51. *Velocity of Elastic Waves in a Granular Substance.*

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1. Introduction.

In his previous papers[1] the writer discussed the physical properties of sand, soils, and rocks that are most intimately related to the propagation of elastic waves in the earth's crust. The results of these studies showed that the elastic properties of these substances are greatly affected by the water contained in them, the closeness of packing, porosity, and the size of the particles that form the substances, and the amount of binding material which they contain. A number of the substances that compose the earth's crust have a granular structure, either microstatic or macrostatic. Sand is a typical example of a macroscopic granular substance. As already explained in the previous paper,[2] a mass of sand may be regarded as an elastic substance, seeing that the elastic waves are propagated through the aggregate which is composed of many sand grains. Moreover it was found that elastic waves seem to pass with greater velocity through fine grained sand than through coarse grained sand. This study is a continuation of the previous ones.

Although, in order to fully investigate the phenomenon attending the passage of elastic waves through the substances just mentioned, it is necessary to consider the structure of the substance as well as its elastic properties, it is not possible to attain this object with substances (or granular substances) found in the natural state, because their structure may not strictly comply with the necessary conditions. We therefore decided to study the properties of granular masses that we had prepared artificially, the internal structure and the closeness of packing of which are macroscopically known. For this purpose we selected for

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1) M. ISHIMOTO and K. IIDA, *Bull. Earthq. Res. Inst.*, **14** (1936), 632; **15** (1937), 671.

K. IIDA, *Bull. Earthq. Res. Inst.*, **15** (1937), 838; **16** (1938), 131; 991; **17** (1939), 59; 79.

2) K. IIDA, "The Velocity of Elastic Waves in Sand," *Bull. Earthq. Res. Inst.*, **16** (1938), 131~144.

our study readily accessible material, such as rubber and lead, the shape of whose individual particles are spherical and of small, uniform diameter. Since, moreover, we had selected sands with round grains, we should thus be in a position to know how these granular masses will behave when elastic waves pass through them and to ascertain how the foregoing elements, that is, the size of grain, the closeness of packing, moisture content, and the cementing material that bind the individual particles, will be affected.

2. Elastic Wave-Velocity and Grain-Size.

(a) *Granular masses.*

In order to observe the effect of the size of grain of a granular substance on elastic wave-velocity, three kinds of granular substances of uniformly sized spherical particles, namely, rubber, lead, and sand were selected. The dimensions of the tiny spherical particles and the component elements of these three substances are shown in Table I. In the present investigation, tiny spheres of rubber and lead ranging from

Table I. Dimensions of *Grains.*

Substance	No.	Diameter	Density	Component	
Rubber *grain*	1	0·29 cm	1·229	Smoked sheet, FAQ	Wt 90·9%
	2	0·41	1·228	Sulphur	2·7%
	3	0·64	1·230	Stearic acid	2·7%
	4	0·93	1·227	Accel. D. M.	0·8%
	5	1·20	1·233	Nocceler D.	0·2%
Solid rubber block		3·90	1·238	Lead white	2·7%
Lead *grain*	1	0·30	11·32		
	2	0·48	11·33	Pb	99·96%
	3	0·65	11·31	As	trace
	4	0·87	11·32	Fe	trace
	5	1·02	11·30		
Solid lead block		2·50	11·30		
				grain-size	grain-size
Sand *grain*	1	0·030	2·70	0·30 mm 98%,	0·15 mm 2%
	2	0·059	2·69	0·59 mm 97%,	0·30 mm 3%
	3	0·117	2·71	1·17 mm 98%,	2·30 mm 2%

about 3 mm to 1·2 cm diameter, for lack of a better word, will hereinafter be called *grains.*

We also prepared cylinders of solid rubber and lead of the same material as those of the *grains*. It will be seen from Table I that these substances are comparatively pure.

(*b*) *Methods of experiments.*

The methods of the present experiments as well as the apparatus used were the same as those used in studying the velocity of elastic waves in sand. To make the cylindrical specimens we used at first a cellophane tube, 0·024 mm thick, as before, to hold the *grains*.

As already explained, vibrations of various torsional and longitudinal frequencies are imparted to the foot of the specimen of the granular substance as well as the solid block where it touches the vibrating plate, and the fundamental resonance frequency thus ascertained, after which the velocities of the elastic waves in it are computed.

The velocities of the elastic waves were also measured with a thin elastic rubber-tube, 0·02 mm thick, and a sort of paper-tube, 0·03 mm thick. When, at times, these receptacles were not used, the experiments were carefully carried out under no boundary conditions, the schematic arrangements of the *grains* being shown in Fig. 1. Within the limits of experimental accuracy, the results of our experiments are almost the same as those in the case of our cellophane-tube experiments.

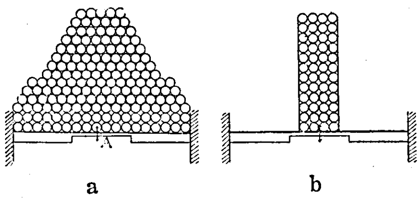


Fig. 1. Schematic arrangements of *grains* on the vibrating plate.

A Vibrating plate.
a Without boundary wall.
b With cellophane wall.

The effect of the cellophane wall on the wave-velocity was also noted, to observe which effect we used cellophane cylinders of various diameters. The curves in Fig. 2 show the relation between the resonance frequency and the diameter of the cellophane cylinder. The values of the resonanace frequencies in the case of no boundary wall are also plotted as ordinates. In all these cases, great care is exercised to see that the closeness with which the *grains* are packed, that is, the porosity is kept constant. In this case, as will be seen from the figures, so far as the present experiments are concerned, the values of the frequency are almost constant. Thus, since the resonance frequencies of vibration are independent of the diameter, the results of our experiments may practically be extended to the case of no boundary wall, but in the case of no wall, the *grains* are apt to be crumbled and unstable. Therefore, for convenience in dealing with a granular mass, cellophane tubes, 4 cm diameter, were always used for packing the *grains*.

In order to observe the effect of size of *grain* on the velocity of the elastic wave, we experimented with various forms of the *grains*, the diameters of which are given in Table I. First, we repeated the experiment with cylinders of various heights of the granular substance, the diameter being constant. Second, the diameters of the individual particles of the granular substance were varied, but with their heights constant. Third, we repeated the experiment by changing the closeness of packing for the various kinds of *grains*. Fourth, the effect of water on the velocity was investigated by using wet *grains* packed in the receptacles mentioned. These last two experimental procedures will be described in the next section. Besides, we determined the velocity of the wave through a tall solid cylinder of the same material as those of the *grains* by subjecting it to vibration, since the elastic constants would be concerned with those of the individual *grains*. The size of the solid lead cylinder was 2·5 cm diameter and 50 cm long, and of the solid rubber cylinder 3·9 cm diameter and 30 cm long.

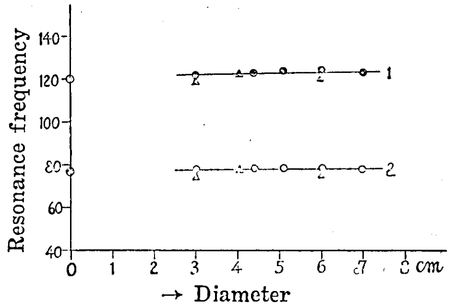


Fig. 2. Relation between the resonance frequency of vibration and the diameter of cellophane cylinder.
1 shows the case of longitudinal vibration.
2 shows the case of torsional vibration.
●, ○ Cellophane cylinder.
▲, △ Rubber-tube.
◉, ◎ No wall.
Material of sample is rubber. $h=15$ cm.

(c) *Results of the experiments.*

An example of a diagram showing the fundamental longitudinal and torsional resonance periods T_l, T_t, and the height of the granular substance, h, may be seen in Fig. 3, in which the period T_l or T_t is taken as ordinate and the height h as abscissa.

Since, as already found, the successive resonance frequencies of higher orders are about an odd number of times that of the fundamental resonance frequency, we determined the velocities of the longitudinal and the torsional waves V_l, V_t by means of granular substance, using the simple formula

$$V_l=\frac{1}{T_l}\lambda, \qquad V_t=\frac{1}{T_t}\lambda,$$

$$\lambda=4h,$$

but disregarding internal friction.

