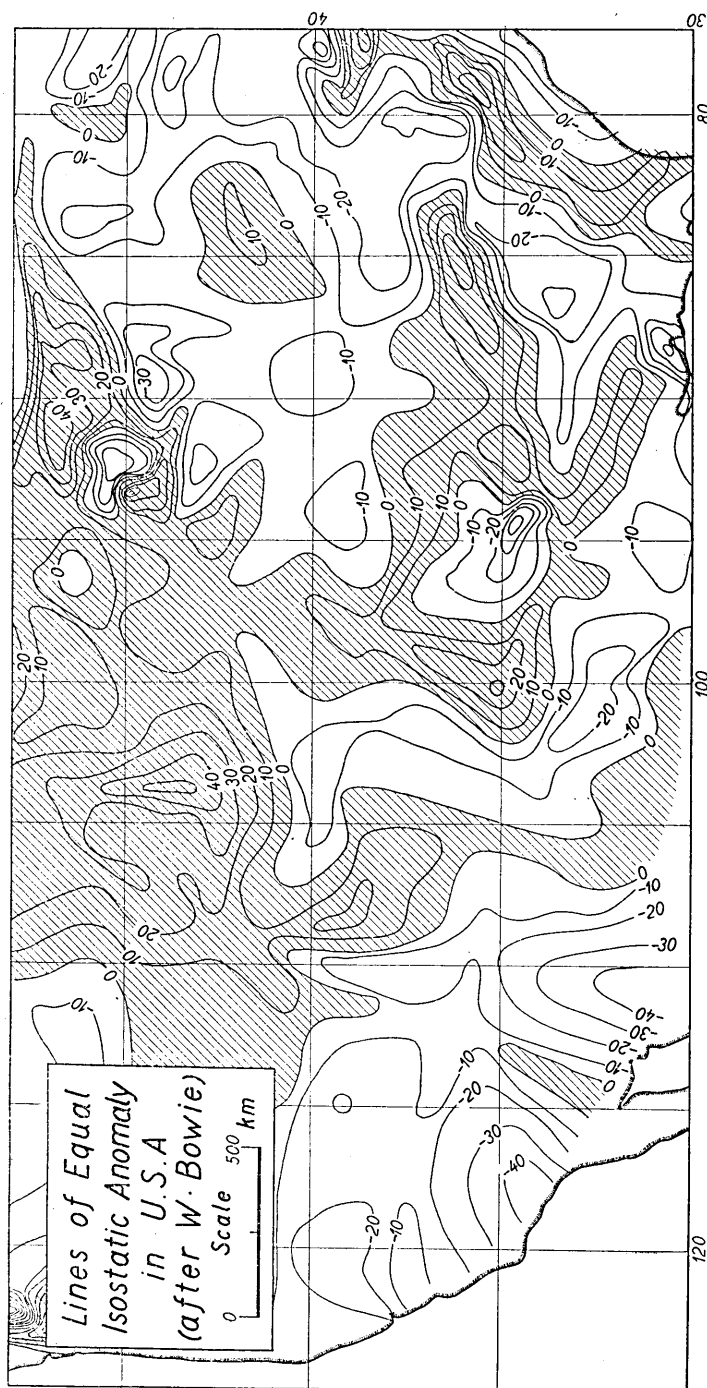


Fig. 2. Isostatic Anomalies in U.S.A (mgal) as calculated by W. Bowie.

area in question shall be interpolated from those at the rather unevenly distributed 925 gravimetric points. For this reason, much accuracy cannot be claimed for B_{mn} , and consequently for the isostatic anomalies derived from them. Notwithstanding this fact, the contour lines shown in Fig. 1 closely resemble those published by W. Bowie⁵⁾, which are reproduced for comparison in Fig. 2. In view of the fact that the latter contour lines are based also on the anomalies at rather scanty and unevenly distributed gravimetric points, and consequently not very accurate, the agreement of the two systems of contour lines must be regarded as very satisfactory.

