8. *The Effect of Distribution of Heat-generating Sources on the Temperature Gradient in the Earth's Crust.*

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1. *The heat-generating sources and temperature gradient in the earth's crust.*

The thermal history of the earth has been investigated theoretically by a number of authors, such as the present Lord Rayleigh,[1] Holmes,[2] Jeffreys[3], etc. From their investigations it appears that the temperature gradient in the present condition of the crust is not a survival of the past molten stage of the earth, a great part of that gradient being the result of the heat generated from the radioactive elements that are practically confined to a thin crustal layer near the surface. Holmes[4] selected the probable average contents (weight percentage) of radioactive substances in rocks as follows (his recent analysis):

Rock group	Uranium	Thorium	Potassium
Sialic	$3{\cdot}75.10^{-6}$	$7{\cdot}5.10^{-6}$	$2{\cdot}7.10^{-6}$
Basaltic	$2{\cdot}1.10^{-6}$	$5{\cdot}0.10^{-6}$	$0{\cdot}8.10^{-6}$
Peridotitic	$1{\cdot}5.10^{-6}$	$3{\cdot}1.10^{-6}$	$0{\cdot}3.10^{-6}$

Since the annual rate of liberation of heat from radioactive elements is $7{,}900.10^{-4}$ cals./gm. U. for the uranium family, $2{,}300.10^{-4}$ cals./gm. T. for the thorium family, and $0{\cdot}2.10^{-4}$ cals./gm. K. for potassium, he obtained the following amounts of heat generation in typical rocks:

Rock group	Cals./gm./sec.	Cals./cc./sec.
Sialic	$3{\cdot}7.10^{-13}$	$8{\cdot}25.10^{-13}$
Basaltic	$1{\cdot}9.10^{-13}$	$5{\cdot}7.10^{-13}$
Peridotitic	$1{\cdot}27.10^{-13}$	$4{\cdot}3.10^{-13}$

Jeffreys,[5] using the data of various writers, obtained the average rates of heat generated by rocks in calories per sec. per cc.; for example, $1{\cdot}3.10^{-12}$ for granite, $0{\cdot}5.10^{-12}$ for basalt, $0{\cdot}36.10^{-12}$ for plateau basalt,

1) LORD RAYLEIGH, *Proc. Roy. Soc.*, **77** (1906), 472–485.

2) A. HOLMES, *Geol. Mag.* (6), **2** (1915), 102–112; *Journ. Wash. Acad. Sci.*, **23** (1933), 169–195.

3) H. JEFFREYS, *Phil. Mag.*, 32 (1916), 575–591; *The Earth* (1929), 148.

4) A. HOLMES, *loc. cit.*

5) H. JEFFREYS, *loc. cit.*

$0{\cdot}13.10^{-12}$ for dunite, and $0{\cdot}17.10^{-12}$ for eclogite. It is possible that the conductivities of typical sedimentary and basaltic rocks are 0·008 C. G. S. and 0·004 C. G. S., respectively, and the specific heat and density of the latter rock 0·20 and 3·3.

Jeffreys, furthermore, assuming that the time elapsed since the solidification of the earth is nearly $1{\cdot}6.10^9$ years, concluded that about 7/8 ths of the temperature gradient of the surface could result from the heat generated from the rocks in the surface layers, 30 km. thick, the problem of heat conduction being thus almost independent of the initial condition of the cooling of the earth.

It is now possible for the relation

$$k\partial V/\partial x=\int_0^\infty P dx \tag{1}$$

to hold, V, P being the temperature and rate of heat generation at depth x. Jeffreys thus indicated that the thicknesses of the upper (granitic) layer and the intermediate (plateau basaltic) layer should be 11 km. and 22 km., respectively, assuming that the lowest layer is almost non-radioactive.

Although in the case of the heat-generating sources being distributed uniformly in the surface layers for different localities, it may be possible for relation (1) to hold approximately, in the case of those sources being distributed with their intensities varying from point to point, equation (1) should be replaced by

$$k\int\int_S (\partial V/\partial x)dS=\int_0^\infty\int\int_S P dx dS, \quad (S\text{: surface area}) \tag{2}$$

which results from the conditions:

$$-k\nabla^2 V=P, \quad \int\int\int \nabla^2 V dx dy dz=\int\int(\partial V/\partial x)dS. \tag{3),(4}$$

Equation (2) shows that although the mean value of the temperature gradient and that of heat generation satisfy a relation of type (1), the actual values of the two quantities do not. Thus, we are now in a position to know the distribution of the temperature gradient for different distributions of heat-generating sources in the surface layer.

The cases discussed in the present paper are the source distribution of sinusoidal type, that of rectangular type, and that of spot type, the distribution being in the plane of the surface layer. In distribution with depth, uniformity in the surface layer was assumed, no source of heat existing in the subjacent layer.