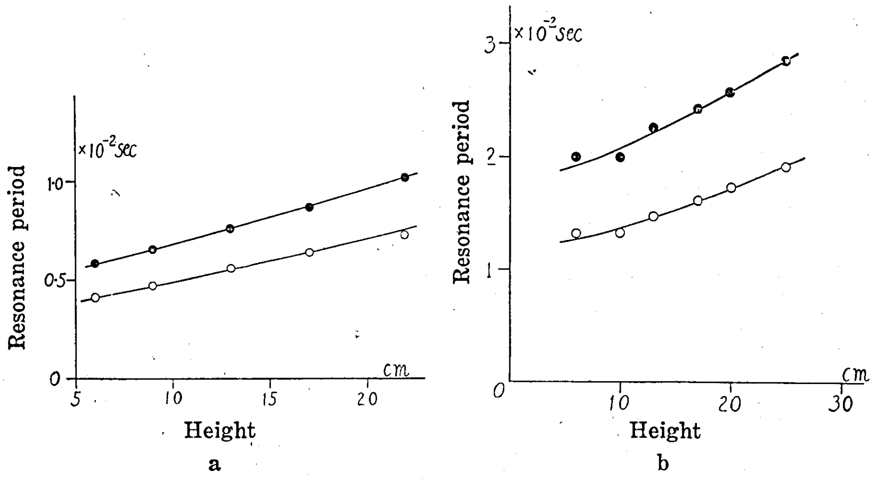


Fig. 3. Relation between the fundamental resonance period and the height of a granular substance.

● Torsional vibration.
○ Longitudinal vibration.
a Lead *grain*, diameter $d=0{\cdot}48$ cm.
b Rubber *grain* $d=0{\cdot}98$ cm.

Table II. V_l, V_t, and h of Cylinders of Various Kinds of *Grains*.

Lead *grain*, $d=0{\cdot}48$ cm.

No.	h	V_l	V_t
1	22·0 cm	123 m/sec.	86 m/sec.
2	17·1	106	78
3	13·0	93	68
4	9·0	76	55
5	6·0	59	41

Lead *grain*, $d=0{\cdot}87$ cm.

No.	h	V_l	V_t
1	20·0 cm	117 m/sec.	87 m/sec.
2	17·0	107	80
8	14·0	98	72
4	11·0	87	64
5	8·0	72	53

Lead *grain*, $d=0{\cdot}30$ cm.

No.	h	V_l	V_t
1	25·0 cm	125 m/sec.	87 m/sec.
2	19·0	109	76
3	15·0	99	68
4	10·0	80	55
5	6·0	56	38

Rubber *grain*, $d=0{\cdot}98$ cm.

No.	h	V_l	V_t
1	25·0 cm	52 m/sec.	35 m/sec.
2	20·0	46	31
3	17·0	42	28
4	13·0	35	23
5	10·0	30	20
6	6·0	18	12

Rubber *grain*, $d=0{\cdot}41$ cm.

No.	h	V_l	V_t
1	24·0 cm	50 m/sec.	34 m/sec.
2	19·0	44	30
3	15·0	39	26
4	11·0	31	21
5	9·0	27	17
6	6·0	18	12

The values thus obtained for each kind of granular substance are shown in Table II and in Figs. 4, 5.

From the curves in Figs. 4, 5

it will be seen that the velocity of the wave varies with the height of the specimen, that is, the greater the height the greater the velocity. Such variation in wave velocity as referred to height differs according to whether the tall cylinder is a solid homogeneous block or whether it is a cylinder whose material consists of more or less loose *grains*, the curves for which are shown in Figs. 4, 5, 6, the former being more concave to the abscissa than the latter.

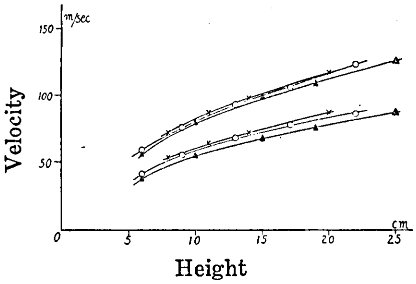
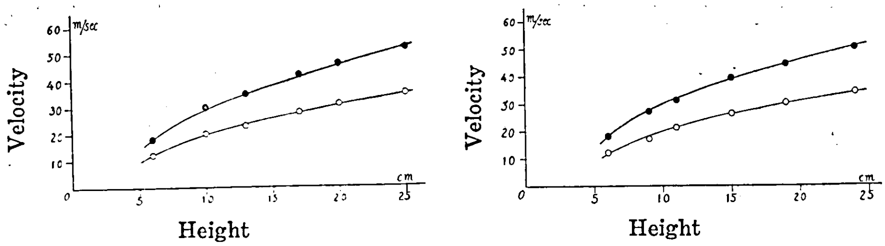


Fig. 4. Relation between the wave-velocity and the height of granular substance.
The upper three curves show the case of longitudinal wave.
The lower three curves show the case of torsional wave.
○ Diameter of lead *grain*, $d=0{\cdot}48$ cm.
▲ $d=0{\cdot}30$ cm. × $d=0{\cdot}87$ cm.

In order to observe the relation between wave-velocity through a granular substance and *grain*-size, the wave-velocity was plotted against the diameter of *grain*, as shown in Fig. 7, in which the diameters were taken as abscissa and the velocities of waves in the granular substance, 20 cm



a b

Fig. 5. Relation between the wave-velocity and the height of granular substance.
● Longitudinal wave.
○ Torsional wave.
a Diameter of rubber *grain*, $d=0{\cdot}98$ cm. b $d=0{\cdot}41$ cm.

height, as ordinate. As will be seen from these curves in Fig. 7 and Table III, the two kinds of wave-velocities are almost constant, as referred to the variation in diameter of the *grains* under identical conditions of packings—a phenomenon that was not observed in the case of sand masses under natural conditions, the results of which appeared in the previous paper.[3] There is a tendency however for the velocity

3) K. IIDA, *loc. cit.*, 2)

to increase slightly with increase in the diameter of the *grains*.

So far as the present experiments are concerned, our conclusion is

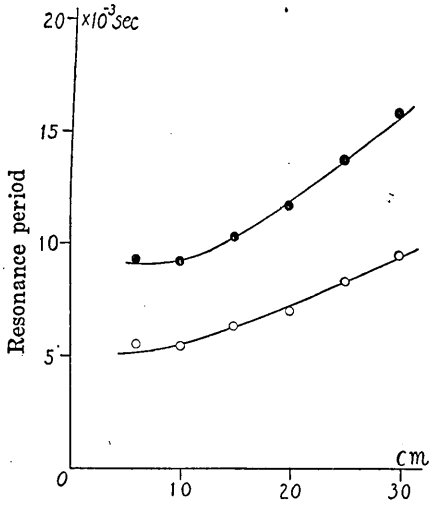


Fig. 6 a. Relation between the fundamental resonance period and the height of the solid rubber block.

● Torsional vibration.
○ Longitudinal vibration.

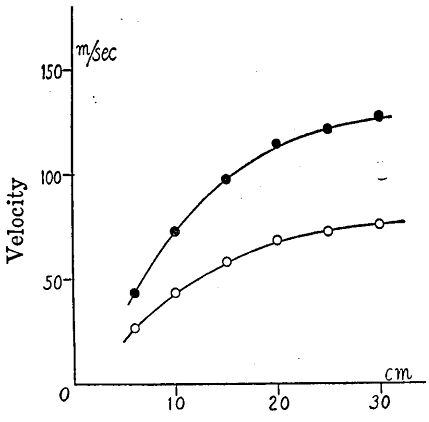


Fig. 6 b. Relation between the velocity of elastic wave and the height of the solid rubber block.

● Longitudinal wave.
○ Torsional wave.

Table III. Velocity of Elastic Waves in Mass of Various Kinds of *Grains*.

