1. *Dispersive Rayleigh-waves of Positive or Negative Orbital Motion, and Allied Problems.*

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1. *Introduction.*

Some years back,[1] we showed that there are generally two dispersion curves for Rayleigh-waves that are transmitted through a body so stratified that the density or elasticity in the subjacent layer far exceeds that in the superficial layer. It was our belief until recently that the sense of the orbital motion of such Rayleigh-waves is always opposite to that of gravitational waves. On the other hand, we studied quite recently the problem of waves that are propagated along the surface of a semi-infinite body on which masses are distributed, and found that the sense of the orbital motion of Rayleigh-waves is not always opposite to that of gravitational waves.

As in the case of Rayleigh-waves through a stratified body, there are two dispersion curves for waves on a semi-infinite body with mass distribution on its surface. In every case, whereas the orbital motion of the waves corresponding to one dispersion curve is in the sense opposite to that of gravitational waves, that corresponding to another dispersion curve is of the same sense as that of gravitational waves. These conditions were found from complex numerical calculation, and not immediately from the types of the mathematical expressions.

Although the question of waves on a semi-infinite body, with mass distribution on its surface, is not concerned with the nature of orbital motion, our study was not restricted to the case of Rayleigh-type waves, but extended to the case of Love-type waves, from which it was ascertained that in the latter type of waves there is no more than one dispersion curve.

2. *Orbital motion of Rayleigh-waves in a stratified body.*

It was already found that the velocity equation of Rayleigh-waves

1) K. SEZAWA and K. KANAI, *Bull. Earthq. Res. Inst.*, **13** (1935), 237~244.

transmitted through a stratified body is expressed by

$$
\begin{aligned}
&\frac{4r's'}{f^2}\left(2-\frac{k'^2}{f^2}\right)\eta-\frac{r's'}{f^2}\left\{4\vartheta+\left(2-\frac{k'^2}{f^2}\right)^2\zeta\right\}\cosh r'H\cos s'H \\
&\quad+\frac{r'}{f}\varphi\left\{\frac{4rs'^2}{f^3}+\frac{s}{f}\left(2-\frac{k'^2}{f^2}\right)^2\right\}\cosh r'H\sin s'H \\
&\quad+\frac{s'}{f}\varphi\left\{\frac{r}{f}\left(2-\frac{k'^2}{f^2}\right)^2-\frac{4r'^2s}{f^3}\right\}\sinh r'H\cos s'H \\
&\quad+\left\{\left(2-\frac{k'^2}{f^2}\right)^2\vartheta-\frac{4r'^2s'^2}{f^4}\zeta\right\}\sinh r'H\sin s'H=0,
\end{aligned}
\tag{1}
$$

where

$$
\left.
\begin{aligned}
&\varphi=\frac{\mu'k^2k'^2}{\mu f^4},\quad \zeta=\frac{4rs}{f^2}\left(\frac{\mu'}{\mu}-1\right)^2-\alpha^2,\quad \eta=\frac{2rs}{f^2}\left(\frac{\mu'}{\mu}-1\right)\beta-\alpha\gamma, \\
&\vartheta=\frac{rs}{f^2}\beta^2-\gamma^2,\quad \alpha=\frac{2\mu'}{\mu}-\left(2-\frac{k^2}{f^2}\right),\quad \beta=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-2, \\
&\gamma=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-\left(2-\frac{k^2}{f^2}\right), \\
&r^2=f^2-h^2,\quad s^2=f^2-k^2,\quad r'^2=f^2-h'^2,\quad s'^2=k'^2-f^2, \\
&h^2=\rho p^2/(\lambda+2\mu),\quad h'^2=\rho'p^2/(\lambda'+2\mu'),\quad k^2=\rho p^2/\mu,\quad k'^2=\rho'p^2/\mu',
\end{aligned}
\right\}
\tag{2}
$$

in which H is the thickness of the stratum, $2\pi/f$ the wave length, and ρ, λ, μ, ρ', λ', μ' the densities and elastic constants of the stratum and

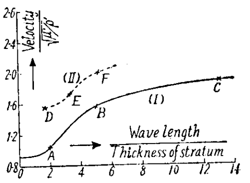

Fig. 1. Dispersion curves for $\mu'/\mu=1/5$, $\lambda'=\mu'$, $\lambda=\mu$, $\rho'/\rho=1$.