In the case of periodic distribution or even in that of irregular distribution of the sources, it is convenient to separate the distribution into a uniform part and a fluctuating part, for which reason the case of uniform areal distribution was additionally studied.

2. *The case of uniform areal distribution of sources in the surface layer.*

Let P_0 be the quantity of heat evolved per unit volume of surface layer of thickness H, and k_1, k_2 be the conductivities of the same layer and the subjacent medium. Let the axis of x be drawn vertically downward. Since the equations $k_1\partial^2V_1/\partial x^2=-P_0$, $\partial^2V_2/\partial x^2=0$ hold for $0<x<H$ and $x>H$, respectively, we get $V_1=-P_0x^2/2k_1+Ax+B$, $V_2=Cx+D$. The boundary conditions are such that $V_1=0$ at $x=0$, $V_1=V_2$, $k_1\partial V_1/\partial x=k_2\partial V_2/\partial x$ at $x=H$, and V_2 is finite at $x=\infty$, from which we obtain $V_1=(P_0/k_1)(-x^2/2+Hx)$, together with

$$(\partial V_1/\partial x)_{x=0}=P_0H/k_1, \tag{5}$$

which, although merely a special case of (1), has been put down as a type of other different cases.

3. *The case of sinusoidal areal distribution of sources in the surface layer. The first method of solving the problem.*

Let the distribution of sources in the surface layer be

$$P_0+P\sin\alpha y, \tag{6}$$

the problem being two-dimensional. Then, the gradient of the heat due to P_0 is the same as that in the preceding section. The solution of the part corresponding to $P\sin\alpha y$ will now be discussed. The equations of stationary conduction in the surface layer and in the subjacent layer are

$$-k_1\left(\frac{\partial^2V_1}{\partial x^2}+\frac{\partial^2V_1}{\partial y^2}\right)=P\sin\alpha y,\quad \frac{\partial^2V_2}{\partial x^2}+\frac{\partial^2V_2}{\partial y^2}=0, \tag{7),(8}$$

respectively. The solutions of these equations are

$$V_1=\left(\frac{P}{\alpha^2k_1}+Ae^{-\alpha x}+Be^{\alpha x}\right)\sin\alpha y,\quad V_2=(Ce^{-\alpha x}+De^{\alpha x})\sin\alpha y. \tag{9),(10}$$

The first term on the right-hand side of (9) is a particular solution and the remaining two terms are complementary solutions. Using boundary conditions $V_1=0$ at $x=0$, $V_1=V_2$, $k_1\partial V_1/\partial x=k_2\partial V_2/\partial x$ at $x=H$, and V_2 is finite at $x=\infty$, we get

$$V_1=\frac{P}{\alpha^2k_1}\left[1+\frac{\{k_2e^{-\alpha H}-(k_1+k_2)\}e^{-\alpha x}+\{e^{-\alpha H}(k_2-k_1)-k_2\}e^{\alpha(x-H)}}{(k_1+k_2)+e^{-2\alpha H}(k_1-k_2)}\right]\sin\alpha y, \tag{11}$$

$$\left.\frac{\partial V_1}{\partial x}\right|_{x=0}=\frac{P\sin\alpha y}{\alpha k_1}\,\frac{(1-e^{-\alpha H})\{(k_1+k_2)+(k_1-k_2)e^{-\alpha H}\}}{(k_1+k_2)+(k_1-k_2)e^{-2\alpha H}}. \tag{12}$$

For obtaining the resultant gradient of heat at the surface, the term $(\partial V_1/\partial x)_{x=0}=P_0H/k_1$ should be added to (12). The results of calculation for various conditions, with their interpretation, will be shown in the next section.

4. *The case of sinusoidal areal distribution of sources in the surface layer. The second method of solving the problem.*

The second type of solution of equations (7), (8) is obtained by writing

$$V_1=\left\{\sum_{m=1}^{\infty} A_m \sin\frac{m\pi x}{H}+B\,\mathrm{sh}\,\alpha x\right\}\sin\alpha y, \quad V_2=Ce^{-\alpha x}\sin\alpha y. \tag{13), (14}$$

Although the first and second terms in (13) are particular and complementary solutions of (7), they never correspond to the particular and complementary solutions shown in (9). Whereas every solution in (13) itself satisfies the boundary condition $V_1=0$ at $x=0$, every solution in (9) does not. All the terms in (9), combined together, will satisfy the condition in question. On the other hand, although the treatment of solutions in (13) is rather complex, that of the solution in (9) is very simple.