Substance	No.	Diameter	Wave-velocity	
			Longitudinal	Torsional
Rubber *grain*	1 2 3 4 5	0·29 cm 0·41 0·64 0·98 1·20	43·5 m/sec 45·5 45·2 46·0 52·0	28·2 m/sec 30·5 30·2 31·0 32·9
Solid rubber block		3·90	130·0	77·5
Lead *grain*	1 2 3 4 5	0·30 0·48 0·65 0·87 1·02	116·3 116·5 116·2 117·0 119·0	78·2 83·0 84·5 86·0 88·1
Solid lead block		2·50	1224·0	500·0
Sand *grain*	1 2 3	0·030 0·059 0·117	105·5 100·0 104·8	68·4 65·0 67·8

that the wave-velocities through a granular mass are much smaller than those through a column that is a solid homogeneous block.

(*d*) *Discussion.*

In the present study it is necessary to inquire how the elastic waves are propagated through those simplest aggregates of uniform spheres, so arranged that they support one another. Since, at any rate, the first thing to be noted is that the downward pull of gravity is the only force acting on the system after aggregation of the particles has been initially attained, this system may be initially in the strained state, so that if the repulsive force between the grains is small compared with that of the force due to gravity, the elastic waves should be propagated through the granular mass under the foregoing condition within the limits of the force at work on the system.

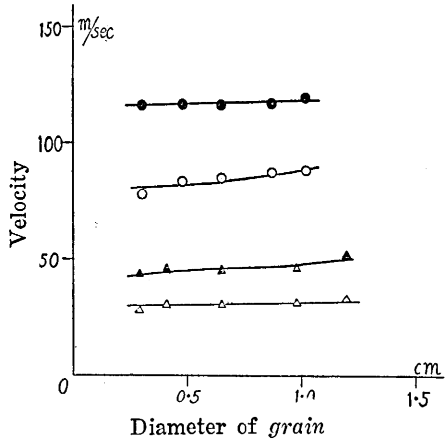


Fig. 7. Relation between the velocity of elastic waves and the diameter of *grains*.

●, ○ Lead *grain*.
▲, △ Rubber *grain*.
●, ▲ Longitudinal wave.
○, △ Torsional wave.

For a general idea of the phenomenon, we consider the ideal case of one-dimensional problem, in which an aggregate of uniform spheres, R in diameter, is arranged in a row with their centers along a straight line, the distance between the centers of the spheres being $2R$, so that each sphere along this line is tangent to its neighbours on either side.

When the diameter of the individual sphere is small compared with the wave length of an elastic wave, it is possible to take the elements, that is, the elasticity of the contacting parts of the spheres, the mass of the individual sphere, and the force acting on the spheres in the problem of the propagatiom of elastic waves, and to consider this system of aggregation as one in which the mass of every sphere is concentrated at its center of gravity, and imagine that the elasticity of the contacting parts of the sphere takes the form of the effective modulus of elasticity of an elastic spring, the mass of which spring is negligible.

The equation of motion for the longitudinal wave referred to this system is expressed by

$$m\frac{\partial^2\xi}{\partial t^2}=sl^2\frac{\partial^2\xi}{\partial x^2}, \tag{1}$$

in which ξ represents the displacement in the direction of the x-axis, s the effective modulus of elasticity of a spring of length l, m the mass of the sphere, and t the time. Therefore the velocity of propagation of a longitudinal wave V_l in the aggregate of spheres is given by

$$V_l=l\sqrt{\frac{s}{m}}. \tag{2}$$

With this formula, we can determine the wave-velocity by substituting the actual quantities of s and l; the last mentioned represents, in practice, the vertical distance from the horizontal plane that passes through the centers of the spheres of one layer to the corresponding plane of the next layer, in which, by "layer", is meant the parallel arrangement of aggregates of uniform spheres arranged with their centers along a straight line in the same plane (Fig. 8). As the quantity s depends on the elasticity of the spheres and the external force, it may be written

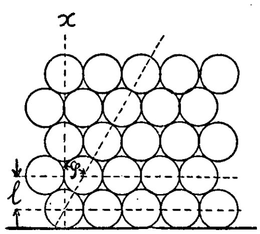


Fig. 8. Schematic arrangement of uniform sphere.

$$s=\frac{\delta F}{\delta\varepsilon}. \tag{3}$$

From a simple calculation, we obtain

$$\varepsilon=2R\left\{1-\sqrt{1-\left(\frac{a}{R}\right)^2}\right\}, \tag{4}$$

in which a is the radius of the contacting circle in the contact plane of two spheres (Fig. 9). According to Hertz's theory[4], a is given by

$$a=k\sqrt[3]{FR}, \tag{5}$$

where k is the constant depending on the elasticity of the spheres, and is written

$$k=\sqrt[3]{\frac{3}{4}\frac{(1-\sigma^2)}{E}}, \tag{6}$$

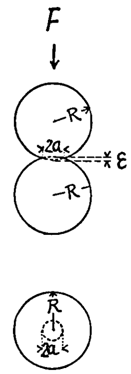


Fig. 9. Contact of two spheres.

in which E is the Young's modulus and σ the Poisson's ratio of the

4) H. HERTZ, *Gesammelte Werke*, **1** (1885), 155. S. TIMOSHENKO, *Theory of Elasticity*, (1933).

sphere. From eqdutions (3), (4), (5), we obtain

$$s=\frac{3(FR)^{\frac{1}{3}}}{2k^2}\left\{1-k^2\left(\frac{F}{R^2}\right)^{\frac{2}{3}}\right\}^{\frac{1}{2}}. \tag{7}$$

Substituting (7) in (2), V_l is expressed by

$$V_l=\frac{3}{\sqrt{2\pi}}\frac{1}{k\sqrt{\rho}}F^{\frac{1}{6}}R^{-\frac{1}{3}}\left\{1-k^2F^{\frac{2}{3}}R^{-\frac{4}{3}}\right\}^{\frac{3}{4}}. \tag{8}$$

In the present case, we take the value of F that depends on the height h of the aggregation of spheres as the mean value of the external force, such as

$$F=\pi R^2\rho g\frac{h}{2}. \tag{9}$$

Thus we obtain approximately the formula representing the longitudinal wave-velocity, neglecting terms higher than the second order, such that

$$V_l\propto h^{\frac{1}{6}}(1-\sigma^2)^{-\frac{1}{3}}\left(\frac{E}{\rho}\right)^{\frac{1}{3}}. \tag{10}$$

This would be useful for the simplest ideal case of systematic packing of the spherical grains, but since, as will be explained later, there are various degrees of packing, it is necessary to modify it as needed.

Generally, l, which has been already defined, takes values from $2R$ to $2R\sqrt{\frac{2}{3}}$ [5], that of F differing with the number of spheres that tangentially touch a particular sphere. Since, as is clear, the minimum number of tangent neighbours in the foregoing simple systematic scheme of packing is 6, and the possible maximum number is 12, the plane of deformation of that particular sphere by the external force is divided into 1∼3. Thus in each plane of deformation we take $F\frac{\cos\varphi}{n}$, notwithstanding the former value of F, in which n is 1∼3 and φ represents an angle between the direction of propagation of the wave and the line passing through the centers of spheres (Fig. 8.). Assuming φ to be 0° at minimum and 35·3° at the maximum value, from equation (10) the mean longitudinal wave-velocity may be approximately expressed by

$$V_l=C_m h^{\frac{1}{6}}(1-\sigma^2)^{-\frac{1}{3}}\left(\frac{E}{\rho}\right)^{\frac{1}{3}}, \tag{11}$$

$$C_m=3^{\frac{2}{3}}\pi^{-\frac{1}{3}}n^{\frac{1}{3}}g^{\frac{1}{6}}\sin^{\frac{1}{3}}\varphi\cos^{\frac{7}{6}}\varphi. \tag{12}$$

5) L. C. GRATON and H. J. FRASER, "Systematic Packing of Spheres—with Particular Relation to Porosity and Permeability," *Jour. Geol.*, **43** (1935), 785∼909.

It may thus be concluded that the velocity of the longitudinal wave is proportional to the sixth root of the height of the granular cylinder, to the elastic constant as well as to the density of the sphere, and to constant due to the condition of packing.

G. Hara[6] has also studied the foregoing simple case in connexion with the propagation of acoustic waves through the particles of carbon in the carbon-microphone. He obtained for the mean velocity C_m of the waves propagated through the carbon particles, the expression

$$C_m = KR^{\frac{1}{2}} h^{\frac{1}{6}} \left(\frac{E}{\rho}\right)^{\frac{1}{3}}, \tag{13}$$

where R, h, E, ρ are the same value as before, and K the constant. There are some differences between equation (11) and (13).

