45. *On Shallow Water Waves Transmitted in the Direction Parallel to a Sea Coast, with Special Reference to Love-waves in Heterogeneous Media.*

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1. *Possible transmission of shallow water boundary waves in the direction parallel to a sea coast.*

A few years ago[1] we showed that, although it is possible for seiches to exist in an epicontinental sea, these seiches decay very rapidly owing to scattering of wave energy into the outer sea. It was only recently that our attention was called to the fact that Proudman[2] had also arrived at the same conclusion after dealing with the problem in a totally different way. Whether in Proudman's problem or in ours, since the momentum of the seiches in the epicontinental sea either comes from the outer sea or is imparted to it, it is impossible for the seiches under consideration to remain undamped unless the depth of the outer sea just outside the epicontinental sea is very great.

On the other hand, Goldsbrough[3] showed the possible existence of seiches in a shallow sea surrounding a circular island, the momentum of the waves in this case being imparted in the sense along the circumference of that island. This suggested to us the probability of shallow water waves in an epicontinental sea being transmitted in the direction parallel to the coast. The waves in question must then be of boundary type, that is to say, the amplitudes of the waves would be discernible only near the coast, otherwise the whole energy of the waves would be infinitely great, which condition is physically inconsistent. Since the waves in the present paper are, as a matter of fact, of boundary type, scattering can scarcely occur, as the results of which it follows that the waves, if formed, will never be damped.

A mathematically sufficient (and not necessary) condition that the waves shall be of boundary type is that the form of the basin of the

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1) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **13** (1935), 476.

2) J. PROUDMAN, *M. N. R. A. S., Geophys. Suppl.*, **1** (1925), 247.

3) G. R. GOLDSBROUGH, *ibid.*, 4 (1939), 404.

epicontinental sea shall differ from that of the outer sea. In the present paper, we shall specially consider two cases, namely, that in which the depth of the epicontinental sea and the depth of the outer sea are both uniform, and that other in which, the depth of the epicontinental sea being uniform, the depth of the outer sea, on the other hand, increases as the square of the distance from the coast. Although the latter condition is rather improbable in an actual sea, it is still possible for its calculation to show the features of waves in a sea of varying depth. Although there are a number of other cases that are actually probable, since the treatment of the problem is fairly complicated, they will be dealt with on another occasion.

2. *The case in which the depth of the epicontinental sea and that of the outer sea are both uniform.*

Let l, h'; h be the breadth and depth of the epicontinental sea and the depth of the outer sea, respectively. Let also the axes of x and y be parallel to and perpendicular to the coast line. If ζ' and ζ be the surface elevations of the waves in the epicontinental sea and the outer sea, we have

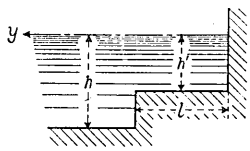

Fig. 1.

$$\frac{\partial^2\zeta'}{\partial t^2}=gh'\left(\frac{\partial^2\zeta'}{\partial x^2}+\frac{\partial^2\zeta'}{\partial y^2}\right), \qquad \frac{\partial^2\zeta}{\partial t^2}=gh\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right). \qquad (1), (2)$$

The solutions of (1) and (2) that correspond to the shallow water waves of boundary type are

$$\zeta'=e^{i(pt+fx)}(A\cos s'y+B\sin s'y), \qquad \zeta=Ce^{i(pt+fx)-sy}, \qquad (3), (4)$$

where $s'^2=p^2/gh'-f^2$, $s^2=f^2-p^2/gh$.

The components of the displacement velocity of the waves in x and y directions are expressed by

$$\frac{\partial u'}{\partial t}=-g\frac{\partial\zeta'}{\partial x}, \qquad \frac{\partial v'}{\partial t}=-g\frac{\partial\zeta'}{\partial y} \qquad (5), (6)$$

for the epicontinental sea; similar expressions hold also for the outer sea.

The boundary condition at the coast line is

$$v'=0 \qquad (7)$$

at $y=0$. The conditions satisfying the boundary between the epicontinental sea and the outer sea are that the surface elevation and displacement velocity in x-direction shall both be continuous, besides the continuity of flux of the water in y-direction, from which it follows that

$$\zeta'=\zeta\,,\quad u'=u\,,\quad v'h'=vh \qquad (8), (9), (10)$$

at $y=l$. The values of ζ, u, v naturally vanish at $y\rightarrow\infty$.

