33. *The Formation of Boundary Waves at the Surface of a Discontinuity within the Earth's Crust. II.*

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1. *Introduction.*

In the previous paper[1] we showed how bodily waves generated from a point source in a solid excite boundary waves of Stoneley type[2] at any discontinuous surface in the same solid. The method of calculation was somewhat similar to that which Sommerfeld[3] used in his paper on the transmission of electromagnetic waves. The case discussed in the previous paper was, however, merely two-dimensional; the case of the three dimensional condition will now be shown. Since the method of treatments does not essentially differ from that in the previous case, the present paper will be as brief as possible.

2. *General equations for the original dilatational disturbance.*

Let us use the axes of r, z in the same way as for x, y in the previous paper, the definitions of ρ_1, λ_1, μ_1, ρ_2, λ_2, μ_2 being also the same. In the case of spherical primary waves

$$\phi_0=\mathfrak{A}e^{\alpha_1 z-ipt}J_0(fr), \tag{1}$$

the reflected and refracted dilatational and distortional waves assume the forms

$$\left.\begin{aligned}\phi_1&=Ae^{-\alpha_1 z-ipt}J_0(fr), &\quad \psi_1&=Ae^{-\beta_1 z-ipt}J_1(fr),\\ \phi_2&=Ce^{\alpha_2 z-ipt}J_0(fr), &\quad \psi_2&=De^{\beta_2 z-ipt}J_1(fr),\end{aligned}\right\} \tag{2}$$

where

$$\left.\begin{aligned}&\alpha_1^2=f^2-h_1^2, \quad \beta_1^2=f^2-k_1^2, \quad \alpha_2^2=f^2-h_2^2, \quad \beta_2^2=f^2-k_2^2,\\ &h_1^2=\frac{\rho_1 p^2}{\lambda_1+2\mu_1}, \quad k_1^2=\frac{\rho_1 p^2}{\mu_1}, \quad h_2^2=\frac{\rho_2 p^2}{\lambda_2+2\mu_2}, \quad k_2^2=\frac{\rho_2 p^2}{\mu_2}.\end{aligned}\right\} \tag{3}$$

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1) K. SEZAWA and K. KANAI, *Bull. Earthq. Res. Inst.*, **16** (1938), 504~526.
2) R. STONELEY, *Proc. Roy. Soc.*, **106** (1924), 416~428.
3) A. SOMMERFELD, *Ann. Phys.*, **28** (1909), 665~738.

The displacements in both media are then expressed by

$$\left.\begin{aligned} u_1&=\frac{\partial}{\partial r}(\phi_0+\phi_1)+\frac{\partial\psi_1}{\partial z}, & v_1&=\frac{\partial}{\partial z}(\phi_0+\phi_1)-\frac{\partial\psi_1}{\partial r}, \\ u_2&=\frac{\partial\phi_2}{\partial r}+\frac{\partial\psi_2}{\partial z}, & v_2&=\frac{\partial\phi_2}{\partial z}-\frac{\partial\psi_2}{\partial r}. \end{aligned}\right\} \tag{4}$$

Using the boundary conditions at $z=0$

$$\left.\begin{aligned} u_1=u_2, \qquad v_1=v_2, \\ \lambda_1\left(\frac{\partial u_1}{\partial r}+\frac{\partial v_1}{\partial z}\right)+2\mu_1\frac{\partial v_1}{\partial z}=\lambda_2\left(\frac{\partial u_2}{\partial r}+\frac{\partial v_2}{\partial z}\right)+2\mu_2\frac{\partial v_2}{\partial z}, \\ \mu_1\left(\frac{\partial v_1}{\partial r}+\frac{\partial u_1}{\partial z}\right)=\mu_2\left(\frac{\partial v_2}{\partial r}+\frac{\partial u_2}{\partial z}\right), \end{aligned}\right\} \tag{5}$$

we get the values of the constants A, B, C, D, the expressions of which are almost the same as those in the previous paper, with the exceptions that in the present case

$$\frac{B\Phi}{\mathfrak{A}}=-2f\alpha_1\Big[\cdots\cdots, \quad \frac{D\Phi}{\mathfrak{A}}=-2\mu_1 fk_1^2\alpha_1\Big\{\cdots\cdots,$$

in lieu of

$$\frac{B\Phi}{\mathfrak{A}}=2if\alpha_1\Big[\cdots\cdots, \quad \frac{D\Phi}{\mathfrak{A}}=2i\mu_1 fk_1^2\alpha_1\Big\{\cdots\cdots \tag{6}$$

in the previous paper.