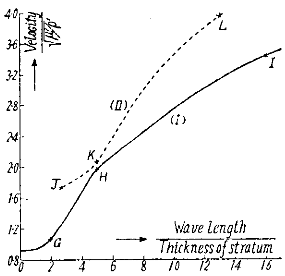

Fig. 2. Dispersion curves for $\mu'/\mu=1/20$, $\lambda'=\mu'$, $\lambda=\mu$, $\rho'/\rho=1$.

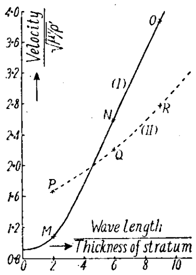

Fig. 3. Dispersion curves for $\mu'/\mu=1/\infty$, $\lambda'=\mu'$, $\lambda=\mu$, $\rho'/\rho=1$.

the subjacent medium respectively. Dispersion curves of cases $\mu'/\mu=1/5$, $\mu'/\mu=1/20$, $\mu'/\mu=1/\infty$ ($\lambda=\mu$, $\lambda'=\mu'$, $\rho'=\rho$) calculated by means of the above equation are shown in Figs. 1, 2, 3.

As will be seen later, the nature of the orbital motion of the waves corresponding to the dispersion curves as shown by full lines differs from that corresponding to the dispersion curves shown by broken lines. These two dispersion curves will be called the first and second dispersion curves, respectively.

Next, the horizontal and vertical components of the surface displacement may be written

$$u'_{II} = -\frac{is'k'^2}{f^4}\left\{\frac{sr'}{f^2}\varphi\cosh r'H\cos s'H + \frac{r's'}{f^2}\zeta\cosh r'H\sin s'H \right.$$
$$\left. + \vartheta\sinh r'H\cos s'H - \frac{rs'}{f^2}\varphi\sinh r'H\sin s'H\right\},$$

$$v'_{II} = -\frac{1}{f}\left[\frac{r's'}{f^2}\left(4-\frac{k'^2}{f^2}\right)\eta - \frac{r's'}{f^2}\left\{2\vartheta+\left(2-\frac{k'^2}{f^2}\right)\zeta\right\}\cosh r'H\cos s'H\right.$$
$$+\frac{r'}{f}\varphi\left\{\frac{2rs'^2}{f^3}+\frac{s}{f}\left(2-\frac{k'^2}{f^2}\right)\right\}\cosh r'H\sin s'H$$
$$+\frac{s'}{f}\varphi\left\{\frac{r}{f}\left(2-\frac{k'^2}{f^2}\right)-\frac{2sr'^2}{f^3}\right\}\sinh r'H\cos s'H$$
$$\left.+\left\{\left(2-\frac{k'^2}{f^2}\right)\vartheta-\frac{2r'^2s'^2}{f^4}\zeta\right\}\sinh r'H\sin s'H\right], \qquad (3)$$

which expressions are obtained by putting $y=H$ in (4) of the previous paper.[2] Using the relation in (3), it is possible to determine the orbital

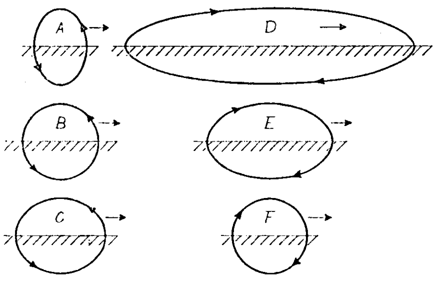

Fig. 4. Orbital motions of conditions A, B, ..., F for the case $\mu'/\mu=1/5$, $\lambda'=\mu'$, $\lambda=\mu$, $\rho'/\rho=1$.