Substituting (13), (14) in the boundary conditions that $V_1=V_2$, $k_1\partial V_1/\partial x=k_2\partial V_2/\partial x$ at $x=H$, and $V_2=$finite at $x=\infty$, we have

$$\left.\begin{aligned} A_m\left\{\left(\frac{m\pi}{H}\right)^2+\alpha^2\right\}\frac{m\pi}{2}&=-\frac{P}{k_1}(\cos m\pi-1),\\ B\alpha\left(ch\alpha H+\frac{k_2}{k_1}sh\alpha H\right)&=-\sum_{m=1}^{\infty}\frac{m\pi}{H}A_m\cos m\pi, \end{aligned}\right\} \tag{15}$$

using the idea of Fourier's series. Thus, we get

$$\frac{\partial V_1}{\partial x}\bigg|_{x=0}=\frac{2HP}{\pi^2 k_1}\left\{\sum_{m=1}^{\infty}\frac{(1-\cos m\pi)}{m^2+\left(\frac{\alpha H}{\pi}\right)^2} -\frac{1}{ch\,\alpha H+\frac{k_2}{k_1}sh\,\alpha H}\sum_{m=1}^{\infty}\frac{\cos m\pi(1-\cos m\pi)}{m^2+\left(\frac{\alpha H}{\pi}\right)^2}\right\}. \tag{16}$$

Although the form of (12) differs considerably from that of (16), the results of numerical treatment show that they are equal.

Using (12) or (16), we calculated the maximum temperature gradient (at $y=0$) of the surface at $\alpha y=\pi/2$ for various ratios of k_2/k_1 and for different H/l's, $l(=2\pi\alpha)$ being the wave length of the sinusoidal distribution of the sources. The results of calculation are shown in Fig. 1. The gradient at any y can be obtained by multiplying the results in Fig. 1 by the factor $\sin\alpha y$. It will be seen that with increase in the ratio of H/l, that is, with decrease in wave length, the temperature gradient tends to vanish, which condition indicates that the greater the fluctuation in the intensity of the heat sources in the layer, the more diminution in the effect of distribution of those sources on the uniform temperature gradient. For example, if the wave length of the sinusoidal distribution of the sources be half the thickness of the layer of the sources, the gradient due to P is less than ten percent of that in the case of wave length of P being infinitely great.

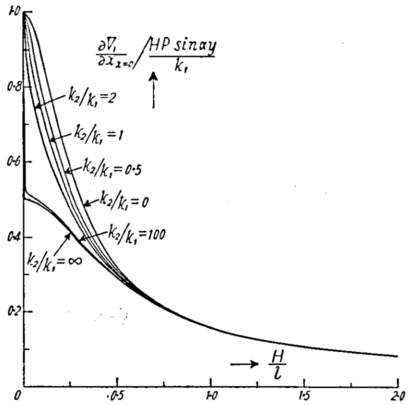

Fig. 1. The case of sinusoidal areal distribution sources in the surface layer.

With increase in the ratio of k_2/k_1, the value of $k_1 \partial V_1/\partial x_{x=0}$ naturally decreases, but if wave length l is greater than half the thickness of the layer of the sources, the value of $k_1 \partial V_1/\partial x_{x=0}$ is almost independent of the ratio k_2/k_1—a condition that is obvious from the nature of the problem.

5. *The case of rectangular areal distribution of sources in the surface layer. The first method of solving the problem.*

Let the distribution of sources in the surface layer be of the type shown in Fig. 2, the problem being again two-dimensional. If we consider the range AB, the equations of stationary conduction for that range in the surface layer and in the subjacent layer are respectively

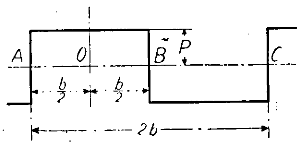

Fig. 2.

$$-k_1\left(\frac{\partial^2 V_1}{\partial x^2}+\frac{\partial^2 V_1}{\partial y^2}\right)=P, \quad \frac{\partial^2 V_2}{\partial x^2}+\frac{\partial^2 V_2}{\partial y^2}=0. \tag{17),(18}$$

The use of Fourier's series naturally implies that the value of P in the neighbouring range (say, BC) should be negative, the problem being therefore a periodic one. The solutions of the above equations are

$$V_1=\sum_{n=1}^{\infty}\left\{A_n+B_n e^{-\frac{n\pi x}{b}}+C_n e^{\frac{n\pi x}{b}}\right\}\cos\frac{n\pi y}{b}, \tag{19}$$

$$V_2=\sum_{n=1}^{\infty} D_n e^{-\frac{n\pi x}{b}}\cos\frac{n\pi y}{b}. \tag{20}$$