Let us next consider the condition in which the primary dilatational waves are radiated from a point source at $r=0$, $z=\xi$. Then

$$\left.\begin{aligned} \phi_0&=\mathfrak{A}e^{-ipt}\frac{e^{ih_1R}}{R} & & [R^2=r^2+(z-\xi)^2] \\ &=\frac{-i\mathfrak{A}}{\pi}e^{-ipt}\int_0^\infty dj\int_{-\infty}^\infty\frac{e^{\alpha_1(z-\xi)+ifrchj}}{\alpha_1}fdf, & & [z<\xi] \\ &=\frac{-i\mathfrak{A}}{\pi}e^{-ipt}\int_0^\infty dj\int_{-\infty}^\infty\frac{e^{-\alpha_1(z-\xi)+ifrchj}}{\alpha_1}fdf. & & [z>\xi] \end{aligned}\right\} \tag{7}$$

From (1), (2), (7) the reflected and refracted waves assume the forms

$$\left.\begin{aligned} \phi_1&=\frac{-i}{\pi}e^{-ipt}\int_0^\infty dj\int_{-\infty}^\infty A\frac{e^{-\alpha_1(z+\xi)+ifrchj}}{\alpha_1}fdf, \\ \psi_1&=\frac{-i}{\pi}e^{-ipt}\frac{\partial}{\partial r}\int_0^\infty dj\int_{-\infty}^\infty B'e^{-\beta_1 z-\alpha_1\xi+ifrchj}fdj, \quad \left[B'=\frac{-1}{f\alpha_1}B\right] \end{aligned}\right\} \tag{8}$$

$$
\left.\begin{aligned}
\phi_2 &= \frac{-i}{\pi} e^{-ipt} \int_0^\infty dj \int_{-\infty}^\infty C' e^{\alpha_2 z - \alpha_1 \xi + ifrchj} f df\,, && \left[C' = \frac{C}{\alpha_1}\right] \\
\psi_2 &= \frac{-i}{\pi} e^{-ipt} \frac{\partial}{\partial r} \int_0^\infty dj \int_{-\infty}^\infty D' e^{\beta_2 z - \alpha_1 \xi + ifrchj} f df\,. && \left[D' = \frac{-D}{f\alpha_1}\right]
\end{aligned}\right\} \qquad (9)
$$

The displacements at any point are obtained by using (7), (8), (9), (4).

3. *Evaluation of the integrals.*

To evaluate the integral expressions of the displacements, we shall first consider such integrals in which the f-value in (7), (8), (9) is replaced by a complex quantity $Z=X+iY$, the type of integral being

$$
\int_0^\infty dj \int_{-\infty}^\infty U \frac{e^{-\sqrt{Z^2-\eta^2}z + iZrchj}}{\sqrt{Z^2-\eta^2}} Z dZ\,. \qquad (10)
$$

The paths of the second integral in (10) are (i) the real axis in the Z plane from $-\infty$ to ∞, (ii) a circular arc of infinite radius in the first and second quadrants, (iii) an arc around the pole $Z=\kappa$, which satisfies the condition $\Phi(\kappa)=0$ in the previous paper, (iv) four branch lines connecting the respective branch points with $Z=i\infty$. The sum of

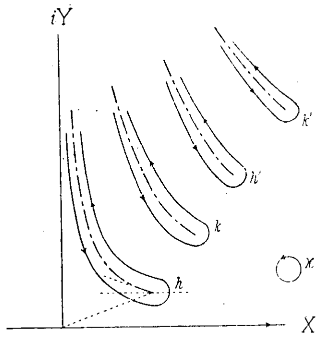

Fig. 1.

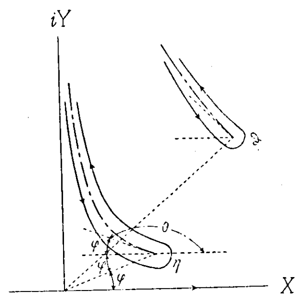

Fig. 2.

the four kinds of contour integrals vanishes, from which it is possible to determine the integrals performed along the real axis from $-\infty$ to ∞. The integrals along the large radius vanish. The integral around the pole $Z=\kappa$ is merely written in the form $-2\pi i\chi(\kappa)/\Phi'(\kappa)$.