Thus from equations (11), (12), it is possible to determine the velocity of the longitudinal wave.

Since equation (11) shows that the wave-velocity is proportional to the sixth root of the height h, in order to ascertain whether or not the experimental results do show such a relation, the wave-velocity measured by experiments was plotted as ordinate and the sixth root of the height h of the granular mass as abscissa. Figs. 10～11 show these relations. As will be seen from these diagrams, it was ascertained that all the points lie on a straight line. It is interesting that the results by simple calculation are in good accord with the experimental results. However, upon obtaining the relation between the diameter of the *grain* and its velocity, we get the result shown in Fig. 7, from which it will be seen that there is a tendency for the velocity to increase with increase in the diameter of the *grain*, thus disagreeing with what we should expect from simple theory.

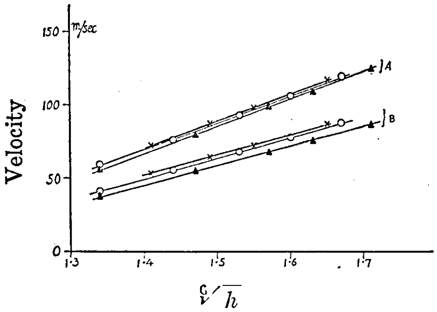


Fig. 10. Relation between the velocity and $\sqrt[6]{h}$.

A Longitudinal wave.
B Torsional wave.
○ Diameter of lead *grain*. $d=0{\cdot}48$ cm.
▲ $d=0{\cdot}30$ cm. × $d=0{\cdot}87$ cm.

Next, the torsional vibration referred to the foregoing system is considered.

6) G. HARA, "Theorie der akustischen Schwingungsausbreitung in gekörnten Substanzen und experimentelle Untersuchungen an Kohlepulver," *Elektrische Nachrichten Technik*, **12** (1935), 191～200.

As will be seen from the curves in Figs. 10, 11, all the points in the case of torsional vibration lie also on a straight line. Consequently the nature of the curve for the torsional vibration is thought to be similar to that of the longitudinal vibration.

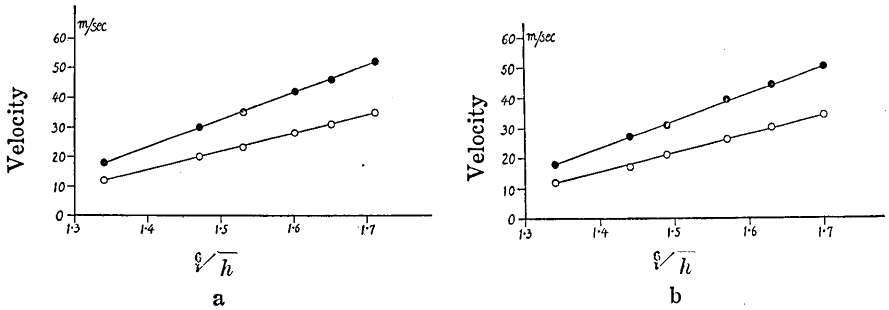


Fig. 11. Relation between the velocity and $\sqrt[6]{h}$.
a Diameter of rubber *grain*, $d=0{\cdot}98$ cm. b $d=0{\cdot}41$ cm.
● Longitudinal wave. ○ Torsional wave.

In the case of torsional vibration, the equation of motion can be used under the assumption that it is similar to that in the case of the longitudinal vibration, such that

$$I\frac{\partial^2\theta}{\partial t^2}=\tau\frac{\partial^2\theta}{\partial x^2}\,, \tag{12}$$

in which θ is the angular displacement, τ the moment of a couple of a spring of length l just mentioned when one end of it is rotated through one radian, I the moment of inertia around the axis of rotation of an unit system of aggregation such as that already mentioned, and practically equal to that of the sphere. The velocity of propagation of the torsional wave in this system V_t is given by

$$V_t=l\sqrt{\frac{\tau}{I}}\,, \tag{13}$$

in which

$$\tau=\frac{\pi\mu a^4}{2l}\,, \tag{14}$$

$$I=\frac{2}{5}mR^2\,, \tag{15}$$

where μ is the modulus of rigidity of a spring, that is, the constant of the torsional elasticity of the contacting parts of two spheres, and depends on the elastic constant of the spheres.

In the present case we can derive the equation of wave-velocity by treating it similarly to that in the case of longitudinal waves.

Finally we obtain

$$V_t = C_m h^{\frac{1}{6}} (1-\sigma)^{\frac{2}{3}} \left(\frac{\mu}{\rho}\right)^{\frac{1}{3}}, \tag{16}$$

$$C_m = \sqrt{5}\left(\frac{3}{8}\right)^{\frac{7}{6}}\left(\frac{1}{4}\right)^{\frac{1}{3}} \pi^{\frac{2}{3}} g^{\frac{1}{6}} n^{\frac{1}{3}} \sin^{\frac{4}{3}}\varphi \cos^{\frac{7}{6}}\varphi \,.$$

It is consequently possible not only to determine the proportional constant between the wave-velocity and the height from the inclination of the straight line in the diagrams, Figs. 10, 11, but also to determine the elastic constants E, μ, σ of a *grain* by using the calculated formula, assuming the probable values such as $\varphi=35°$ and $n=3$, the results of which are summarized in Table IV.

Table IV. The Computed Values of E, μ, σ of a *Grain.*

Substance.	Diameter cm	E (C. G. S.)	μ (C. G. S.)	σ
A rubber *grain*	0·29	3·30×10⁸	1·20×10⁸	0·38
	0·41	3·26 〃	1·17 〃	0·40
	0·64	3·27 〃	1·18 〃	0·39
	0·98	3·60 〃	1·33 〃	0·35
	1·20	3·50 〃	1·28 〃	0·37
Solid rubber block		2·18×10⁸	1·30×10⁸	0·40
A lead *grain*	0·30	6·60×10¹⁰	2·54×10¹⁰	0·30
	0·48	6·57 〃	2·44 〃	0·35
	0·65	6·40 〃	2·44 〃	0·31
	0·87	6·32 〃	2·31 〃	0·37
	1·02	7·00 〃	2·63 〃	0·33
Solid lead block		1·62×10¹¹	5·62×10¹⁰	0·44

In the case of a rubber *grain,* the elastic constants are comparable with those of a solid block of material (shown in Table III), whereas in the case of a lead *grain* the former values are much smaller than the latter. Therefore, the foregoing simple theory may be useful for the propagation of elastic waves through a granular mass in the first approximation, but it cannot yet be regarded as conclusive until further experiments have been made. Seeing that, although in the foregoing calculation, the velocity is independent of the diameter of the *grains,* it was found that there is a tendency for the velocity to increase slightly with increase in diameter of the *grains,* hence difficult to completely explain the phenomenon of wave propagation in a granular mass.

3. Relation between Elastic Wave-Velocity and Closeness of Packing.

Since as already mentioned, the closeness with which the *grains* are packed is an important factor affecting its elastic properties, the granular masses were examined for possible ranges of packing. Since, in the present case, the shape of an individual *grain* is spherical, the manner in which the *grains* are arranged in the solid masses that contact with their neighbours is completely obtained by analyzing cases of systematic packing of uniform spheres.

The present experimental procedure was as follows; the *grains* were slowly poured into a cellophane bag in the form of a cylinder, and allowed to pack naturally until the bag was full. After weighing, the vibration-experiments were made, and the velocities of elastic waves determined by the foregoing methods. In this case the degree of packing is called the loosest. Next the experiments were repeated with the masses in varying degrees of packing, the desired packing being obtained by tapping the receptacle. As the total volume of the receptacle was known, the porosity was readily computed from the ratio of the density of the granular mass to that of the individual *grain*.

According to L. C. Graton and H. J. Fraser,[7] there are several modes of packing, the porosity values of which are shown in Table V.

Table V. The Porosity Values of Several Modes of Packing.
(after L. C. Graton and H. J. Fraser.)