The first term on the right-hand side of (19) represents a particular solution, while the remaining two give complementary solutions. Fourier's analysis shows that

$$A_n=(P_2/k_1)(4b^2/n^3\pi^3)\sin\frac{n\pi}{2}. \tag{21}$$

Substituting (19), (20), (21) in the boundary conditions that $V_1=0$ at $x_1=0$; $V_1=V_2$, $k_1\partial V_1/\partial x=k_2\partial V_2/\partial x$ at $x=H$; $V_2=$finite at $x=\infty$, we get

$$\left.\begin{aligned} &\frac{B_n\Phi}{A_n}=\frac{k_2}{k_1}-\left(1+\frac{k_2}{k_1}\right)e^{\frac{n\pi H}{b}},\quad \frac{C_n\Phi}{A_n}=-\frac{k_2}{k_1}+\left(\frac{k_2}{k_1}-1\right)e^{-\frac{n\pi H}{b}},\\ &\frac{D_n\Phi}{A_n}=2\left(\mathrm{ch}\frac{n\pi H}{b}-1\right)e^{\frac{n\pi H}{b}}, \end{aligned}\right\} \tag{23}$$

where

$$\Phi=2\left(\mathrm{ch}\frac{n\pi H}{b}+\frac{k_2}{k_1}\mathrm{sh}\frac{n\pi H}{b}\right), \tag{23}$$

from which it follows that

$$\left(\frac{\partial V_1}{\partial x}\right)_{x=0}=\frac{4}{\pi^2}\frac{P_2H}{k_1}\left(\frac{b}{H}\right)\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\frac{-\frac{k_2}{k_1}+\frac{k_2}{k_1}\mathrm{ch}\frac{n\pi H}{b}+\mathrm{sh}\frac{n\pi H}{b}}{\mathrm{ch}\frac{n\pi H}{b}+\frac{k_2}{k_1}\mathrm{sh}\frac{n\pi H}{b}}\cos\frac{n\pi y}{b}. \tag{24}$$

In order to get the total gradient of heat at the surface, $(\partial V_1/\partial x)_{x=0}=P_0H/k_1$ should be added to (24). The results of calculation for the various conditions will be shown in the next section.

6. *The case of rectangular areal distribution of sources in the surface layer. The second method of solving the problem.*

The second type of solution of equations (17), (18) may be obtained by writing

$$V_1=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{H}\cos\frac{n\pi y}{b}+\sum_{n=1}^{\infty}B_n\mathrm{sh}\frac{n\pi x}{H}\cos\frac{n\pi y}{b}, \tag{25}$$

$$V_2=\sum_{n=1}^{\infty}C_n\cos\frac{n\pi y}{b}e^{-\frac{n\pi x}{b}}. \tag{26}$$

In this case, too, although the first and second terms in (25) are particular and complementary solutions of (17), they are not equivalent respectively to the particular and complementary solutions shown in (19). Although every solution in (26) fits itself into the boundary condition $V_1=0$ at $x=0$, every solution in (19) does not. All the terms in (19) combined, will satisty the condition in question. The treatment of the solutions in (25) is rather complex, while that of the solutions in (19) is not.

Substituting (25), (26) in the boundary conditions $V_1=V_2$, $k_1\partial V_1/\partial x=k_2\partial V_2/\partial x$ at $x=H$; and $V_2=$finite at $x=\infty$, we get

$$A_{mn}=\frac{P_2H^2}{k_1}\frac{8(1-\cos m\pi)\sin\frac{n\pi}{2}}{mn\left\{m^2+\left(\frac{H}{b}n\right)^2\right\}}, \tag{27}$$

$$B_n = \frac{-\sum_{m=1}^{\infty} \frac{m\pi}{H} A_{mn} \cos m\pi}{\frac{n\pi}{b}\left(\mathrm{ch}\frac{n\pi H}{b} + \frac{k_2}{k_1}\mathrm{sh}\frac{n\pi H}{b}\right)}, \tag{28}$$

Fourier's series having been used in the calculation. We get finally

$$\frac{\partial V_1}{\partial x_{x=0}} = \frac{8PH}{k_1\pi^3}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(1-\cos m\pi)\sin\frac{n\pi}{2}}{n\left\{m^2 + \left(\frac{H}{b}n\right)^2\right\}}\right.$$

$$\left. - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(1-\cos m\pi)\sin\frac{n\pi}{2}}{n\left\{m^2 + \left(\frac{H}{b}n\right)^2\right\}} \frac{\cos m\pi}{\mathrm{ch}\frac{n\pi H}{b} + \frac{k_2}{k_1}\mathrm{sh}\frac{n\pi H}{b}}\right]. \tag{29}$$

Although the form of (24) differs considerably from that of (29), the results of numerical treatment show that they are virtually the same.