The pole $Z=\kappa$ exists only when there is a solution of the Stoneley waves, which condition has been discussed in a previous paper.[4)]

The branch lines as shown in Fig. 1 are so chosen that the values

4) K. SEZAWA and K. KANAI, *Bull. Earthq. Res. Inst.*, **17** (1939), 1~8.

of $\sqrt{Z^2-h_1^2}$, $\sqrt{Z^2-k_1^2}$, $\sqrt{Z^2-h_2^2}$, $\sqrt{Z^2-k_2^2}$ along the respective branch lines are purely imaginary for any Z on the same branch lines; the inclinations of the branch lines thus being adjusted as shown in Fig. 2.

4. *Evaluation of the integrals along the branch lines.*

As in the previous paper there are four cases to be discussed for every branch line.

$$\text{(i)}\quad \int_{i\infty}^{\eta} \frac{U_1 e^{\sqrt{Z^2-\eta^2}z+iZrchj}}{-\sqrt{Z^2-\eta^2}} ZdZ + \int_{\eta}^{i\infty} \frac{U_2 e^{-\sqrt{Z^2-\eta^2}z+iZrchj}}{\sqrt{Z^2-\eta^2}} ZdZ \tag{11}$$

along the branch line through $Z=\eta$. We change the variable such that

$$\pm\sqrt{Z^2-\eta^2} = \pm i\tau, \qquad Z = i\sqrt{\tau^2-\eta^2}, \qquad dZ = \frac{i\tau d\tau}{\sqrt{\tau^2-\eta^2}}. \tag{12}$$

The integrals in (11) then assume the forms

$$\int_{\infty}^{0} U_1' e^{i\tau z-\sqrt{\tau^2-\eta^2}rchj}(-i)\,d\tau + \int_{0}^{\infty} U_2' e^{-i\tau z-\sqrt{\tau^2-\eta^2}rchj}\,id\tau\,. \tag{11$'$}$$

Writing $\tau=-\tau'$,

$$\int_{-\infty}^{0} U_1'' e^{-i\tau' z-\sqrt{\tau'^2-\eta^2}rchj}\,id\tau' + \int_{0}^{\infty} U_2' e^{-i\tau z-\sqrt{\tau^2-\eta^2}rchj}\,id\tau\,, \tag{11$''$}$$

U_1'' being equal to U_2' for $\tau=0$. If U_1'' is important merely in the vicinity of the branch point $\tau=0$, (11$''$) reduces to

$$iU_1''\int_{-\infty}^{\infty} e^{-i\tau z-\sqrt{\tau^2-\eta^2}rchj}\,d\tau\,,$$

that is,

$$U_1''\int_{-\infty}^{\infty} \frac{e^{-i\sqrt{Z^2-\eta^2}z+iZrchj}}{\sqrt{Z^2-\eta^2}} ZdZ\,, \tag{11$'''$}$$

so that the contribution from the branch point $Z=\eta$ to the integral (10) is

$$\int_0^{\infty} dj \int_{-\infty}^{\infty} \frac{Ue^{-\sqrt{Z^2-\eta^2}z+iZrchj}}{\sqrt{Z^2-\eta^2}} ZdZ = i\pi U_1'' \frac{e^{i\eta R}}{R}\,, \tag{13}$$

where $R^2=r^2+z^2$, $r\neq 0$, $z\neq 0$.

$$\text{(ii)}\quad \int_{i\infty}^{\kappa} \frac{U_1 e^{\sqrt{Z^2-\eta^2}z+iZrchj}}{-\sqrt{Z^2-\eta^2}} ZdZ + \int_{\kappa}^{i\infty} \frac{U_2 e^{-\sqrt{Z^2-\eta^2}z+iZrchj}}{\sqrt{Z^2-\eta^2}} ZdZ \tag{14}$$

along the branch line through $Z=\kappa$ (as an example). In this case, we write

$$\pm\sqrt{Z^2-\kappa^2}=\pm i\tau\,,\qquad Z=i\sqrt{\tau^2-\kappa^2}\,,\qquad dZ=\frac{i\tau d\tau}{\sqrt{\tau^2-\kappa^2}}\,, \tag{15}$$