On the other hand, the Fourier coefficients H_{mn} for the topographical elevation are more reliable than B_{mn} , seeing that the elevation throughout the whole of the United States is known with sufficient accuracy for the present purpose. The difference in the Bouguer anomalies, that is, the one actually observed at the point (xy) and the other computed for that point according to

$$\Delta g_0'' = -2\pi k^2 \rho \sum_m \sum_n H_{mn} \exp(-\sqrt{m^2 + n^2}d) \frac{\cos mx}{\sin} \frac{\cos ny}{\sin}, \quad (3)$$

will therefore give the isostatic anomaly at (xy) , which is more accurate than that computed according to (4). But none of the gravimetric points actually occupied exactly agrees in position with any of the 1296 points for which the Bouguer anomalies were computed according to (3), which reason made it necessary for us to compare the observed Bouguer anomaly at a gravimetric point with that at one of the 1296 points nearest that point. Since the distance between the two points of comparison does not exceed 60 km, the difference between these two Bouguer anomalies will not differ much from the isostatic anomaly at the point (xy) . Where several neighbouring gravimetric points have a common point for comparison, the average of the observed Bouguer anomalies was compared with the anomaly computed for that point. The isostatic anomaly at (xy) obtained in this way may be compared with the same anomaly at this point, as given in the publications of the U. S. Coast and Geodetic Survey.

As an example to illustrate the comparison, the Bouguer and isostatic anomalies at the gravimetric point No. 768 ($\varphi = 32^\circ 16' \text{ N}$, $\lambda = 107^\circ 46' \text{ W}$) are given respectively as -166 mgal and -16 mgal . On the other hand, the point ($\varphi = 32^\circ 30' \text{ N}$, $\lambda = 107^\circ 40' \text{ W}$) is one of the 1296 points for which the Bouguer anomalies were computed according to (3) and is nearest the gravimetric point No. 768. The Bouguer anomaly at this point, computed by the present method assuming perfect isostasy,

5) W. BOWIE, Spec. Publ. No. 40, U. S. Coast and Geodetic Survey, (1917).

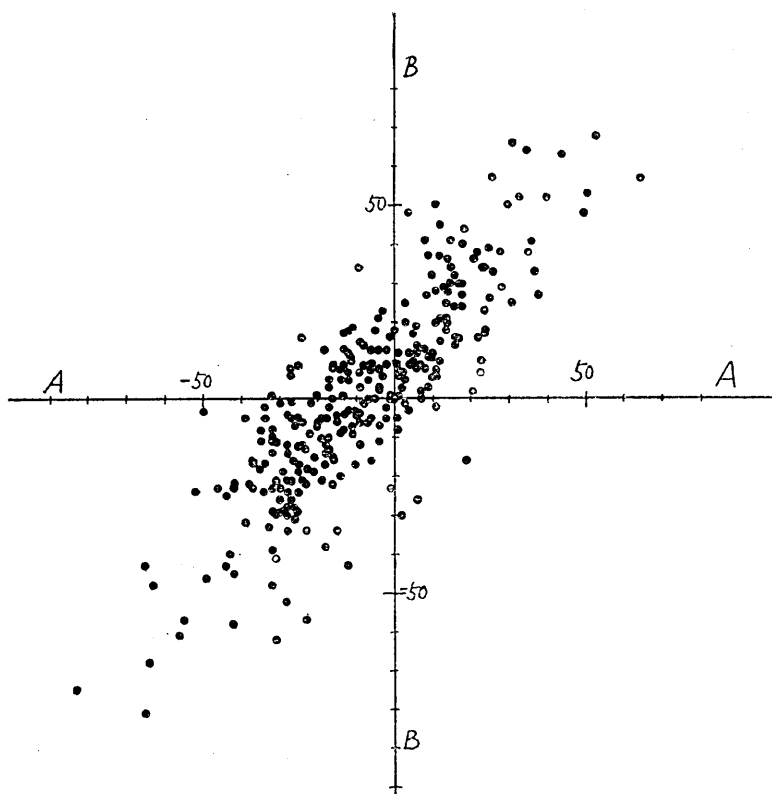


Fig. 3. Correlation of the Isostatic Anomalies (mgal).

A....According to U. S. Coast and Geodetic Survey.