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Name (Crystal analogue)	Cubic	Ortho-rhombic	Rohmbo-hedral	Ortho-rhombic	Tetragonal-sphenoidal	Rhombo-hedral
Tangent neighbours	6	8	12	8	10	12
Spacing of layers	$R\sqrt{4}$	$R\sqrt{3}$	$R\sqrt{2}$	$R\sqrt{4}$	$R\sqrt{3}$	$2R\sqrt{2/3}$
Volume of unit cell	$8{\cdot}00R^3$	$6{\cdot}93R^3$	$5{\cdot}66R^3$	$6{\cdot}93R^3$	$6{\cdot}00R^3$	$5{\cdot}66R^3$
Volume of unit void	$3{\cdot}81R^3$	$2{\cdot}74R^3$	$1{\cdot}47R^3$	$2{\cdot}74R^3$	$1{\cdot}81R^3$	$1{\cdot}47R^3$
Porosity	47·64%	39·54%	25·95%	39·54%	30·19%	25·95%

Although it will be noticed that the porosity values vary from the maximum number of 47·64 per cent to the minimum number of 25·95 per cent, our experiment shows that the porosity values vary from 48·7 to 36·2 per cent in the case of tiny rubber spheres, and from 48·0 to 36·0 per cent in the case of tiny lead spheres. Both these maximum and minimum values slightly exceed those obtained by Graton and

7) L. C. GRATON and H. J. FRASER, *loc. cit.*, 4)

Fraser, which is probably due to the fact that some of the *grains* form more or less imperfect spheres, and that the *grains* are constrained by the cellophane walls. Since irregularities in the shape of the *grains* ought to result in a wider range of porosity, the irregular forms can be packed either more loosely or more tightly than uniform spheres. The results of our experiments are summarized in Table VI.

Table VI. Porosity and Wave-Velocity of a Granular Substance.

Lead *grain*, d=0·48 cm.			Rubber *grain*, d=0·98 cm.			Sand *grain*, d=0·59 mm.		
P %	V_l m/sec.	V_t m/sec.	P %	V_l m/sec.	V_t m/sec.	P %	V_l m/sec.	V_t m/sec.
36·0	150	105	36·2	86	57	40·0	100	65
36·5	137	95	37·1	74	50	41·3	93	61
37·5	125	87	37·9	73	50	42·6	85	54
38·4	120	83	39·5	62	41	44·0	84	54
39·7	111	78	40·6	56	38	45·5	77	50
41·2	106	74	41·7	52	35	46·6	73	46
42·3	98	68	42·8	44	30	47·6	70	44
43·0	96	68	43·9	39	26	49·0	69	44
43·8	93	65	45·2	33	22			
44·9	88	60	46·1	31	19			
46·3	85	59	47·4	28	16			
47·5	80	55	48·7	20	13			

To obtain the relation between wave-velocity and closeness of packing, the velocities V_l, V_t were plotted as ordinates and the corresponding porosities of the granular masses as abscissae, as shown in Fig. 12. The diagram, Fig. 12, shows that the velocities vary with the arrangement of the *grains*, regardless of the manner, and increase with tightness of packing as a function of porosity—phenomena already ascertained in the previous investigation of sand masses.[8] From

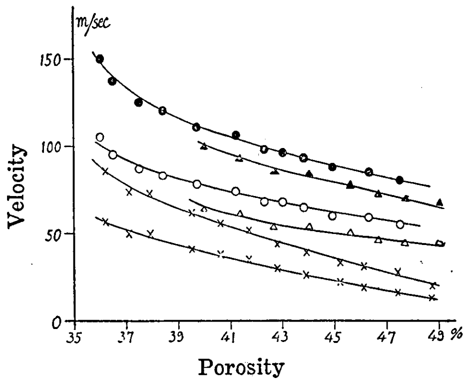


Fig. 12. Relation between the velocity of elastic wave and porosity.
●, ▲, × (upper) Longitudinal wave.
○, △, × (lower) Torsional wave.
●, ○ Lead *grain*. △, ▲ Sand *grain*.
× Rubber *grain*.

8) K. IIDA, *loc. cit.*, 2)
K. YONETA, "On Wave Propagation on the Surface of a Sand Mass," *Memo. Fac. Eng. Hokkaido Imp. Univ.*, 4 (1938), 265~273.

these results we get for the empirical formula

$$V_l = v_l e^{-B_1 P},$$

$$V_t = v_t e^{-B_2 P},$$

in which P is the porosity in percentage, v_l, v_t velocities for $P=0$, B_1, B_2, constants depending on the kind of substance used and the condition, whether dry or wet. The values of these constants that were determined from our experiments are shown in Table VII. In the case of *grains* of lead and sand, this formula accords well with the experimental results, whereas in the case of a rubber *grain* the accordance is not so perfect.

Table VII.
The Values of v_l, B_1, v_t, B_2.

Material	v_l m/sec	B_1	v_t m/sec	B_2
Rubber	4020	0·106	4190	0·117
Lead	596	0·042	565	0·049
Sand	480	0·040	338	0·042

4. Relation between Water Content and Velocity.

To ascertain whether or not the effect of water contained in granular masses could be observed in the elastic wave-velocity, we experimented with wet granular masses in varying degrees of water. For wet packing, the *grains* were first immersed in a water tank to soak it thoroughly and remove the airbubbles adhering to the *grains*. The receptacle was filled with water and the *grains* slowly poured in and allowed to settle, after which the water was expelled through a small hole in the base of the receptacle. In this way, the water in the granular substance was reduced to a minimum in order to permit

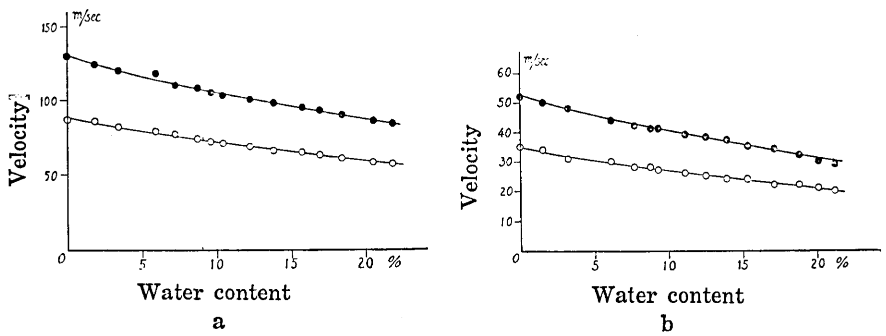


Fig. 13. Relation between the velocity of elastic waves and the water content.
● Longitudinal wave. a Lead *grain*, $d=0{\cdot}48$ cm.
○ Torsional wave. b Rubber *grain*, $d=0{\cdot}98$ cm.

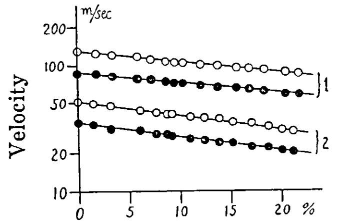
it of being set upright on the vibrating plate. It was thus possible to obtain it in various degrees of water by drying it in ordinary room temperature, and to determine the velocity of elastic waves through it. From the results of measurement shown in Fig. 13 and Table VIII, it was ascertained that the relations between the elastic properties of these kinds of granular masses and the water content are similar to those that we had previously found. The terms V_l, V_t have a tendency to diminish somewhat, exponentially, with increase in water content, to illustrate which, the logarithms of the wave-velocity were plotted as ordinate and the water content as abscissa, as shown in Fig. 14. It will be seen that all the points lie on a straight line, whence it may be concluded that these velocities V_l, V_t may be expressed by

Table VIII.
Wave-velocity and Water Content.