so that (14) becomes

$$\int_{\infty}^{0}\frac{U_1' e^{-i\sqrt{\tau^2-\varkappa^2+\eta^2}z-\sqrt{\tau^2-\varkappa^2}rchj}}{\sqrt{\tau^2-\kappa^2+\eta^2}}(-\tau)d\tau$$
$$+\int_0^{\infty}\frac{U_2' e^{-i\sqrt{\tau^2-\varkappa^2+\eta^2}z-\sqrt{\tau^2-\varkappa^2}rchj}}{\sqrt{\tau^2-\kappa^2+\eta^2}}(-\tau)d\tau\,, \tag{14'}$$

which vanishes if $U_1'=U_2'$ for $\tau=0$.

$$\text{(iii)}\quad \int_{i\infty}^{\varkappa} U_1 e^{\sqrt{Z^2-\varkappa^2}z+iZrchj}\,ZdZ+\int_{\varkappa}^{i\infty} U_2 e^{-\sqrt{Z^2-\varkappa^2}z+iZrchj}\,ZdZ \tag{16}$$

along the branch line through $Z=\kappa$, for example. Write

$$\pm\sqrt{Z^2-\kappa^2}=\pm i\tau\,,$$

then

$$\int_{\infty}^{0} U_1' e^{i\tau z-\sqrt{\tau^2-\varkappa^2}rchj}(-\tau)d\tau+\int_0^{\infty} U_2' e^{-i\tau z-\sqrt{\tau^2-\varkappa^2}rchj}(-\tau)d\tau\,. \tag{16'}$$

Consider the condition that $U_1'=U_2'$ for $\tau=0$; and put $\tau=-\tau'$, then (16′) reduces to

$$U_1'\int_{-\infty}^{\infty} e^{-i\tau z-\sqrt{\tau^2-\varkappa^2}rchj}(-\tau)d\tau\,, \tag{16''}$$

that is,

$$U_1'\int_{-\infty}^{\infty} e^{-\sqrt{Z^2-\varkappa^2}z+iZrchj}\,ZdZ\,, \tag{16'''}$$

so that the contribution from the branch point $Z=\kappa$ to the integrals concerning ψ_1, ψ_2 in (8), (9) is

$$\int_0^{\infty}dj\int_{-\infty}^{\infty} U e^{-\sqrt{Z^2-\varkappa^2}z+iZrchj}\,ZdZ=i\pi U_1'\frac{\partial}{\partial z}\frac{e^{i\varkappa R}}{R}\,. \tag{17}$$

5. *General expressions of the waves.*

From the results shown in the previous sections it is now possible to get the general expressions of waves at relatively large values of r, z.

The parts of ϕ_1, ϕ_2, ψ_1, ψ_2 corresponding to the surface waves are such that

$$\phi_{1\mathrm{I}}=\frac{A(\kappa)}{\Phi'(\kappa)}e^{-\sqrt{\kappa^2-h_1^2}\,z-\sqrt{\kappa_1^2-h_1^2}\,\xi-ipt}\frac{\kappa}{\pi i\sqrt{\kappa^2-h_1^2}}\int_0^\infty e^{i\kappa r ch j}dj$$

$$=\frac{A(\kappa)}{\Phi'(\kappa)}e^{-\sqrt{\kappa^2-h_1^2}\,z-\sqrt{\kappa_1^2-h_1^2}\,\xi-ipt}\frac{\kappa}{2\sqrt{\kappa^2-h_1^2}}H_0^{(1)}(\kappa r)\,, \tag{18}$$

$$\Phi_{2\mathrm{I}}=\frac{C(\kappa)}{\Phi'(\kappa)}e^{-\sqrt{\kappa^2-h_2^2}\,z-\sqrt{\kappa_1^2-h_1^2}\,\xi-ipt}\frac{\kappa}{2\sqrt{\kappa^2-h_1^2}}H_0^{(1)}(\kappa r)\,, \tag{19}$$

$$\psi_{1\mathrm{I}}=\frac{B(\kappa)}{\Phi'(\kappa)}e^{-\sqrt{\kappa^2-k_1^2}\,z-\sqrt{\kappa^2-h_1^2}\,\xi-ipt}\frac{\kappa}{\pi\sqrt{\kappa^2-h_1^2}}\int_0^\infty e^{i\kappa r ch j}\,chjdj$$