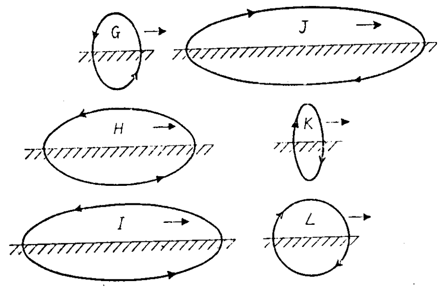

Fig. 5. Orbital motions of conditions G, H, ..., L for the case $\mu'/\mu=1/20$, $\lambda'=\mu'$, $\lambda=\mu$, $\rho'/\rho=1$.

motions corresponding to conditions A, B, C, ... of the dispersion curves

2) *loc. cit.* 1).

for the three cases $\mu'/\mu=1/5$, $\mu'/\mu=1/20$, $\mu'/\mu=1/\infty$ in Figs. 1, 2, 3, respectively. The results of calculation are shown in Figs. 4, 5, 6.

It will be seen that although the ratio of the minor and major axes of the orbit differs with difference in wave length, the sense of the same orbit is invariably the same as that of gravitational waves for every condition that corresponds to the second dispersion curve (indicated by a broken line), the sense being reversed for every condition that corresponds to the first dispersion curve (the curve indicated by full line).

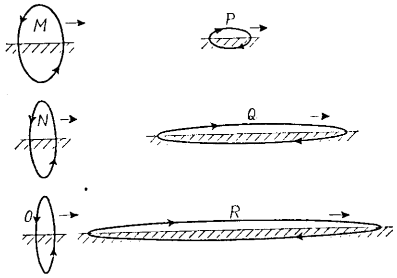

Fig. 6. Orbital motions of conditions M, N, ..., R for the case $\mu'/\mu=1/\infty$, $\lambda'=\mu'$, $\lambda=\mu$, $\rho'/\rho=1$.

Since the velocity of transmission of waves corresponding to the second dispersion curve is higher than that of waves corresponding to the first dispersion curve, if it were actually possible to observe the two kinds of Rayleigh-waves now under consideration, the orbital motion of the waves that arrive first would be of the same sense as that of gravitational waves, whereas that of the waves that arrive next would be reversed.

3. *Orbital motion of Rayleigh-waves in a semi-infinite body, on which concentrated masses are distributed.*

The present condition of the problem represents an extreme case in which either masses of very great rigidity and without mutual connection between them are distributed on the surface of a semi-infinite body or masses of very small rigidity and with mutual connection between them are distributed on the same surface. Although it is scarcely possible for such a case actually to exist, a treatment of the case may serve to acquaint us with the features of the orbital motion of general Rayleigh-waves.

The dilatational and distortional movements of the elastic part are expressed by

$$\left.\begin{aligned} u_1&=\frac{if}{h^2}Pe^{-rz+i(pt-fx)}, & v_1&=\frac{r}{h^2}Pe^{-rz+i(pt-fx)}, \\ u_2&=\frac{is}{k^2}Qe^{-sz+i(pt-fx)}, & v_2&=\frac{f}{k^2}Qe^{-sz+i(pt-fx)}, \end{aligned}\right\} \qquad (4)$$

where

$$\Delta=\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}=Pe^{-rz+i(pt-fx)},\quad r^2=f^2-h^2,\quad s^2=f^2-k^2. \tag{5}$$

If the intensity of the masses distributed per unit area of the surface be M, the boundary conditions would be such that

$$\mu\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}\right)=M\frac{\partial^2 u}{\partial t^2},\quad \lambda\Delta+2\mu\frac{\partial v}{\partial z}=M\frac{\partial^2 v}{\partial t^2}. \tag{6}$$

Substituting (4), (5) in (6), we get the equation for determining the velocity of transmission of the waves, as follows:

$$\left(\frac{k}{f}\right)^4\left(\frac{fM}{\rho}\right)^2\left(1-\frac{rs}{f^2}\right)-\left(\frac{k}{f}\right)^4\frac{fM}{\rho}\left(\frac{r}{f}+\frac{s}{f}\right)+\left\{\frac{4rs}{f^2}-\left(1+\frac{s^2}{f^2}\right)^2\right\}=0, \tag{7}$$

$(k/f)\sqrt{\mu/\rho}$ representing the velocity of transmission in question. If, especially, $fM/\rho=0$, that is, the case without surface masses, equation (7) then reduces to

$$4rs/f^2-(1+s^2/f^2)^2=0, \tag{8}$$

which is of the same form as that of the transmission of Rayleigh-waves on a semi-infinite body.