Lead *grain*, d=0·48 cm.			Rubber *grain*, d=0·98 cm.		
Water content %	V_l m/sec.	V_t m/sec.	Water content %	V_l m/sec.	V_t m/sec.
0	132	89	0	52	35
1·8	124	86	1·5	50	34
3·4	120	82	3·2	48	31
5·9	118	79	6·1	44	30
7·2	110	77	7·7	42	28
8·7	108	74	8·8	41	28
9·6	105	72	9·3	41	27
10·4	103	71	11·1	39	26
12·2	100	69	12·5	38	25
13·8	98	66	13·9	37	24
15·7	95	65	15·3	35	24
16·9	93	63	17·1	34	22
18·4	90	61	18·8	32	22
20·5	86	58	20·1	30	21
21·8	84	57	21·2	29	20

$$V_l = A_1 e^{-B_1 w}, \quad V_t = A_2 e^{-B_2 w},$$

in which A_1, A_2 are velocities for $w=0$, B_1, B_2 constants depending on the kind of substance, and w the water content in percentage. In the case of a rubber *grain* $A_1=52$ m/sec, $B_1=0{\cdot}027$, $A_2=35$ m/sec, $B_2=0{\cdot}026$, while in the case of a lead *grain* $A_1=132$ m/sec, $B_2=0{\cdot}027$, $A_2=89$ m/sec, $B_2=0{\cdot}020$. In a sand mass $A_1=62$ m/sec, $B_1=0{\cdot}025$, $A_2=45$ m/sec, $B_2=0{\cdot}024$.



Water content

Fig. 14. Relation between the log. of velocity and water content.
○ Longitudinal wave.
● Torsional wave.
1 Lead *grain*, d=0·48 cm.
2 Rubber *grain*, d=0·98 cm.

We must now see how the velocity diminishes with increase in the water content. Wet material generally packs more loosely than dry material. According to H. J. Fraser,[9] wet pack-

9) H. J. FRASER, "Experimental Study of the Porosity and Permeability of Clastic Sediments," *Jour. Geol.*, **43** (1935), 910~1010.

ing increases the effect of angularity, flat and needle-like shapes having the greatest effect on porosity. For example, the porosities of particles of crushed quartz increase from 48 per cent in dry packing to 54 per cent in wet packing. Water thus increases the porosity. As already found, the greater the porosity the smaller the velocity. It is, therefore, thought that the velocity diminishes with increase in water content. Moreover, the water that surrounds the *grains* seems to be useful in binding them and acting as a lubricant between the *grains* when the water increases, so that the binding forces that act on the *grains* seem to diminish compared with those in dry *grains*. This causes a decrease in the velocity.

5. Effect on Velocity of the Binding Material contained in the Granular Mass.

To see how the binding material in granular masses affect the wave-velocity, aggregates of either parrafin or mizuame, the latter a very thick viscous jelly made from rice, and the foregoing various *grains* were carefully prepared.

These aggregates, examples of which are shown in Fig. 15, were made as follows: In the case of the aggregate of mizuame and *grains*, the *grains* were thoroughly immersed in mizuame and the adhering air removed, after which these *grains*, with the mizuame, were slowly poured into the cellophane bags and allowed to settle. In the case of the

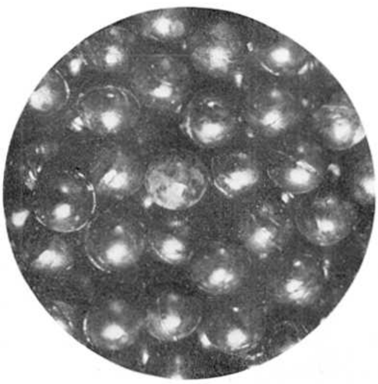


Fig. 15 a. Rubber *grain*. (Dia. of a *grain*, $d=0{\cdot}41$ cm.)

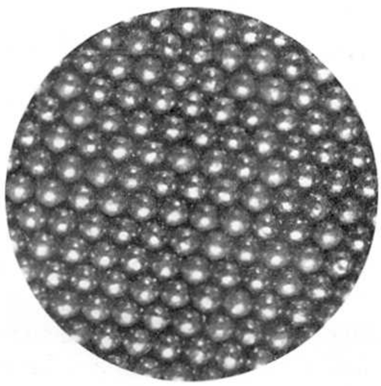


Fig. 15 b. Lead *grain*. (Dia. of a *grain*, $d=0{\cdot}48$ cm.)

aggregate of paraffin and the various *grains*, the latter were first packed in a brass tube to the desired density, and molten paraffin was carefully poured into the brass receptacle and allowed to cool at room temperature. As soon as the paraffin solidified, the aggregate was emptied out, and in this way an aggregate of paraffin and *grains* were obtained.

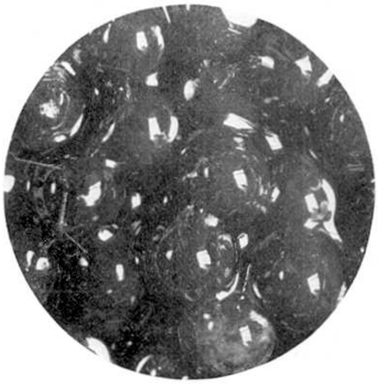


Fig. 15c. Aggregate of mizuame and rubber *grain*. (dia., $d=0{\cdot}98$ cm.)

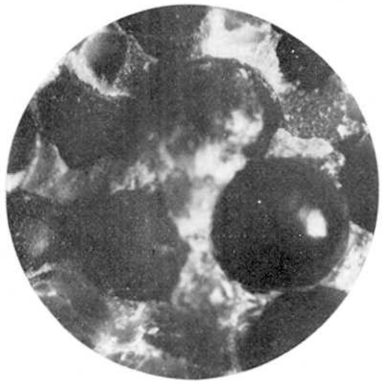


Fig. 15d. Aggregate of paraffin and rubber *grain*. (dia., $d=0{\cdot}98$ cm.)

Fig. 15. Photographs of aggregate of *grains* and either paraffin and *grain* or mizuame and *grain*.

To investigate the elastic properties of these aggregates it is important to study the physical properties of both mizuame and paraffin. In this case we obtained the viscosity and the elasticity of these substances. Although mizuame in the liquid state has no elasticity, it acquires it at low temperature, because it assumes the solid state.

(*1*) *Viscosity coefficient of mizuame.*

The viscosity coefficients of mizuame were measured by the falling tiny sphere method in the mizuame at various temperatures. The coefficient of viscosity of mizuame η can be obtained by

$$\eta=K\frac{r^2}{V},$$

$$K=\frac{2}{9}(\rho-\rho')g,$$

in which r is the radius of sphere, V the falling velocity of the tiny metal sphere, $0{\cdot}3\sim0{\cdot}6$ cm diameter, in the mizuame; ρ, ρ' the density of the sphere ($\rho=8{\cdot}35\sim11{\cdot}32$) and mizuame ($\rho'=1{\cdot}44$) respectively. The results are shown in Table IX and in Fig. 16, in which are plotted the viscosity coefficient against the temperature. The viscosity coefficient at 20°C is 4×10^3 in C. G. S. Units — a value smaller than that obtained by N. Miyabe.[10] This value just given however, seems reasonable, seeing

10) N. MIYABE, *Bull. Earthq. Res. Inst.*, **12** (1934), 199.

Table IX. Viscosity Coefficient of Mizuame.

Temp. °C	Viscosity coefficient. (C. G. S.)	Temp. °C	Viscosity coefficient. (C. G. S.)	Temp. °C	Viscosity coefficient. (C. G. S.)	Temp. °C	Viscosity coefficient. (C. G. S.)
12·4	$3{\cdot}20 \times 10^4$	14·8	$1{\cdot}04 \times 10^4$	18·8	$4{\cdot}40 \times 10^3$	25·1	$1{\cdot}06 \times 10^3$
12·6	2·60 〃	15·0	$9{\cdot}00 \times 10^3$	19·4	3·81 〃	26·8	$8{\cdot}57 \times 10^2$
12·8	2·48 〃	15·7	6·94 〃	19·9	4·23 〃	28·0	6·20 〃
13·0	2·10 〃	16·0	6·23 〃	22·0	2·42 〃	28·8	4·91 〃
13·4	1·89 〃	16·2	6·21 〃	22·4	2·00 〃	29·8	5·00 〃
13·5	1·60 〃	16·5	5·53 〃	23·6	1·64 〃	30·0	4·19 〃
14·2	1·10 〃	18·0	4·80 〃	24·5	1·60 〃	30·3	4·00 〃

that we used a rather softer material than the ordinary commercial article. As will be seen from the figure, the viscosity coefficient diminishes with increase in temperature.