Using (24) or (29), we calculated the temperature gradient of the surface at $y=0, (1/8)b, (2/8)b, (3/8)b$, for various ratios of k_2/k_1 and for different $H/2b$'s, $2b$ being the period of rectangular fluctuation in the intensity of the heat sources. The results of calculation are shown in Figs. 3–6. It will be seen that in this case too, with increase in the ratio of $H/2b$, the temperature gradient tends to vanish, from which it is possible to conclude that local fluctuations in the intensity of the

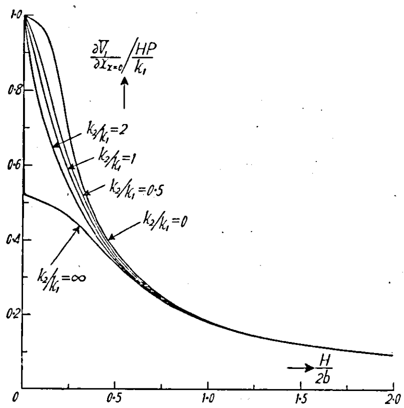

Fig. 3. The case of rectangular areal distribution of sources in the surface layer; $y=0$.

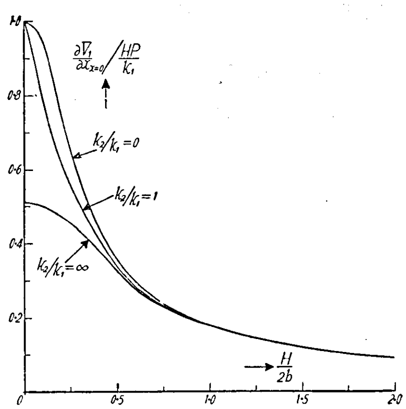

Fig. 4. The case of rectangular areal distribution of sources in the surface layer; $y=(1/8)b$.

sources do not seriously affect the general temperature gradient of the surface. When H is equal to $2b$, the gradient due to P is merely ten percent of that in the case of the wave length of P being infinitely great.

With increasing ratio of k_2/k_1, the value of $k_1(\partial V_1/\partial x)_{x=0}$ tends to decrease, but if b is greater than H, the value of $k_1(\partial V_1/\partial x)_{x=0}$ does not change with change in the ratio of k_2/k_1.

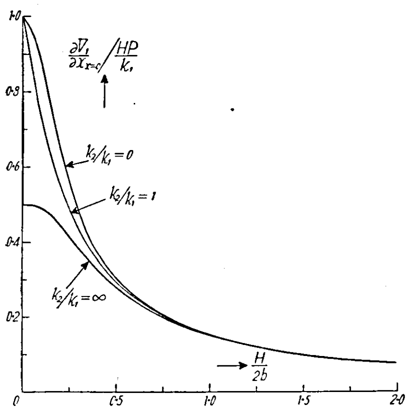

Fig. 5. The case of rectangular areal distribution of sources in the surface layer; $y=(2/8)b$.

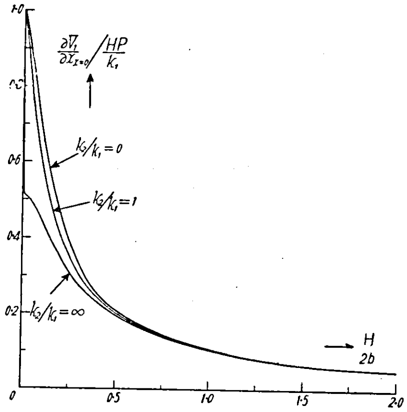

Fig. 6. The case of rectangular areal distribution of sources in the surface layer; $y=(3/8)b$.

Although in the sinusoidal distribution of heat sources, the temperature gradient changes sinusoidally in consonance with the distribution of sources, in the present distribution of the sources, the temperature gradient is nearly the same for a wide range of y, at least for relatively small value of $H/2b$. For a relatively large ratio of $H/2b$, the change in gradient with y is somewhat appreciable, but not very marked. This results from the condition that the distribution is rectangular for the range $0<y<b$.

In confirmation of the foregoing feature, we shall draw curves showing the relation between y/b and $k_1\partial V_1/\partial x$ for various ratios of k_2/k_1 in two cases, $H/2b=1/4$ and 1, the results being shown in Figs. 7, 8. It will be seen that since all

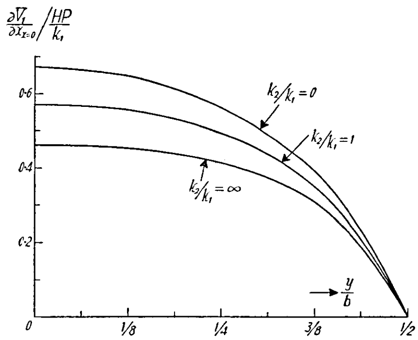

Fig. 7. The case of rectangular areal distribution of sources in the surface layer; $H/2b=1/4$.