Substituting (3), (4) in (7), (8), (9), (10), we get the relation

$$\tan\sqrt{p^2/gh'-f^2}\,l=\frac{h\sqrt{f^2-p^2/gh}}{h'\sqrt{p^2/gh'-f^2}}\,, \qquad (11)$$

from which it is possible to get the velocity of transmission of the waves. Equation (11) is of the same form as that for the velocity equation of Love waves trasmitted through a surface layer of thickness l and rigidity proportional to h', resting on a subjacent medium of rigidity proportional to h, the densities of both media being equal. It holds then that if h' were less than h, waves of boundary type would invariably be transmitted through the epicontinental sea in the direction parallel to the coast line. If, on the other hand, h' were greater than h, it would be impossible for boundary waves to exist. Although, by ordinary reasoning, it may appear that the waves are transmitted in the case that h is less than h', mathematical relation (11) indicates that the condition of the possible transmission is the reverse.

Since expression (11) is the same as the velocity equation of Love-waves, we shall not here solve that expression. The condition of the problem in the case of Love-waves applies immediately to the present case. For example, the velocity of transmission of the extremely short waves is expressed by $\sqrt{gh'}$ and that of the extremely long waves by $\sqrt{gh}$. At all events, the energy of the waves accumulates near the epicontinental sea or even near the coast.

3. *The case in which the depth of the epicontinental sea is uniform and that of the outer sea increases as the square of the distance from the coast.*

Let l and h' be the depth and breadth of the epicontinental sea. We shall assume that the axes of x and y are drawn in the same sense as those mentioned in the preceding section, and that the depth of the outer sea varies as

$$h=ay^2\,, \qquad (12)$$

so that the depth at the boundary of the outer sea is $h=al^2$. The surface elevations of the epicontinental sea and the outer sea satisfy the equations

$$\frac{\partial^2\zeta'}{\partial t^2}=gh'\left(\frac{\partial^2\zeta'}{\partial x^2}+\frac{\partial^2\zeta'}{\partial y^2}\right), \tag{13}$$

$$\frac{\partial^2\zeta}{\partial t^2}=g\left\{h\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial}{\partial y}\left(h\frac{\partial\zeta}{\partial y}\right)\right\}. \tag{14}$$

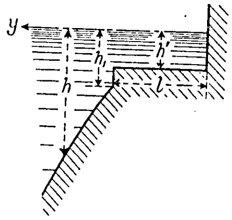

Fig. 2.

The horizontal and vertical components of the displacement velocity in the outer sea are determined by

$$\frac{\partial u}{\partial t}=-g\frac{\partial\zeta}{\partial x}, \qquad \frac{\partial v}{\partial t}=-g\frac{\partial\zeta}{\partial y}. \tag{15), (16}$$

The boundary conditions are such that

$$v'=0 \tag{17}$$

at $y=0$ and

$$\zeta'=\zeta, \quad u'=u, \quad v'h'=vh \tag{18), (19), (20}$$

at $y=l$. The values of ζ, u, v at $y\to\infty$ tend to vanish.

The solution of (13) is the same as that of (1). To solve (14), we write $\zeta=\zeta_1e^{i(pt+fx)}$, when (14) changes to

$$\left(\frac{p^2}{g}-f^2h\right)\zeta_1+\frac{\partial}{\partial y}\left(h\frac{\partial\zeta_1}{\partial y}\right)=0. \tag{21}$$

Since $h=ay^2$, (21) transforms to

$$\left(\frac{p^2}{ga}-f^2y^2\right)\zeta_1+2y\frac{\partial\zeta_1}{\partial y}+y^2\frac{\partial^2\zeta_1}{\partial y^2}=0. \tag{22}$$