The dispersion curves for the case $\lambda=\mu$, that is, the case of Poisson's ratio being 1/4, are shown in Fig. 7. This figure generally represents dispersion curves of waves (of length $L=2\pi/f$) through a medium whose Poisson's ratio is given. It will be seen that, whereas one dispersion curve is in the range between zero and infinity in the value of $L\rho/M$, another dispersion curve is in a narrow range, namely, in the range of relatively small values of $L\rho/M$. The former and the latter dispersion curves will again be called the first and second dispersion curves, respectively.

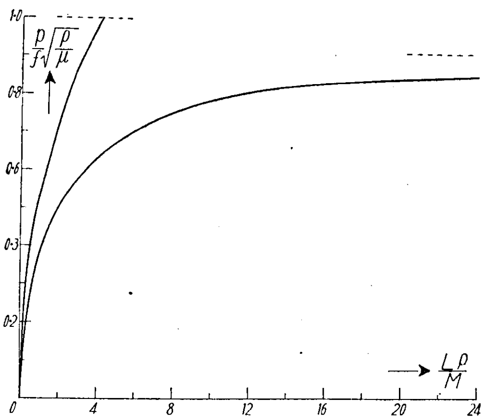

Fig. 7. Dispersion curves for the case of a semi-infinite body with masses distributed on its upper surface.

The reason for the second dispersion curve being in a certain range, is very simple. If we assume that the amplitude of the waves at depth tends to vanish, the velocity of the waves for the value of $L\rho/M$ beyond a certain limit assumes a complex form, as the result of which the waves, if formed, would either decay or grow without limit. If, on the other hand, we assume that the velocity is real and stationary, the amplitude of the waves at depth becomes infinitely great. At all events, unless the value of $L\rho/M$ is below a certain limit, waves of surface type cannot exist.

Next, the expressions for the horizontal and vertical components of displacements become

$$\left.\begin{aligned} u &= P\frac{if}{h^2}\left\{e^{-rz}+\frac{\dfrac{s}{f}\left(\dfrac{2r}{f}-\dfrac{Mp^2}{\mu f}\right)}{\dfrac{Mp^2s}{\mu f^2}-2+\dfrac{k^2}{f^2}}e^{-sz}\right\}e^{i(pt-fx)}, \\ v &= P\frac{f}{h^2}\left\{\frac{r}{f}e^{-rz}+\frac{\dfrac{2r}{f}-\dfrac{Mp^2}{\mu f}}{\dfrac{Mp^2s}{\mu f^2}-2+\dfrac{k^2}{f^2}}e^{-sz}\right\}e^{i(pt-fx)}, \end{aligned}\right\} \tag{9}$$

from which the ratio of horizontal and vertical amplitudes on the surface is written

$$\frac{u_{z=0}}{v_{z=0}}=\frac{i\left(\dfrac{k^2}{f^2}-2+\dfrac{2rs}{f^2}\right)}{\dfrac{k^2}{f^2}\left\{\dfrac{fM}{\rho}\left(\dfrac{rs}{f^2}-1\right)+\dfrac{r}{f}\right\}}. \tag{10}$$

The results of calculation are shown in Fig. 8. When $L\rho/M=0$, the ratio of the horizontal and vertical amplitudes is unity for waves corresponding to both dispersion curves. With increase in the value of $L\rho/M$, the ratio in question for waves corresponding to the first dispersion curve tends to decrease, assuming finally the value for Rayleigh-waves on a semi-infinite body, whereas, with the same increase, the ratio corresponding to the second dispersion curve increases. It is thus possible to conclude that the waves with higher velocity are orientated more in

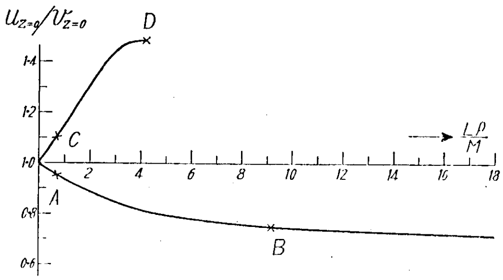

Fig. 8. The ratio of horizontal and vertical amplitudes of the upper surface for the case shown in Fig. 7. Curves AB, CD correspond to the dispersion curves of long and short ranges, respectively, in Fig. 7.

the direction of propagation of the same waves, while waves with lower velocity, on the other hand, orientate more in the direction normal to that of propagation.