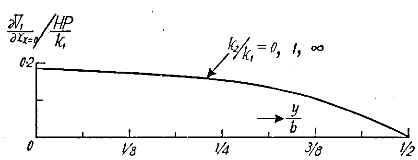

Fig. 8. The case of rectangular areal distribution of sources in the surface layer; $H/2b=1$.

the curves are of parabolic form, rather than sinusoidal, the value of $k_1\partial V_1/\partial x$ does not change much for a relatively wide range of y/b.

Comparing Figs. 7, 8 it will also be seen that the greater the ratio of $H/2b$, the more the diminution in the effect of the change in the ratio of k_2/k_1 on the value of $k_1\partial V_1/\partial x$, in which case the amplitude of $k_1\partial V_1/\partial x$ itself also diminishes.

7. *The case of concentrated sources at anti-symmetrical periodic points.*

The sources in this case lie at anti-symmetrical periodic points of interval b as shown in Fig. 9. If we consider range AB, the equations of stationary conduction for that range are

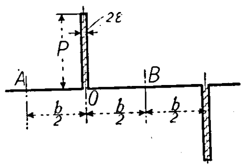

Fig. 9.

$$-k_1\left(\frac{\partial^2 V_1}{\partial x^2}+\frac{\partial^2 V_1}{\partial y^2}\right)=P, \quad \frac{\partial^2 V_2}{\partial x^2}+\frac{\partial^2 V_2}{\partial y^2}=0 \qquad (30), (31)$$

for $-\varepsilon<y<\varepsilon$, ε being small, and

$$-k_1\left(\frac{\partial^2 V_1}{\partial x^2}+\frac{\partial^2 V_1}{\partial y^2}\right)=0, \quad \frac{\partial^2 V_2}{\partial x^2}+\frac{\partial^2 V_2}{\partial y^2}=0 \qquad (32), (33)$$

for $-b/2<y<-\varepsilon$, $\varepsilon<y<b/2$. The solutions of (30)—(33) are mostly of the type

$$V_1=\sum_{n=1}^{\infty}\{A_n+B_n e^{-\frac{n\pi x}{b}}+C_n e^{\frac{n\pi x}{b}}\}\cos\frac{n\pi y}{b}, \quad V_2=\sum_{n=1}^{\infty}D_n e^{-\frac{n\pi x}{b}}\cos\frac{n\pi y}{b}, \qquad (34), (35)$$

which are quite similar to those in (19), (20). To determine the coefficients A_n of the particular solution, we use again Fourier's analysis like

$$\int_{-b/2}^{b/2}A_n\left(\frac{n\pi}{b}\right)^2\cos^2\frac{n\pi y}{b}dy=\int_{-\varepsilon}^{\varepsilon}\frac{P}{k_1}\cos\frac{n\pi y}{b}dy=\frac{2\varepsilon P}{k_1}=\frac{|P|}{k_1}, \qquad (36)$$

where $|P|=2\varepsilon P$, from which we get

$$A_n=\frac{|P|}{k_1}\Big/\left(\frac{n\pi}{b}\right)^2\frac{b}{2}, \qquad (37)$$

so that

$$\frac{\partial V_1}{\partial x_{x=0}}=\frac{2\varepsilon P}{\pi k_1}\sum_{m=1}^{\infty}\frac{1}{n}\frac{-\frac{k_2}{k_1}+\frac{k_2}{k_1}\text{ch}\frac{n\pi H}{b}+\text{sh}\frac{n\pi H}{b}}{\text{ch}\frac{n\pi H}{b}+\frac{k_2}{k_1}\text{sh}\frac{n\pi H}{b}}\cos\frac{n\pi y}{b}. \quad [n=2m+1] \qquad (38)$$

Using (38), we calculated the temperature gradient of the surface at $y=(1/8)b$, $(1/4)b$, $(3/8)b$ for various ratios of k_2/k_1 and for different $H/2b$'s, b being the periodic interval between the positive and negative sources. The results of calculation are shown in Figs. 10, 11, 12. In this case the temperature gradient increases with increase in the ratio of $H/2b$, tending to an asymptotic value. This asymptotic value however decreases with distance from the sources, that is to say, with increase in the value of y/b. Although the gradient $k_1\partial V_1/\partial x_{x=0}$ tends to decrease with increase in k_2/k_1, the decrement becomes insignificant for relatively large values of $H/2b$.