Putting

$$\zeta_1=y^{-1}Z, \qquad 2fy=\rho, \qquad \frac{p^2}{ga}=\frac{1}{4}-m^2, \tag{23), (24), (25}$$

we get the equation

$$\frac{d^2Z}{d\rho^2}+\left(-\frac{1}{4}+\frac{\frac{1}{4}-m^2}{\rho^2}\right)Z=0, \tag{26}$$

whose solution, which converges at $\rho=\infty$, is of the type

$$Z = W_{k,m}(\rho) = \frac{\Gamma(-2m)}{\Gamma\left(\frac{1}{2}-m-k\right)} M_{k,m}(\rho) + \frac{\Gamma(2m)}{\Gamma\left(\frac{1}{2}+m-k\right)} M_{k,-m}(\rho), \quad (27)$$

where $2m$ is not an integer, and $k=0$. This is the confluent hypergeometric function due to Whittaker.[4] The values $M_{k,m}(\rho)$ and $M_{k,-m}(\rho)$ are of the forms[5]

$$\left.\begin{aligned} M_{k,m}(\rho) &= \rho^{\frac{1}{2}+m} e^{-\frac{\rho}{2}} M\left(\frac{1}{2}+m-k,\ 2m+1,\ \rho\right), \\ M_{k,-m}(\rho) &= \rho^{\frac{1}{2}-m} e^{-\frac{\rho}{2}} M\left(\frac{1}{2}-m-k,\ -2m+1,\ \rho\right), \end{aligned}\right\} \quad (28)$$

in which the numerical values of M were first obtained by Airey and Webb[6] and examined more closely by Nicholson[7] and others. These results were also shown graphically in Jahnke and Emde's Tables (latest edition)[8].

Substituting (27), (23), (24) in (17), (18), (19), (20), it is possible to get the velocity equation for the transmission of the waves under consideration, the results of which are

$$\frac{1}{\nu}\sqrt{\left(\frac{1}{4}-m^2\right)\nu-\frac{\rho_1^2}{4}}\tan\sqrt{\left(\frac{1}{4}-m^2\right)\nu-\frac{\rho_1^2}{4}} = \frac{\left(1+k-\frac{\rho_1}{2}\right)W_{k,m}(\rho_1)+W_{k+1,m}(\rho_1)}{W_{k,m}(\rho_1)}, \quad (29)$$

where $\nu=h_1/h'(=al^2/h')$, $k=0$, and $\rho_1=2fl$; h_1 being the depth of the outer sea at $x=l$. It should be borne in mind that, in treating $W_{k,m}$, we used the recurrence formula such that

$$W'_{k,m}(\rho) = -\frac{1}{\rho}\left\{W_{k+1,m}(\rho)+\left(k-\frac{\rho}{2}\right)W_{k,m}(\rho)\right\}, \quad (30)$$

as obtained by Goldstein.[9]

Using the expression in (29) we calculated the velocity of transmission of the waves for various ratios of $\nu=h_1/h'$ and $\rho_1=2fl$; $2\pi/f(=L)$

4) E. T. WHITTAKER and G. N. WATSON, *Modern Analysis*, (1920), 346.
5) *ibid.*, 337.
6) H. A. WEBB and J. R. AIREY, *Phil. Mag.*, **36** (1918), 129—141
7) *Brit. Assoc. Rep.* (1926), 276.
8) JAHNKE u. EMDE, *Funktionentafeln*, (1938), 275.
9) S. GOLDSTEIN, *Proc. London Math. Soc.*, **34** (1932), 110.

being the wave length. Since the intervals of the numerical values shown in the tables just given are too far apart, the method of trial and error was resorted to. To be more precise, we first assumed tentative values of $m=\sqrt{1/4-p^2/ga}$, $\rho_1=2fl$; then from (29) we found the corresponding value of $\nu=h_1/h'$. It is then possible to get the velocity of the waves as follows:

$$V=\frac{p}{f}=\sqrt{\left(\frac{1}{4}-m^2\right)\frac{4}{\rho_1^2}\frac{h_1}{h'}gh'}$$

$$=\sqrt{\frac{(1-4m^2)\nu}{\rho_1^2}gh'}. \qquad (31)$$

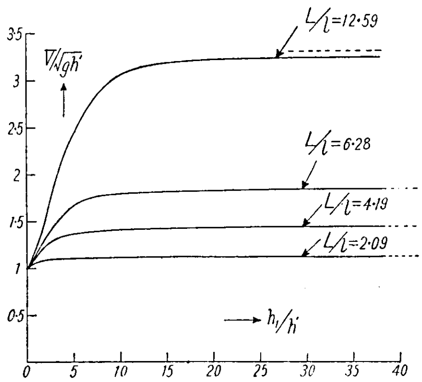

Fig. 3. Case $L/l=12{\cdot}59$ and parts of cases $L/l=6{\cdot}28$, $4{\cdot}19$ were obtained in the usual way; case $L/l=2{\cdot}09$ was determined by the method of asymptotic expansion.