B....According to the Present Method.

is -151 mgal. The difference of -166 mgal and -151 mgal, which is -15 mgal, agrees well with -16 mgal, which is the given isostatic anomaly for this point. Fig. 3 shows the correlation between the two isostatic anomalies, the one computed by the present method and the other according to the U. S. Coast and Geodetic Survey.

Most of the differences in the two isostatic anomalies are less than 20 mgal, as Table II shows. At several points, the differences amount to a few tens of a milligal. As these points are found to be situated where the spatial variation in Bouguer anomaly is large, these large differences are regarded as being due mainly to the fact that the points of comparison are not the same.

Now that the distribution of the isostatic anomalies has been determined, it is not difficult to get an approximate idea of the undulation of the isostatic geoid in the area concerned. The potential theory⁶⁾ shows that if the gravity anomaly is given by

6) *Loc. cit.* (1).

Table II.

Diff.	Number	Diff.	Number
39—30	6	—10—20	72
29—20	5	—21—30	45
19—10	11	—31—40	15
9—0	46	—41—50	3
—1—10	96	—51—60	1

$$\Delta g(xy) = \sum_m \sum_n G_{mn} \frac{\cos mx}{\sin} \frac{\cos ny}{\sin}, \quad (5)$$

the undulation of the corresponding geoid is given by

$$h(xy) = \sum_m \sum_n \frac{1}{\sqrt{m^2 + n^2}} G_{mn} \frac{\cos mx}{\sin} \frac{\cos ny}{\sin}. \quad (6)$$

In the present case, since

$$G_{mn} = B_{mn} + 2\pi k^2 \rho \exp(-\sqrt{m^2 + n^2} d) H_{mn}, \quad (7)$$

the undulation is given by

$$h(xy) = \sum_m \sum_n \frac{1}{\sqrt{m^2 + n^2}} \{B_{mn} + 2\pi k^2 \rho \exp(-\sqrt{m^2 + n^2} d) H_{mn}\} \frac{\cos mx}{\sin} \frac{\cos ny}{\sin}, \quad (8)$$

which is easily obtained. Table III gives the coefficients of the series. With these coefficients, the series (8) was evaluated also for $36 \times 36 = 1296$ points at every 10° interval of x and y . The undulation of the geoid obtained in this way is given by mere number, expressed in radians. In order to express it in length, it should be multiplied by $L/2\pi$, where L is the length that was taken as 2π in the Fourier analysis which, in this case, is 4000 km. The contour lines of the elevation of the geoid are shown in Fig. 4.

The maximum elevation and depression of the isostatic geoid are respectively 6 m and 8 m. It is natural for the geoid to be elevated where the gravity anomaly is widely positive and depressed where it is widely negative. Usually the undulation of the geoid is measured from the niveau spheroid, which corresponds to the normal gravity formula used for the purpose of deriving the gravity anomalies. In the present case, from the nature of the theory on which the above calculations are based, it is evident that the undulation of the geoid should be measured from a geometrical plane.

The deflection of the vertical is the inclination of the geoid. In order to calculate the deflection of this vertical from the elevation of

Table III B. Fourier Coefficients for the Undulation of the Isostatic Geoid (10^{-6} radian). cos-sin.

Table III D. Fourier Coefficients for the Undulation of the Isostatic Geoid (10^{-6} radian). $\sin - \sin$.