$$=\frac{B(\kappa)}{\Phi'(\kappa)}e^{-\sqrt{\kappa^2-k_1^2}\,z-\sqrt{\kappa^2-h_1^2}\,\xi-ipt}\frac{\kappa}{2\sqrt{\kappa^2-h_1^2}}H_1^{(1)}(\kappa r)\,, \tag{20}$$

$$\psi_{2\mathrm{I}}=\frac{D(\kappa)}{\Phi'(\kappa)}e^{-\sqrt{\kappa^2-k_2^2}\,z-\sqrt{\kappa^2-h_1^2}\,\xi-ipt}\frac{\kappa}{2\sqrt{\kappa^2-h_1^2}}H_1^{(1)}(\kappa r)\,, \tag{21}$$

where $A(\kappa)$, $B(\kappa)$, $C(\kappa)$, $D(\kappa)$, $\Phi'(\kappa)$, h_1, k_1, h_2, k_2 are the same as $A_1\Phi_1$, $B_1\Phi_1$, $C_1\Phi_1$, $D_1\Phi_1$, Φ_1, h, k, h', k' in (35) of the previous paper[5], $\Phi(\kappa)$ being the same as Φ_1 in (36) of the same paper.

It should be borne in mind that no surface wave exists when there is no root of

$$\Phi(\kappa)=0, \tag{22}$$

that is to say, when no Stoneley wave exists, the condition of which has been shown in a previous paper.[6]

The parts ϕ_1, ψ_1, ϕ_2, ψ_2 corresponding to the bodily waves are such that

$$\phi_{1\mathrm{II}}=A'(h_1)e^{-ipt}\frac{e^{ih_1R}}{R}, \qquad [R^2=r^2+(z+\xi)^2] \tag{23}$$

$$\psi_{1\mathrm{II}}=B'(k_1)e^{-ipt}\frac{r}{R}\frac{z}{R}\frac{\partial^2}{\partial R^2}\frac{e^{ik_1R}}{R}e^{-\sqrt{k_1^2-h_1^2}\,\xi}\,, \quad [R^2=r^2+z^2] \tag{24}$$

$$\Phi_{2\mathrm{II}}=C'(h_2)e^{-ipt}\frac{z}{R}\frac{\partial}{\partial R}\frac{e^{ih_2R}}{R}e^{-\sqrt{h_2^2-h_1^2}\,\xi}\,, \qquad [R^2=r^2+z^2] \tag{25}$$

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5) K. SEZAWA and K. KANAI, *loc. cit.* 1). Φ_1 in (35) of the same paper was misprinted and should be replaced by Φ_1'.

6) K. SEZAWA and K. KANAI, *loc. cit.* 4).

$$\psi_{2\text{II}} \doteq D'(k_2)e^{-ipt}\frac{r}{R}\frac{z}{R}\frac{\partial^2}{\partial R^2}\frac{e^{ik_2R}}{R}e^{-\sqrt{k_2^2-k_1^2}\,\xi}, \quad [R^2=r^2+z^2] \tag{26}$$

where $A'(h_1)$, $B'(k_1)$, $C'(h_2)$, $D'(k_2)$ are the same as A_2, B_2, C_2, D_2 in (37)~(41) of the previous paper.[7]

The expressions of displacements for the reflection wave side, for example, are determined by relations of types

$$\left.\begin{aligned} u_1&=\frac{\partial}{\partial x}(\phi_{1\text{I}}+\phi_{1\text{II}}), & v_1&=\frac{\partial}{\partial y}(\phi_{1\text{I}}+\phi_{1\text{II}}), \\ u_1'&=\frac{\partial}{\partial y}(\psi_{1\text{I}}+\psi_{1\text{II}}), & v_1'&=-\frac{\partial}{\partial x}(\psi_{1\text{I}}+\psi_{1\text{II}}). \end{aligned}\right\} \tag{27}$$

It will be seen that, whereas the amplitudes of the boundary surface waves decrease very slowly with changes in the epicentral distance, those of bodily waves decrease very rapidly with increase in horizontal distance, the detailed feature of which will be discussed in Section 6.