The distribution of displacements with depth in the respective conditions of A, B, C, D in the amplitude ratio curves in Fig. 8 are shown

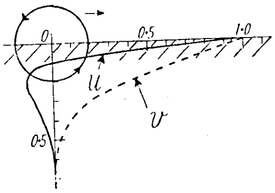

Fig. 9. Distribution of displacements with depth and orbital motion at the upper surface for the case A in Fig. 8.

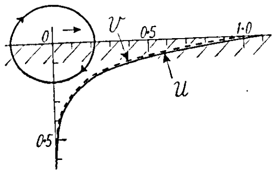

Fig. 11. Distribution of displacements with depth and orbital motion at the upper surface for the case C in Fig. 8.

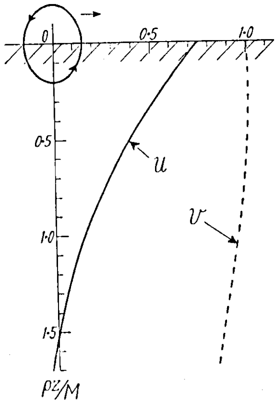

Fig. 10. Distribution of displacements with depth and orbital motion at the upper surface for the case B in Fig. 8.

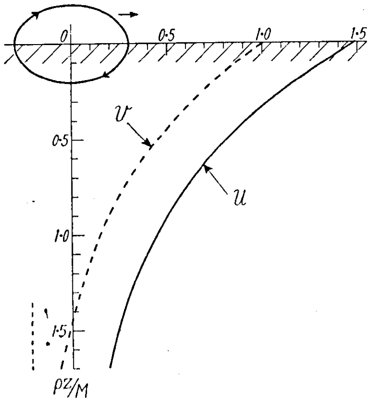

Fig. 12. Distribution of displacements with depth and orbital motion at the upper surface for the case D in Fig. 8.

in Figs. 9∼12. The types of orbital motions of the surface for the conditions corresponding to A, B, C, D are also shown in the same figures.

It will be seen that the orbital motion of the waves corresponding to the second dispersion curve has the same sense as that of the gravitational waves, and the orbital motion of the waves corresponding to the first dispersion curve is reversed. In this case, too, the waves with higher velocity are similar in type to gravitational waves, while the reverse condition holds for waves with lower velocity.

From the present problem, it may be possible to ascertain the reason for there generally existing two dispersion curves for Rayleigh-waves that are transmitted through a stratified body.

4. *Notes on Love-waves in a semi-infinite body on which concentrated masses are distributed.*

The necessary condition for Love-type waves being transmitted is that there shall exist, at least, one surface layer of certain rigidity and density. If, in lieu of it, masses are distributed on the surface of a semi-infinite body, then it is also possible for Love-type waves to be transmitted.

In this case, the solution and the boundary condition are such that

$$v=Ae^{sz+i(pt-fx)}, \tag{11}$$

$$\mu\frac{\partial v}{\partial z}=M\frac{\partial^2 v}{\partial t^2}, \tag{12}$$

where

$$s^2=f^2-k^2, \quad k^2=\rho p^2/\mu, \tag{13}$$

from which it is possible to determine the velocity equation, its form being

$$\sqrt{1-\frac{k^2}{f^2}}-\frac{k^2}{f^2}\frac{fM}{\rho}=0. \tag{14}$$

Using this equation, we obtain the dispersion curve shown in Fig. 13.

It will be seen that, in the present case, only one dispersion curve exists. Since, furthermore, the displacement is horizontal and normal to the direction of wave transmission, there is no need to discuss the orbital motion.