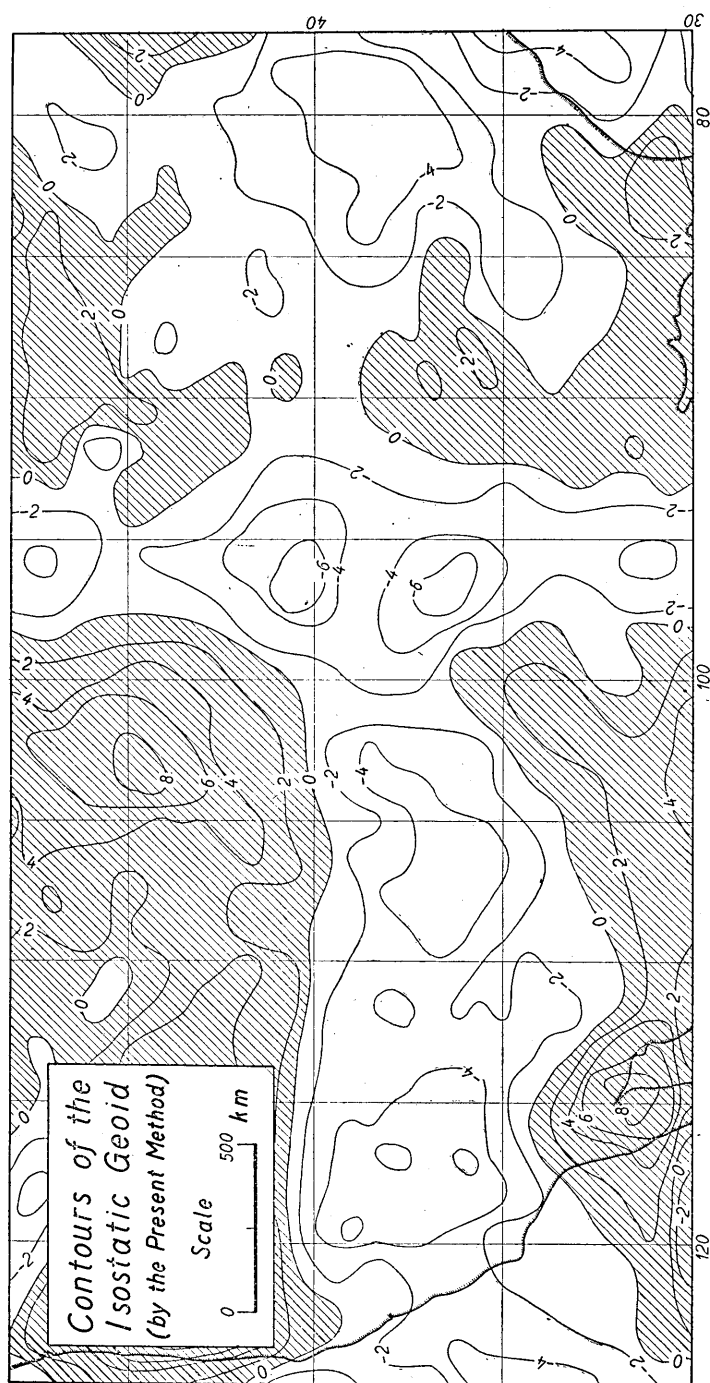


Fig. 4. Contour Lines of the Isostatic Geoid in U.S.A. (m.) as calculated by the Present Method.

the geoid obtained, the following method was used. Take a small rectangle which is formed by four adjacent points from the 1296 points, and let the elevation of the geoid at its four corners, which were computed according to (8), be respectively h_1 , h_2 , h_3 and h_4 .

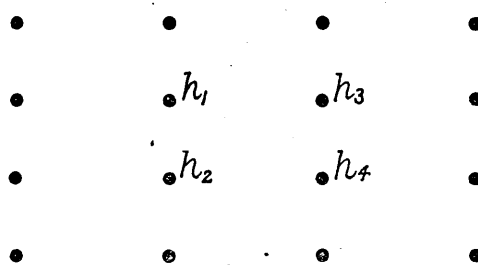


Fig. 5.

Then the east- and northward components of the deflection of the vertical at the centre of the rectangle are respectively

$$\left. \begin{aligned} \xi &= \frac{1}{2} \{ (h_3 - h_1) + (h_4 - h_2) \} \\ \eta &= \frac{1}{2} \{ (h_1 - h_2) + (h_3 - h_4) \}. \end{aligned} \right\} \quad (9)$$

Fig. 6 shows the deflections of the vertical as calculated by this method, each arrow indicating the direction in which the nadir end of the vertical is pointed. It will be seen that the deflection of the vertical rarely exceeds 5 seconds of arc.

On the other hand, J. Hayford⁷⁾ in his classical work on isostasy, gave the deflections of the vertical in the United States, corrected for the Pratt 113.7 km isostasy. For the reason already given, these deflections may be compared with those computed in this paper on the assumption of the Airy 56 km isostasy. The deflections of the vertical as given by Hayford and the undulation of the isostatic geoid as calculated in the present paper from the data of gravimetry are entered on the same map sheet, Fig. 7, an inspection of which figure will show that they are concordant.