The results of calculation for various ratios of $\nu=h_1/h'$ for the cases $L/l=12{\cdot}59$, $6{\cdot}28$, $4{\cdot}19$, $3{\cdot}142$, $2{\cdot}093$, $1{\cdot}256$ are shown in Figs. 3 $\sim$7. The ordinates in these figures represent the velocity of transmission of the waves. For obtaining the results in Figs. 4$\sim$7, we used the asymptotic expansion of the function $W_{k,m}(\rho)$ such that

$$W_{k,m}(\rho)\sim e^{-\frac{1}{2}\rho}\rho^k$$

$$\cdot\left\{1+\sum_{n=1}^{\infty}\frac{\left\{m^2-\left(k-\frac{1}{2}\right)^2\right\}\left\{m^2-\left(k-\frac{3}{2}\right)^2\right\}\cdots\left\{m^2-\left(k-n+\frac{1}{2}\right)^2\right\}}{n!\,\rho^n}\right\}. \qquad (32)$$

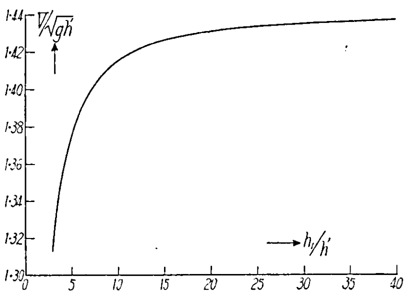

Fig. 4. $L/l=4{\cdot}197$. Calculated by the method of asymptotic expansion.

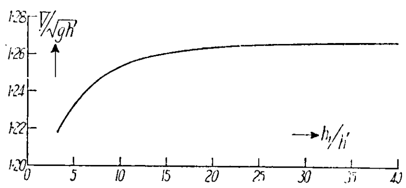

Fig. 5. $L/l=3{\cdot}142$. Calculated by the method of asymptotic expansion.

On the other hand, the asymptotic values of the oridinates for $\nu=h_1/h'\to\infty$ were determined by the condition of the waves of the kind shown in Section 2. The asymptotic values for cases $L/l=12{\cdot}59$, $6{\cdot}28$, $4{\cdot}19$, $2{\cdot}094$ are $3{\cdot}3$, $1{\cdot}862$, $1{\cdot}449$, $1{\cdot}129$, respectively. Although the ratios of h_1/h' at which the slopes of the respective curves are maximum differ with the kind of curve, those ratios are all nearly within the range $h_1/h'=2\sim5$.

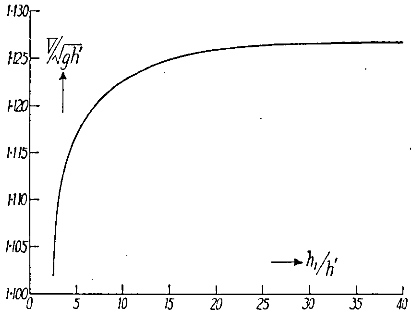

Fig. 6. $L/l=2{\cdot}093$. Calculated by the method of asymptotic expansion.

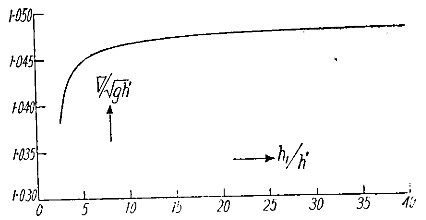

Fig. 7. $L/l=1{\cdot}216$. Calculated by the method of asymptotic expansion.