6. *The case of primary distortional waves with amplitudes orientated horizontally.*

In this case the problem is very simple. The expressions for the incident, reflected, and refracted waves assume the forms

$$v_0=\mathfrak{B}e^{\beta_1 z-ipt}J_0(fr), \tag{28}$$

$$v_1=B_1e^{-\beta_1 z-ipt}J_0(fr), \qquad v_2=B_2e^{\beta_2 z-ipt}J_0(fr), \tag{29}$$

where

$$B_1=\frac{\mu_1\beta_1-\mu_2\beta_2}{\mu_1\beta_1+\mu_2\beta_2}\mathfrak{B}, \qquad B_2=\frac{2\mu_1\beta_1}{\mu_1\beta_1+\mu_2\beta_2}\mathfrak{B}. \tag{30}$$

When the primary waves are radiated from a point source $r=0$, $z=\xi$, the displacements of the three kinds of waves assume the forms

$$v_0=\mathfrak{B}e^{-ipt}\frac{e^{ik_1R}}{R}, \qquad [R^2=r^2+(z-\xi)^2] \tag{31}$$

$$v_1=-\mathfrak{B}e^{-ipt}\frac{e^{ik_1R}}{R}, \qquad [R^2=r^2+(z+\xi)^2,\ z>0] \tag{32}$$

$$v_2=2\mathfrak{B}\frac{e^{-\sqrt{k_2^2-k_1^2}\,\xi}}{\sqrt{k_2^2-k_1^2}}\frac{z}{R}\frac{\partial}{\partial R}H_0^{(1)}(k_2R). \qquad [z>0] \tag{33}$$

In this case the amplitudes of the waves vary as $z/R^{3/2}$.

7) *loc. cit.* 1).

7. *Comparison of all the cases.*

Although the case of distortional waves with amplitudes without being orientated in a plane parallel to the discontinuous surface, is omitted, the problem of that case is somewhat similar to that discussed in Sections 2, 3, 4. The generated waves are then boundary waves and bodily waves. The condition of possible existence of boundary waves is the same as that for Stoneley waves.

At all events, the amplitudes of boundary waves, if the waves were generated, vary as in the law R^{-1}. The variations in the amplitudes of the bodily waves are somewhat complex. If the primary waves were distortional, with amplitudes orientated horizontally, the variation of the generated waves obeys the law

$$\overline{v} \propto \frac{z}{(z^2+r^2)^{3/4}}. \tag{34}$$

If the primary waves were dilatational, the horizontal and vertical displacements of the generated dilatational waves would vary as

$$u_1 \propto \frac{rz}{(z^2+r^2)^{5/4}}, \quad v_1 \propto \frac{z^2}{(z^2+r^2)^{5/4}}, \qquad (35),\ (36)$$

respectively, and the horizontal and vertical displacements of the generated distortional waves would vary as

$$u_2 \propto \frac{rz^2}{(z^2+r^2)^{7/4}}, \quad v_2 \propto \frac{r^2 z}{(z^2+r^2)^{7/4}}, \qquad (37),\ (38)$$

respectively.

From (34)$\sim$(38), it will be seen that, whereas v_1, u_2 decrease very rapidly with increase in epicentral distance, u_1, v_2, $\overline{v}$ decrease relatively slowly with the same increase in that epicentral distance. It is now possible to conclude that although the law of decrease in the amplitudes for u_1, v_2, $\bar{v}$ is the same, the region in which $\bar{v}$ assumes a relatively large value, is somewhat greater than that in which u_1 or v_2 assumes a relatively large value.

33. 地殼內の不連續面に於ける境界波の生成 (第2報)

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地殼內に密度又は彈性の不連續面があると，この面に沿うて英國の Stoneley の述べたやうな地震波が傳播する．前回の第１報に於てこのやうな波が如何に作られるかを２次元の彈性論を以て說明し，且つ，實際にあるやうな３次元のものは研究中といふ事にして置いたところが，英國のある人がある雜誌にこの３次元の方を一層期待するやうに述べてをるので，物理的の本質はあまり變らぬけれども，ともかくも３次元の問題を解いて出して置いた次第である．計算の方法も前回と同樣に Sommerfeld の電磁波の解法に關係あるものを用ひたのであるから．第１報と異なるやうな特徴が殆どない．卽ち前回の研究の擴張を試みたといふのに過ぎないのである．