Thus a link could be established here between the deflections of the vertical and the gravity anomalies that were observed quite independently and which, so far as the writer is aware, could not be connected by means of any other previous method.

According to W. Heiskanen⁸⁾, in his report on isostasy, it is not possible to give an accurate undulation of the geoid by the present method. While it is true that this method cannot be used if very high accuracy is required, the writer does not believe that this drawback will entirely nullify its other advantages.

7) J. HAYFORD, *The Figure of the Earth and Isostasy from Measurements in the United States*, U. S. Coast and Geodetic Survey, (1909); *Supplementary Investigation in 1909 of the Figure of the Earth and Isostasy*, *ibid.*, (1910).

8) W. HEISKANEN, *Report on Isostasy*, International Union of Geodesy and Geophysics, Washington Assembly, (1939).

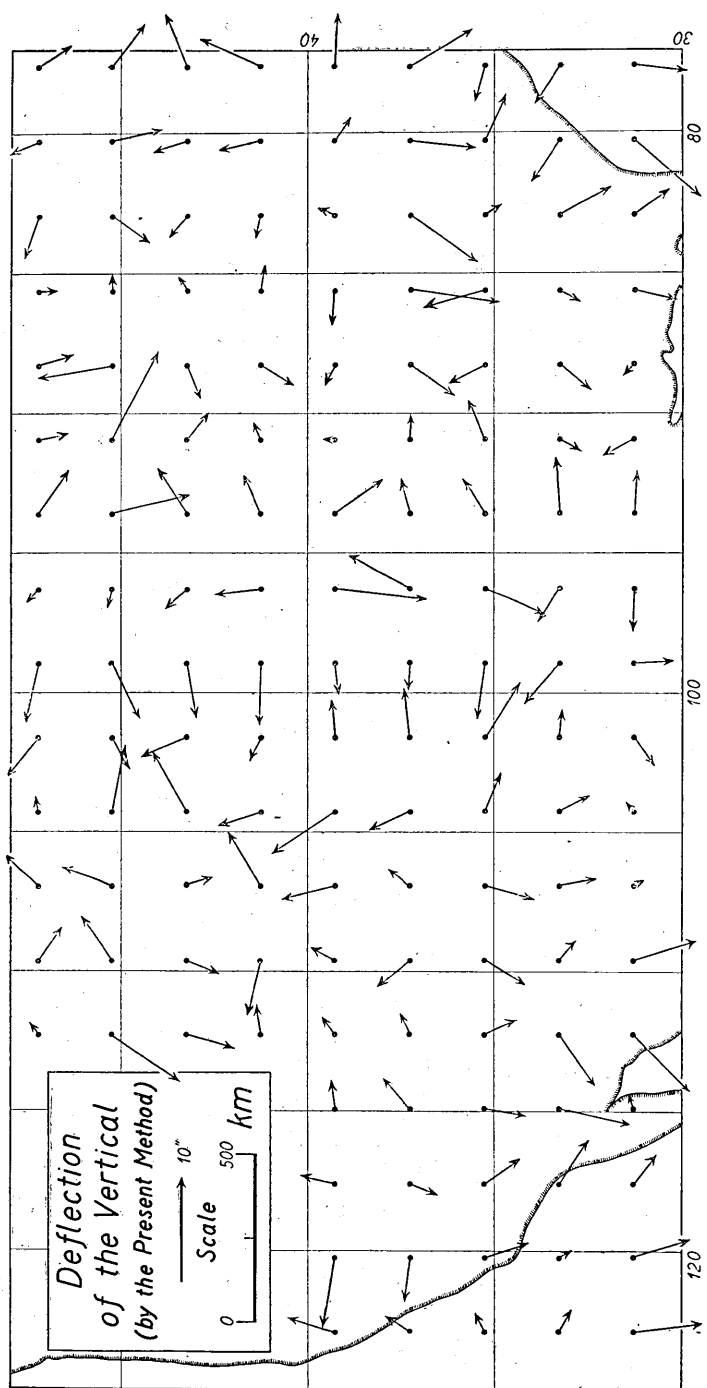


Fig. 6. Deflections of the Vertical in U.S.A. as calculated by the Present Method.