In order to confirm the foregoing features, we plotted the gradient $k_1\partial V_1/\partial x_{x=0}$ as ordinate with y/b as abscissa for two cases $H/2b=1/4$ and 1, the results being shown in Figs. 13, 14. It will be seen that, as just mentioned, the decrease in the gradient with y/b is very marked, the feature thus being the reverse of that of the cases in Sections 5, 6. The condition that the effect of k_2/k_1 on the gradient vanishes for a greater value of $H/2b$, is also obvious from

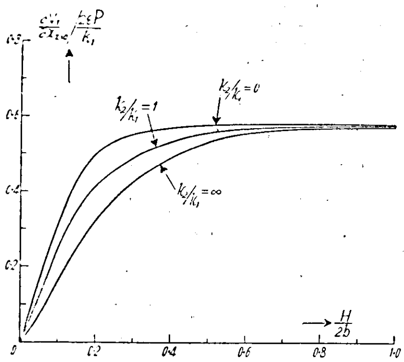

Fig. 10. The case of concentrated sources at anti-symmetrical periodic points; $y=(1/8)b$.

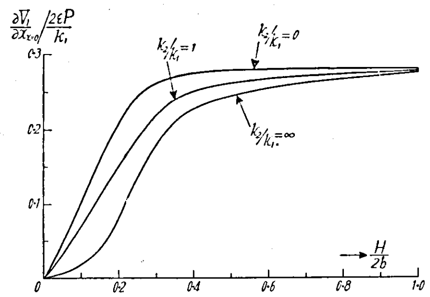

Fig. 11. The case of concentrated sources at anti-symmetrical periodic points; $y=(2/8)b$.

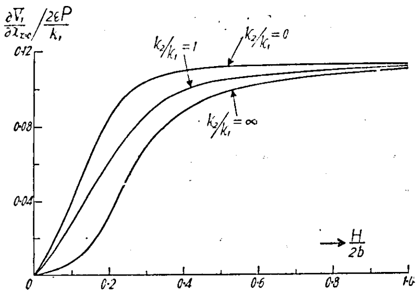

Fig. 12. The case of concentrated sources at anti-symmetrical periodic points; $y=(3/8)b$.

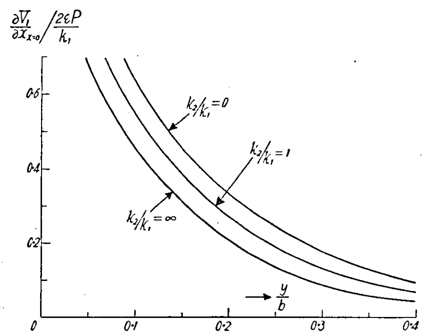

Fig. 13. The case of concentrated sources at anti-symmetrical periodic points; $H/2b=1/4$.

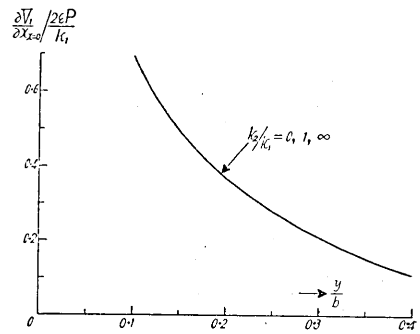

Fig. 14. The case of concentrated sources at anti-symmetrical periodic points; $H/2b=1$.

the results in the same figures.

Although the gradient $k_1\partial V_1/\partial x_{x=0}$ at the point exactly corresponding to any heat source is infinity, since we have assumed that $2\varepsilon P$ is finite, that is to say, P is infinitely great, the condition, if anything, is natural. If P were finite, the gradient there would also be finite and the gradient outside that point would be infinitesimal, for which reason our conception regarding the decrease in the effect of local heat sources with increasing periods of those sources, is still likely to hold.

8. *Summary of the results of mathematical analysis.*

From the mathematical investigations in previous sections, many features of the temperature gradient on the surface were ascertained, of which the important parts that could commonly hold, will be summarized. The greater the ratio of the conductivity in the lower layer to that in the upper layer, the more does the gradient under consideration diminish. But this diminution becomes insensible if the intensities of sources fluctuate and the wave length of the fluctuation in the intensity of the heat sources be less than nearly half the thickness of the layer of the sources. In such a condition, the temperature gradient itself becomes also so feeble that its amount is as small as less than ten percent of that in the case of infinite wave length. Whereas in the case of rectangular distribution of sources, the temperature gradient, even should it be small, varies rather slightly for a wide range; in the case of concentrated sources at periodic points, the gradient decreases very rapidly with distance from every source.

From the above results it is possible to conclude that, although the temperature gradient on the surface of the earth's crust is nearly constant for any place, the intensity of the heat generated in the crust there could differ greatly from that in any other place. Since from Jeffreys's investigation, the thickness of the heat-generating layer is practically 30km, the effect of local heat sources of the range from 10 to 20km on the gradient is less than ten percent of that of the case of uniform sources extending to infinity. It should however be remembered that if there were a point source of great intensity, like a volcano, it would be possible for the gradient exactly at that point to be enormous.