It is now desirable to get the dispersion curves of the waves in the usual sense, to attain which we selected the ordinates of different curves at the given abscissae, $h_1/h'=1$, 5, 30, in Figs. 3∼7, and replotted them as shown in Figs. 8. It will be seen that every dispersion curve is of normal type. Although the velocity of transmission of the shortest possible waves is $\sqrt{gh'}$ for any ratio of h_1/h', that for longer waves increases without limit with increase in wave length. The greater the ratio of h_1/h', the greater the steepness of every dispersion curve.

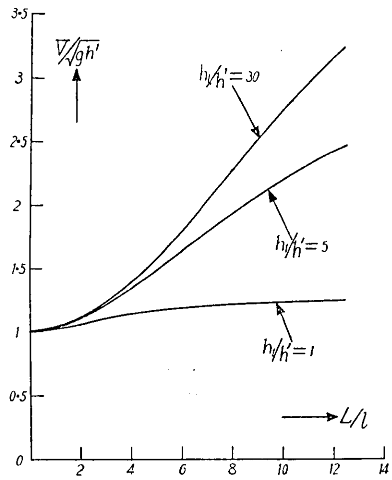

Fig. 8. Dispersion curves for three cases of h_1/h' (=1, 5, 30).

4. *Notes on Love-waves in heterogeneous media.*

We have already shown that the case in which the depth of the epicontinental sea and that of the outer sea are both uniform, is analogous to that of Love-waves transmitted along layers of uniform densities and uniform elasticities. A close examination of the problem in Section 3 shows that the case of varying water depth also corresponds to that of Love-waves transmitted in some heterogeneous media.

Let densities of the upper and subjacent layers be equal, and let elastic constant in the subjacent medium vary as y^2; the equations of motion of both media will then be

$$\frac{\partial^2\zeta'}{\partial t^2}=\frac{\mu'}{\rho}\left(\frac{\partial^2\zeta'}{\partial x^2}+\frac{\partial^2\zeta'}{\partial y^2}\right), \quad \frac{\partial^2\zeta}{\partial t^2}=\frac{\mu}{\rho}\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial}{\partial y}\left(\frac{\mu}{\rho}\frac{\partial\zeta}{\partial y}\right), \qquad (33), (34)$$

where ζ', ζ are horizontal displacements. The boundary conditions are such that

$$\partial\zeta'/\partial y=0 \qquad (35)$$

at $y=0$ and

$$\zeta'=\zeta\,, \quad \mu'\partial\zeta'/\partial y=\mu\partial\zeta/\partial y \qquad (36), (37)$$

at $y=l$. If we write

$$\frac{\mu'}{\rho}=gh', \quad \frac{\mu}{\rho}=gh(=gay^2), \quad \frac{\partial\zeta'}{\partial y}=-\frac{1}{g}\frac{\partial v'}{\partial t}, \quad \frac{\partial\zeta}{\partial y}=-\frac{1}{g}\frac{\partial v}{\partial t}, \qquad (38)$$

equations (33), (34), (35), (36), (37) become of the same forms as equations (13), (14), (17), (18), (20), respectively, from which the treatment of the elasticity problem is virtually the same as that of the shallow water problem in Section 3. The velocity equation for Love-waves can be obtained by replacing h_1/h', gh', m^2 in (31) by μ_1/μ', μ'/ρ, $1/4-p^2/ga$, respectively, μ_1 being the value of μ at $y=l$.

It is thus possible for Figs. 3~7 to represent the wave velocities of Love-waves for various ratios of μ_1/μ' provided that $\sqrt{gh'}$, h_1/h' are replaced by $\sqrt{\mu'/\rho}$, μ_1/μ', respectively. With the same condition, Fig. 8 indicates the dispersion curves for Love-waves.

In the same manner, the result of calculation for Love-waves transmitted through any heterogeneous media applies to the problem of the tidal waves of the present kind; it is also possible for the reverse condition to hold.