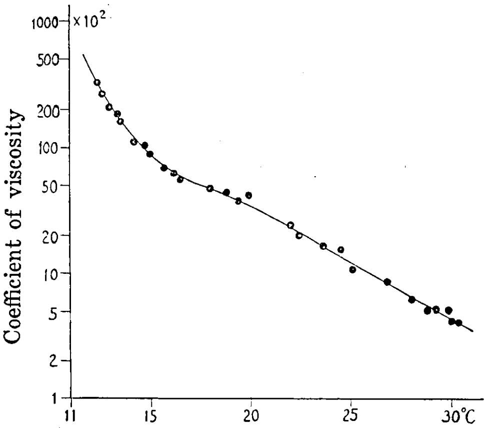


Fig. 16. Relation between the coefficient of viscosity of mizuame and temperature.

(2) *The wave-velocity through mizuame.*

The compressional wave-velocity through mizuame packed in a cellophane cylinder was determined as before. The results obtained at various temperatures are shown in Fig. 17, in which the temperature is taken as abscissa and the velocity as ordinate. The longitudinal wave-velocity increases with decrease in temperature. The torsional wave could not be observed at temperatures exceeding 14°C. It was however, ascertained that at temperature below 13°C, torsional waves are propagated through mizuame. It is interesting that at a temperature lower than 13°C, this substance behaves like a solid body, whence this substance, in such a state,

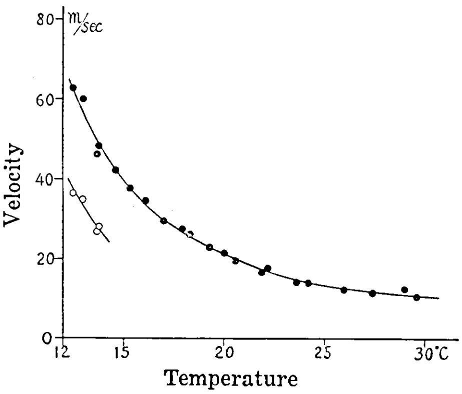


Fig. 17. Showing that the velocity varies with increasing temperature.

● Longitudinal vibration.
○ Torsional vibration.

behaves like an elastic body when subjected to that condition for a short period of time, whereas it usually behaves like a liquid of high viscosity when the forces act for a long period of time.

(3) *The wave-velocity in the aggregate of* grains *and mizuame.*

When the *grains* consist of tiny rubber spheres, the density of the rubber *grains* being smaller than that of mizuame, are suspended in the mizuame. Care was taken to keep the surface of the *grains* submerged in the mizuame, and the two kinds of elastic wave-velocities, longitudinal and torsional, were determined, with results as shown in Figs. 18 a, b, in which the results of the aggregates of either sand or lead *grains* and the mizuame are also shown. The ratio of the total volume of the *grains* in the mizuame to that of the mizuame was plotted as abscissa and the wave-velocity as ordinate. The velocities of the waves in all these aggregates decrease with decrease in the ratio, and rapidly decrease when the ratio is below a value of 0·7. The torsional waves under such conditions below the ratio of 0·5 could scarcely be observed,

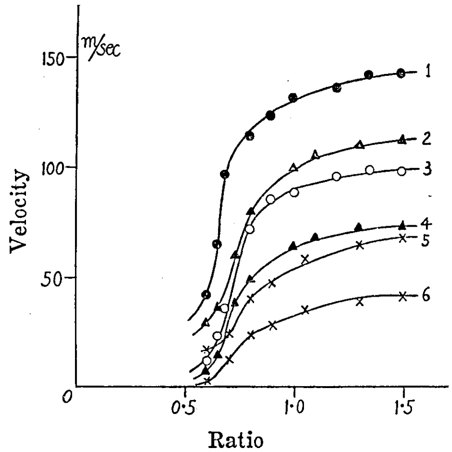


Fig. 18 a. Relation between the velocity of elastic waves and the ratio of the total volume of *grains* to that of mizuame.

1, 2, 5 Longitudinal wave.
3, 4, 6 Torsional wave.
●, ○ Lead *grain*.
▲ Sand *grain*. × Rubber *grain*.
Temperature=19·0°C.

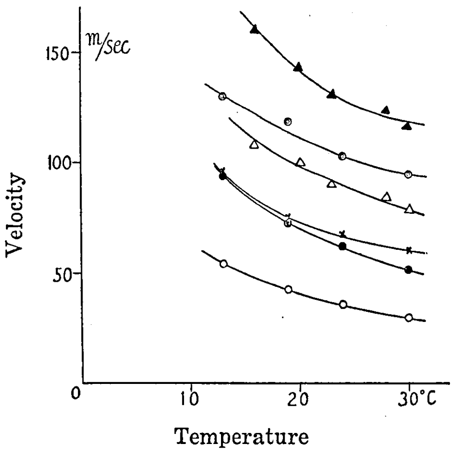


Fig. 18 b. The velocities of elastic waves in the various aggregates decrease with increase in temperature.

▲, △ Lead *grain*.
◎, × Sand *grain*.
●, ○ Rubber *grain*.
▲, ◎, ● Longitudinal wave.
△, ×, ○ Torsional wave.

while the longitudinal waves could usually be observed. This fact is very important in studying the propagation of elastic waves in a substance that is composed of viscous liquid and elastic particles. Fig. 18 b shows that the velocities of elastic waves in these aggregates decrease with increase in temperature.

It will be noticed from the figure that the velocities through the foregoing aggregate are greater than those obtained by the granular mass (mass of the *grains*) alone, and that the velocities exceed the mean values calculated from each part of the two, from which fact it was ascertained that the binding material in the granular mass undoubtedly greatly affects its elastic properties. As to the main reason

Table X. Velocity in Paraffin Cylinder.
(Density, $\rho=0{\cdot}89$.)

Temp. °C	Longitudinal vel. V_l m/sec.	Torsional vel. V_t m/sec.	Young's modulus E (C. G. S.)	Modulus of rigidity μ (C. G. S.)
14·0	760	450	$5{\cdot}15\times10^9$	$1{\cdot}81\times10^9$
15·6	690	420	4·24 "	1·57 "
18·0	690	380	4·24 "	1·29 "
19·1	555	340	2·75 "	1·09 "
20·4	380	230	1·29 "	$4{\cdot}71\times10^8$
22·0	200	115	$3{\cdot}56\times10^8$	1·18 "
24·0	111	62	1·07 "	$3{\cdot}42\times10^7$
28·0	44	26	$1{\cdot}73\times10^7$	$6{\cdot}01\times10^6$
30·0	30	19·5	$8{\cdot}00\times10^6$	3·39 "
31·0	26·0	16·5	6·01 "	2·43 "

for this increase in velocity, it is believed that the *grains* are held together by the adherent substance, and that the binding force thus apparently increases.

(4) *The aggregate of paraffin and the* grains.

Both the elastic and viscous properties of paraffin have already been discussed in our previous papers.[11] In the present case, the elastic constant was obtained by subjecting the paraffin cylinder to vibration. The results are comparable with the

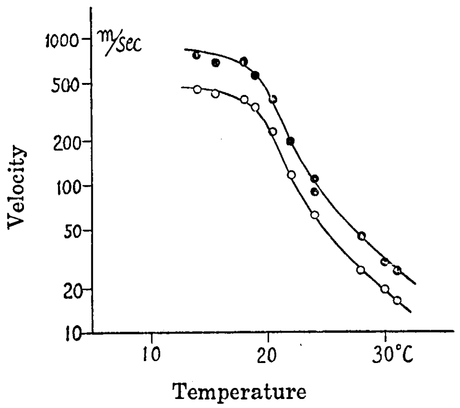


Fig. 19. The velocity of elastic waves in paraffin cylinder varies with increase in temperature.

● Longitudinal wave. ○ Torsional wave.

dynamical values obtained previously,[12] as shown in Table X and in Fig. 19, in which we plotted the wave-velocities obtained by the present experiment against the temperature. It will be seen that the elastic constant decreases at first slowly and then rapidly with increase in temperature. An aggregate of tiny lead or rubber spheres as well as sand *grains* and paraffin, the elastic properties of which are well known, was prepared as before.

(5) *The wave-velocity in the aggregate of paraffin and the* grains.