The two problems shown in Sections 3-6 have been investigated in two ways. In each problem, notwithstanding that the particular solution and the complementary solution that were determined in two ways are not mathematically equivalent, the numerical calculation gives the same result whichever way is used. Previously, we had the idea that the particular and complementary solutions give rise to physically different conditions. In the case of the vibration problem, the particular solution represents the term corresponding to forced vibration and the complementary solution, on the other hand, to that corresponding to free vibration, so that it is generally impossible to change the type of solution. In the present case, although the particular and complementary solutions respectively correspond to the condition affected by heat sources and that free from those sources, the correspondence is merely apparent, the respective solutions being of alterable type. It appears that, generally, particular and complementary solutions should then be used for mathematical convenience and never be restricted to their physical meanings.

9. *Temperature distribution at depth.*

Our investigation here is concerned only with the temperature gradient on the surface due to any local heat source. If the temperature at depth were considered, the present solution would be of little avail. As shown by Jeffreys, the temperature at a relatively great depth depends chiefly on the thermal state of the crust when it solidified and not on the intensity of the heat-generating sources in the surface layer, for which reason the local fluctuation in heat sources is outside that problem.

Authors differ in the temperature distribution within the earth's crust below depths of 20 or 30 km. Jeffreys[6] shows that the temperature increases with depth, reaching 3,500° at 700 km, whereas Gutenberg[7] and McNish[8] conclude that the temperature at such depth should be less than 2,000°. The former results from the idea of increase of melting point with pressure and the latter from considerations with respect to the existence of terrestrial magnetism, besides. On the other hand, Elsasser[9] gave another explanation of the origin of the earth's magnetic field, the restriction to a temperature below 2,000° not being always sufficient.

10. *The prospects of there being conductivity in the lower layer.*

In the present investigation it has been ascertained that, generally speaking, the effect of local heat sources on the temperature gradient is rather small. There remains, however, the question whether or not it is possible to ascertain the conductivity of the lower layer of the earth's crust. The results shown in Figs. 1, 3–6, 10–11 may furnish some answer to this question. In the case of fluctuation of distributed heat sources, if the gradient at the surface changes considerably with change

6) H. JEFFREYS. *The Earth* (1929), 154.

7) B. GUTENBERG, *ZS. f. Geophys.*, 3 (1927), 371–377; *Internal Constitution of the Earth* 7 (1939), 162.

8) A. G. MCNISH, *Geophys. Union Trans.* (1937), 43–50.

9) W. M. ELSASSER, *Phys. Review*, 55 (1939), 489.

in wave length of the fluctuation just mentioned, it indicates that the conductivity in the lower layer is rather small, whereas if the change in that gradient were not very marked, the conductivity in the lower layer would be large. In the case of concentrated sources at points, the reverse condition holds. Since the difference between distributed sources and concentrated sources is not a simple matter, application of the present idea is actually rather difficult.

In conclusion, we wish to express our thanks to Messrs. Watanabe and Kodaira, who assisted us greatly in our calculations. We also wish to express our warmest thanks to the Officials for Scientific Research in the Department of Education for financial aid (Funds for Scientific Research) granted for a series of our investigation, of which this study is a part.

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8. 發熱源の地域的分布が地殼の溫度勾配に及ぼす影響

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地殼の溫度勾配は地球の鎔融時代の痕跡といふよりも大部分が地表層の中にある放射性元素から出る熱のためであるが，この發熱源が地表層に沿うて均一に分布してをる場合は可なり研究されてゐる．ここでは地表層に沿うて均一でない場合卽ち地域的變化のある場合に普通考へられる溫度勾配が如何に變り得るかを研究したのである．

一般に發熱しない下層の熱傳導度と表面層のそれとの比が大きければ大きい程地表の溫度勾配が減ずる譯であるが，發熱源の地域的變化の波長が地表層の厚さの半分以下になると，この變化する發熱源のための溫度勾配の減少が無くなるのみか，溫度勾配それ自身も非常に小さくなる．矩形型に分布せる發熱源のための溫度勾配は廣い範圍に涉り極く少しづゝしか變化しないが集中發熱源の場合には發熱源からの距離によつて著しく變化するのは大體常識通りである．

以上の事から地表の溫度勾配は局所的に著しく變らなくても地中の發熱源の分布は著しく變つてよい事がわかる．Jeffreys によれば發熱源のある地表層は大體 30 km 位であるから，發熱源は 10 km か 20 km 位の波長で變化してもその影響が溫度勾配に現れぬ事になる譯である．しかし火山の如き集中發熱源のときには，その極く近くだけは大きな勾配でもよいのである．

この研究から微分方程式の特異解と補助解とは必しも夫々物理的意味を持たなくてもよい事もわかつた．

尙只今の問題は地表の溫度勾配について成立つのであつて深い箇所のそれは問題の性質上自ら異なるものである．又，只今の研究を進める事により深層部の熱傳導度の想定の可能性のある事をも示して置いた．