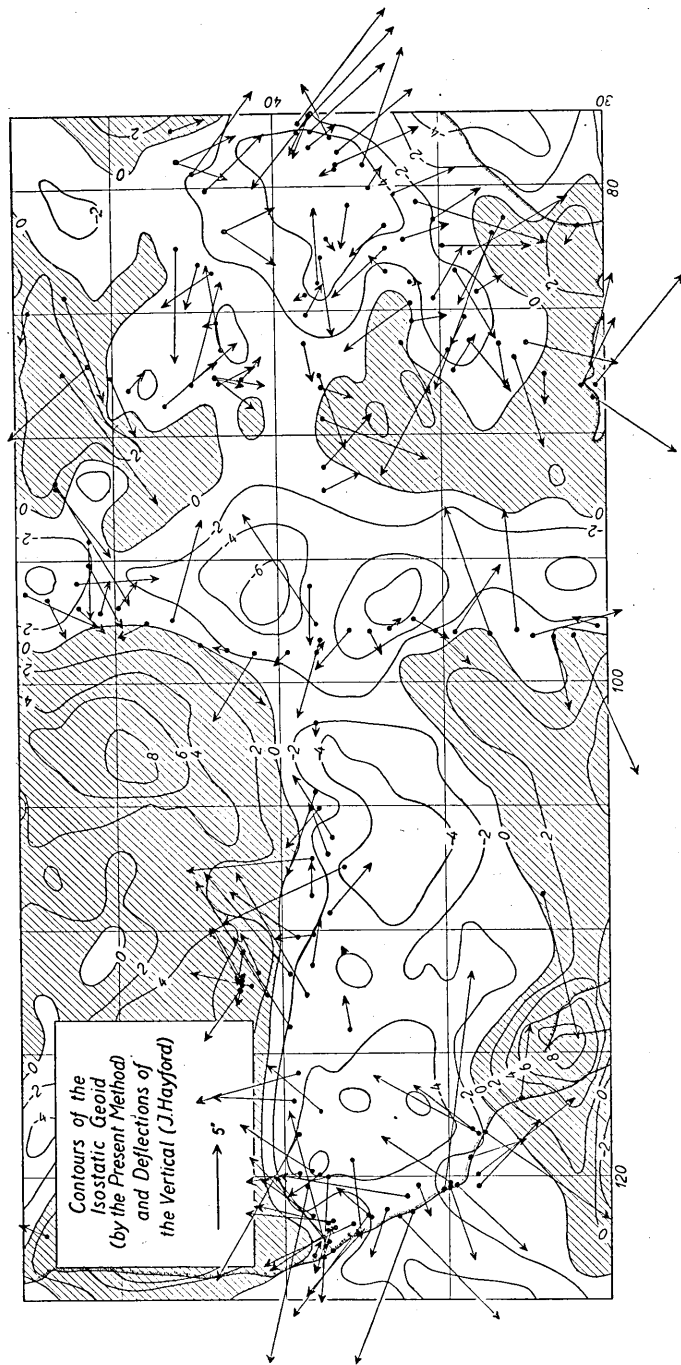


Fig. 7. Contours of the Isostatic Geoid in U.S.A. as calculated by the Present Method and Deflections of the Vertical as calculated by J. Hayford.

Finally the writer wishes to record here his appreciation of the suggestions made him by Prof. W. Heiskanen, Dr. J. de Graaff Hunter, and Prof. Vening Meinesz, and others, at the Washington Meeting of the International Union of Geodesy and Geophysics, 1939, to which this study was presented.

22. 重力異常と地下構造との關係

地震研究所 坪 井 忠 二

北米合衆國に於ける Bouguer 重力異常を材料とし、二重 Fourier 級數を利用し、

1. 均衡異常
2. 鉛直線偏倚

を算出した。その結果は J. Hayford, W. Bowie が算出したものとよく一致した。又ゼオイドの凹凸を求めた。重力異常の材料からゼオイドの凹凸を求めたのはこれを以て嚆矢とする。かくて従來獨立に觀測されて來た重力異常と鉛直線偏倚とを統一して論ずる事が出来る様になつた。