It might now be mentioned that the criterion for the possible existence of Love-waves (also that of shallow water waves of the present kind) is not very simple. As far back as 1931[10] one of us gave the solution for a kind of waves transmitted through a semi-infinite body of varying elasticity. Although the same waves showed the character of surface type for some wave lengths, it is scarcely possible for the waves to be invariably of surface type for any wave length. It is

10) K. SEZAWA, "A Kind of Waves transmitted over a Semi-infinite Solid Body of Varying Elasticity", *Bull. Earthq. Res. Inst.*, **9** (1931), 310~315.

well known that a convenient condition for the transmission of pure Love-waves, is that any discontinuous surface layer shall exist. On the other hand, for special distributions in density as well as in elasticity, even should there be no surface layer, transverse waves (not Love-waves) of nearly surface type possibly exist. Meissner[11] indicated the possible existences of the two kinds of transverse waves in question. Thus, the surface waves in our previous calculation also correspond to those in one of Meissner's examples. As a matter of fact, although the transverse waves that we obtained were of surface type, we never assumed that they immediately concern Love-waves.

There still remains some uncertainty with respect to the criterion for tansverse waves being of surface type. The criterion would be either that the amplitude of the waves at a great depth is infinitesimal, or that the total wave energy integrated through the whole depth is finite. Although the waves in one of Meissner's examples and those in our previous calculation may satisfy the condition that the amplitude tends to diminish at a great depth, it is not certain whether or not they fulfil, besides, the condition for the wave energy just mentioned. From our recent ideas, on the other hand, waves of the surface type satisfy both the conditions in a strict sense, for which reason, in order to show the damping of the periodic waves of Love-type,[12] we specially selected such a case of transverse waves (in plane problems) as is transmitted through discontinuous stratified layers.

5. *Concluding remarks.*

From mathematical calculation we found that it is possible for shallow water waves to be transmitted through an epicontinental sea in the direction parallel to the coast line. The momentum of the waves is also transmitted in the sense of that along the coast. Since scattering of wave energy scarcely occurs, the waves, once formed, are transmitted without any decay, provided no viscous or other frictional damping forces are in operation. Besides, the dispersion of the waves is normal.

Since the velocity equation of the waves is quite similar to that of Love-waves, it is possible that the results of the present calculation for complicated cases may also apply to calculations for similar cases of Love-waves.

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11) E. MEISSNER, "Elastischen Oberflächen Querwellen", *Verh. 2. int. Kongr. f. tech. Mech.* 1926 , 3~11.

12) K. SEZAWA and K. KANAI, "Damping of Periodic Visco-Elastic Waves . . , II", *Bull. Earthq. Res. Inst.*, **17** (1939), 12~25.

It may be doubted whether shallow water waves of the kind assumed here really do exist. As a matter of fact, transmission of sea waves in a sense parallel to the coast is scarcely observable. Since the frictional force of the bottom of a shallow sea could always be very large and since, furthermore, excitation of the waves at such a shallow part of the sea is highly improbable, it would be impossible actually to observe such waves.

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45. 海岸線に沿うて傳播する淺海境界波 附報 不均質層のラブ波に就いて

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金井清

數學的計算によつて海岸線に沿うて傳播し得る陸棚の淺海波のある事がわかつた．海岸線に直角に來たり出たりする淺海波の場合にはモメンタムもその方向に働き勢力の逸散があるけれども只今の波ではモメンタムが海岸に平行に働き振幅の減衰がなく，そのまま傳播するのである．只今の計算では陸棚より外の深海の底が一定の場合と海岸からの距離の2乘に比例する場合との兩方を計算したが，何れも計算が割合に早くできるものを選んだのに過ぎない．しかし之から定量的性質，卽ち海岸に沿うて振幅の變形しない波の理論的存在性がわかつたのである．

只今の計算を考へるとラブ波の計算の行き方と殆ど同じである．故に只今の計算の結果は數量の定義を變へればその儘ラブ波の場合にもつて行くことができる．又，他の種類のラブ波の計算を同樣に只今の問題の例に作り直す事もできる．從て只今の波の存在性を疑ふ事はラブ波の存在性を否定するのと同じことになる譯である．

只今出したやうな淺海波は實際には觀測でき難いやうである．之は恐らく淺海底の摩擦が大きい爲に波が減衰するか，又はそれが海岸に平行に進み得るやうな風力狀態が餘りあり得ないからかも知れぬ．

終りにラブ波の成立の條件について二三の議論を試みた．