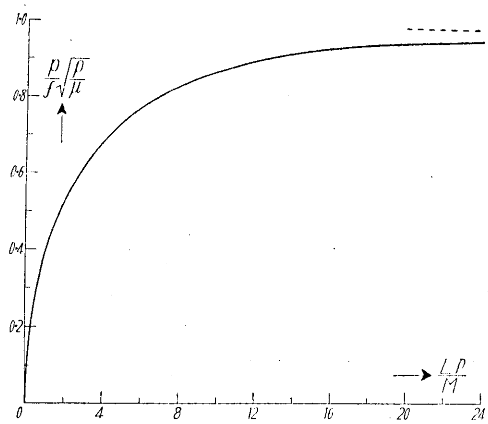

Fig. 13. Dispersion curve for Love-waves through a body with masses distributed on its upper surface.

Comparing the results in Fig. 13 with those in Fig. 7, it will be seen that the velocity of Love-waves is intermediate between the velocities of the two kinds of Rayleigh-waves indicated by the two dispersion curves. There is then the likelihood that the seismic disturbance which corresponds to bodily waves contains Rayleigh-waves that are represented by the second dispersion curve just given. Since the velocity of these Rayleigh-waves is less than that of transverse bodily waves, it is probable that the wave motion corresponding to the second preliminary tremor is modified to a certain extent, that is to say, the same wave motion is not only orientated transversely, but is combined with orbital movements like gravitational waves.

5. *Summary and concluding remarks.*

From mathematical investigation it is ascertained that the two kinds of dispersive Rayleigh-waves, which we previously found, differ entirely in the nature of their orbital motions. The waves corresponding to the first dispersion curve are transmitted with an orbit of sense opposite to that of gravitational waves, while those corresponding to the second dispersion curve, do so with an orbit of the same sense as that of gravitational waves.

That there is such a fundamental difference between the two kinds of dispersive Rayleigh-waves is also confirmed by treating the orbital motion of Rayleigh-waves in a semi-infinite body on which concentrated masses are distributed. It is shown that, in the latter condition of the body, there are two dispersion curves for Rayleigh-waves, one curve being in the range from zero to infinity in wave length and another in a narrow range, namely, in that of relatively small wave lengths. The orbital motion of the waves corresponding to the second dispersion curve is of the same sense as that of gravitational waves, whereas the orbital motion of waves corresponding to the first dispersion curve is reversed. These features agree well with those in the case of dispersive Rayleigh-waves that are transmitted through a stratified body.

After treating the problem of Love-waves in a semi-infinite body on which concentrated masses are distributed, it was found that the velocity of the Love-type waves in question is intermediate between the velocities of the two kinds of Rayleigh-waves indicated by the two dispersion curves last given, as a result of which it may be that transverse bodily waves partly contain Rayleigh-waves (of orbital motion of the same sense as that of gravitational waves) that correspond to the second dispersion curve.

1. 分散性レーレー波の波點運動方向

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數學的研究から，我々が以前に見出した２種の分散性レーレー波はその波點の運動方向に於て互に異なるものである事がわかつた．第１の分散曲線に相當する波の波點運動方向は重力波の場合とは逆の向きになるし，第２の分散曲線に相當する波のそれは重力波の場合と同じ向きになる．第２のものが第１のものよりも速度が高いから，レーレー波の始まりの方は重力波と同じ運動をなし，終りの方は反對の運動をなす事にもなる．

この様に２種のレーレー波に性質の根本的相違があるといふ事は，半無限固體の表面上に質量が分布する場合の計算を試みる事によつて更によく確められる．即ち，このやうな半無限固體の場合にも２個の分散曲線があり，一方の曲線は波長が零から無限大までの範圍にあるし，他の曲線は波長の短いある範圍にのみある．最初の分散曲線に相當する波の波點運動は重力波の場合とは逆向きであるし，第２の分散曲線に相當する波の波點運動は重力波の場合と同じになる．即ち表面層のあるときの分散性レーレー波の性質とよく一致してをる譯である．

半無限體の表面に質量が分布する場合のラブ波の問題を取扱つた結果，このラブ波の速度は２種のレーレー波の速度の中間にある事がわかつた．その結果として固體横波の中には第２の分散曲線に相當するレーレー波をも含み得る事がわかつた．