The velocities of the longitudinal and the torsional elastic waves through the aggregates of paraffin and the various *grains* of lead, rubber, and sand were studied by the ordinary procedure, the results of which are shown in Fig. 20, from which it will be seen that the velocities of these aggregates are expected to exceed those obtained by the granular mass alone. This is chiefly due to the properties of the paraffin binding the *grains*.

To observe the relation between the quantity of paraffin and of the

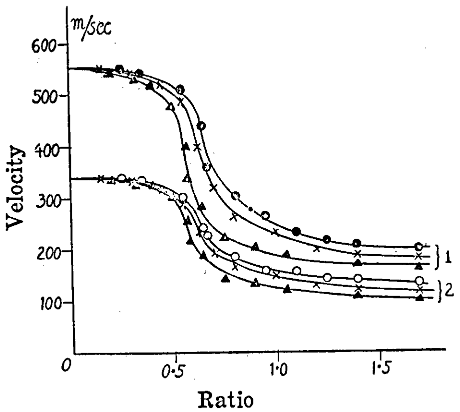


Fig. 20 a. Relation between the velocity of elastic waves and the ratio of the volume of the *grain* to that of paraffin.

●, ○ Lead *grain*.
▲ Rubber *grain*.
× Sand *grain*.
1 Longitudinal wave.
2 Torsional wave.
Temperature=19·0°C.

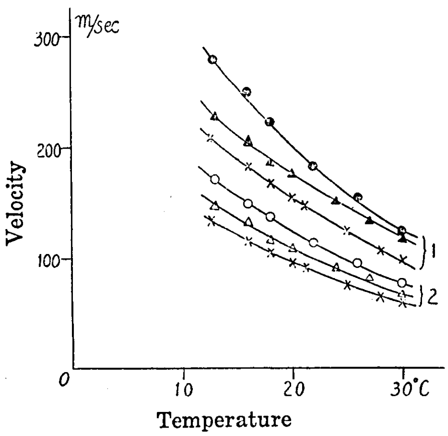


Fig. 20 b. The velocities of elastic waves in the various aggregates vary with increase in temperature.

●, ○ Lead *grain*.
▲, △ Sand *grain*.
× Rubber *grain*.
1 Longitudinal wave.
2 Torsional wave.

rubber, lead, and sand *grains* with respect to elastic wave-velocities,

11), 12) K. IIDA, *Bull. Earthq. Res. Inst.*, **13** (1935), 433~456.

experiments were made also with aggregates of varying degrees of mixture of paraffin and the *grains*. The ratio of the volume of the *grains* to that of paraffin was plotted as abscissa and the wave-velocity as ordinate, the relations between the two thus obtained being shown in Fig. 20 a. All the measured velocities through the aggregates of paraffin and the *grains* are greater than the mean values calculated from the wave path of each part of the two. In the case of lead and sand *grains*, it will be seen that the observed wave-velocity rapidly increases and tends to that of paraffin when the ratio in question is below a value of 0·7 (exactly 10 parts paraffin and 7 parts *grains*), while the more this ratio exceeds 0·7 (about) the smaller the velocity. In the case of rubber *grains*, the wave velocity increases when this critical ratio is 0·6. The wave-velocities for the ratio below a value of 0·5 are comarable with that of paraffin. This phenomenon is explained by considering the wave path that is propagated through these aggregates. Probably the waves act as if passing through a medium consisting of paraffin alone, without anything else imbedded in it. Therefore, with respect to the wave motion, the aggregates behave like a cylinder of paraffin.

Fig. 20 b shows that the velocities of elastic waves in these aggregates decrease somewhat exponentially with increase in temperature.

6. Summary and Conclusion.

In the present experiments, the elastic properties of certain kinds of granular masses are studied. The relations between the wave-velocity and the elements, such as grain-size, closeness of packing, water content, and the binding material held by the *grains*, were also studied.

It was ascertained that the wave-velocity through a granular mass is proportional to the sixth root of its height, to the cube root of the ratio of the elastic constant to the density of the *grains*, and to the constant due to the condition of packing. It was obtained by simple theory that the wave-velocity is independent of the diameter of the individual *grains*. There is a tendency, however for the measured velocity to increase slightly with increase in the diameter of the *grains*, but its value may be considered almost constant. The elastic constants of a *grain* are comparable with those of a solid block of material as deduced by simple theory and experiments.

The wave-velocities through a granular mass are much smaller than those through a column that is a solid block.

It was found that the velocities decrease either with increase in the water content or with increase in porosity. Empirical formulae showing the relation between these two elements were derived.

Model aggregates were prepared of the *grains* and paraffin and mizuame to represent the conditions under which substances are deposited by nature on the earth's surface. The wave-velocity in these model aggregates exceed not only those obtained by the mass of the *grains* alone, but also the mean value of that in each component substance. It was proved, however, that the velocity greatly depends on the ratio of the volume of one substance to another.

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51. 粒狀構造物體中を傳播する彈性波の速度

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地震波に對して示す地殼の性質は主として地殼物質の彈性的性質に因ると考へられる．地殼物質の中には其れを肉眼的に或ひは顯微鏡的に觀察するとき粒狀構造をなしてゐるものが多い．砂は肉眼觀察に於いて見出される粒狀構造物體の代表的なものである．

砂の如き多くの粒子の集團よりなる物體の彈性波に對して示す二，三の物理的性質は既に述べた．而して砂の如き物質もその個々の粒には拘泥せず粒子の集りを一つの物體として考へる時，それは彈性波を通過せしめる事によつて一種の彈性體と見做され，而してその彈性的性質が粒子の配列，緻密度，粒子と粒子とを結合せしめる colloid 物質或ひは水分等によつて著しく影響を受けるのである．砂の粒子の大きさが大となるに從つてその集團中を通過する彈性波の速度が小となる傾向が見出されたのであるが，大きさの一定な砂粒と稱しても種々な形狀があり不規則である關係上色々な性質の闡明には不十分と思はれる．從つて今回は形の不規則より生ずるものを度外視して以上述べた性質を一歩進めて吟味せんと企て，先づ比較的容易に加工の出來る物質を選びそれを球形となし，大きさも種々なるものを作つて實驗したのである．粒子として選んだ物質は彈性ゴム，鉛等であるが，其の他形が割合に球形に近い二，三種類の砂をも選定して用ひた．

以上述べた粒子の集團よりなる物體中を通過する彈性波が粒子の大さと如何なる關係にある

か，又粒子の配列，粒子と粒子とを結合せしめた物質の種類或ひはそれ等の混合物體に對して彈性波の速度が如何な値を示すかを考究した．

實驗によつて得られた結果は彈性波の速度は粒子の大さにより多少の變化をなし，粒子の大さが大となるに從つて僅少ながら增大するが，其の彈性波の速度は粒狀體と同質なる固體中を傳播する彈性波速度に比して著しく小さい．又物體の高さの變化と共に增減し高さの增大と共に速度が增す傾向がある．簡單な計算に於いては高さ h の 6 乘根に比例する値を得たが實驗は大體此れと一致する．又速度と粒子の彈性常數との間にある關係を得たが，此等は尚多くの吟味を必要とする．

速度は粒子の集合の粗密の狀態によつて左右され，間隙率の値が小なる程大であつて其の間には

$$V=Ae^{-BP}$$

なる關係があるやうである．B は常數，P は間隙率，V は速度，A は $P=0$ に於ける速度を表はす．又速度は粒狀物體が含有する含水量の增加と共に小さなり

$$V=ve^{-BW}$$

なる關係が存在する．B は常數，v は含水率零に於ける速度．W は含水率である

次に粒狀物體中に水飴，パラフィン等の物理的性質の豫め知れた物質を入れて，粒子と粒子とを結合或ひは粒子をその粘性物體中に浮游せしめて，人工的に粒子とそれらの物質との集合物體を作り，其の中を通過する彈性波の速度を觀察した．其の速度は兩物體の體積の割合に比例して增減をなし，其の比が或る値を超えると急に一方の物體の速度のみを示す傾向が見られた．其の比は勿論物質により又粒子の大さにより異るが其の間の關係は本文中の圖に於いて示してある．而して飴の如き粘性物體中に彈性粒子が suspend する時，其の全體としての集合物體は彈性波を通過せしめる事によつて彈性體の性質を表はす事が明らかにされた．