18. *On the Packet Velocity of Dispersive Elastic Waves of Irregular Form.*

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(Read March 22, 1939.—Received March 23, 1939.)

1. *Introduction. Velocity of a wave packet.*

It is well known that the energy of waves is transmitted with a group velocity that corresponds to the length of those waves. Let c, U be the velocities of transmission of the wave form and the wave energy respectively, then we have the relation

$$U=\frac{d}{df}(cf), \tag{1}$$

where $2\pi/f$ is the wave length. If we draw a tangent line to a point on the dispersion curve of the waves whose abscissa and ordinate represent wave length $2\pi/f$ and phase velocity c respectively, then the length of that part of the y-axis that is below the point of intersection with the tangent line just given, gives the group velocity of waves of length corresponding to the point on which the tangent line was drawn.

If the disturbance that is transmitted through a dispersive medium is of some irregular form that is not of sine form, the deformation of the waves with time is, usually, not simple, in which case the usual idea of group velocity is not of much avail.

Although a relatively complex case of dispersive elastic waves has been shown in a previous paper,[1] it is impossible for that special case to be used as representing general dispersive elastic waves. On the other hand, Coulomb[2] gave an example of the behaviour of a packet of Rayleigh or of Love waves, in which a special centre of the packet of those waves (that is, a centre of wave assembly) is transmitted with the group velocity of the longest periodic waves pertaining to the disturbance. From another investigation[3] of ours, it has been shown that, in the case of periodic waves, the leading or trailing part of the same waves is transmitted with group velocity. We are thus now in a posi-

1) K. SEZAWA and G. NISHIMURA, *Bull. Earthq. Res. Inst.*, 8 (1930), 330.
2) M. J. COULOMB, *Livre jubilaire de M. Marcel Brillouin*, 1934.
3) K. SEZAWA, *Bull. Earthq. Res. Inst.*, 4 (1928), 107.

tion to ascertain the nature of the dispersive elastic waves, particularly when the wave form is of irregular type.

We shall first solve various different cases, after which our conclusion with respect to the group velocity of the dispersive elastic waves will be given. Although a special case of periodic waves transmitted through a dispersive medium has already been shown in the previous paper,[4] to enable us to compare that case with the case of irregular wave form, we shall also solve the problem of periodic waves in a medium, the dispersion condition of which differs somewhat from that already given.

The law of dispersion is generally very complex. In our examples we assumed such formulae as

$$\text{(a)}\quad c=A+B/f \qquad \text{(b)}\quad c=A+Bf. \qquad (2), (3)$$

Although the dispersion formula that fits the case of Rayleigh-waves or Love-waves is, as we have shown in the previous paper,[5] of the following type

$$\text{(c)}\quad c=A+B/(a^2+f^2), \qquad (4)$$

or, at least, of the type

$$\text{(d)}\quad c=A+B/(a+f), \qquad (5)$$

if f be large compared with a, (d) may be replaced by (a), whereas if f be small compared with a, (d) is transformed to (b). Since, mathematically, the use of (c) or (d) is extremely complex, formulae (a) and (b) have been availed of. Coulomb, in treating dispersive Love-waves and Rayleigh-waves, assumed formulae·

$$c=A-Bf^2, \qquad c=A-Bf, \qquad (2')$$

for the respective waves. As a matter of fact, the type of the dispersion equation is not the most important part of our problem; our aim being rather to show the relation between the dispersion equation and the change in wave form, for which formulae (a), (b) are still fairly well adapted.

2. *Periodic waves of finite extent that obey dispersion formula (b).*

Let the initial form of the waves that are transmitted in the positive sense of x be $F(x)$; then the wave form at $t=t$ is expressed by

4) *ibid.*
5) *loc. cit.*, 1).

$$v=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ifct}df\int_{-\infty}^{\infty}F(\sigma)e^{if(\sigma-x)}d\sigma, \tag{6}$$

where c is of type (a) in the present particular case. Since the periodic waves at $t=0$ is of finite extent, we put

$$\left.\begin{aligned} F(\sigma) &= \cos\lambda\sigma, &\quad [0<x<a] \\ &=0. &\quad [0>x,\ a<x]. \end{aligned}\right\} \tag{7}$$

It is restricted that a should be finite, otherwise it is impossible to get the integral. When the dispersion formula (b), that is, $c=A+Bf$, is used, (6) then reduces to

$$v=\frac{i}{4\pi}\int_{-\infty}^{\infty}e^{ift(A+Bf)-ifx+ifa}\left\{\frac{-e^{ia\lambda}}{\lambda+f}+\frac{e^{-ia\lambda}}{\lambda-f}\right\}df$$
$$+\frac{i}{4\pi}\int_{-\infty}^{\infty}e^{ift(A+Bf)-ifx}\left\{\frac{1}{\lambda+f}-\frac{1}{\lambda-f}\right\}df. \tag{8}$$

To evaluate this, we consider the integrals

$$\int e^{iZt(A+BZ)-iZx+iZa}\left\{\frac{-e^{iaZ}}{\lambda+Z}+\frac{e^{-iaZ}}{\lambda-Z}\right\}dZ, \tag{9}$$

$$\int e^{iZt(A+BZ)-iZx}\left\{\frac{1}{\lambda+Z}-\frac{1}{\lambda-Z}\right\}dZ, \tag{10}$$

taken round the contour shown in the sketch, the singular points lying at $Z=-\lambda$, $Z=\lambda$. The part of the integral that is taken along the real axis corresponds to the expression in (8).

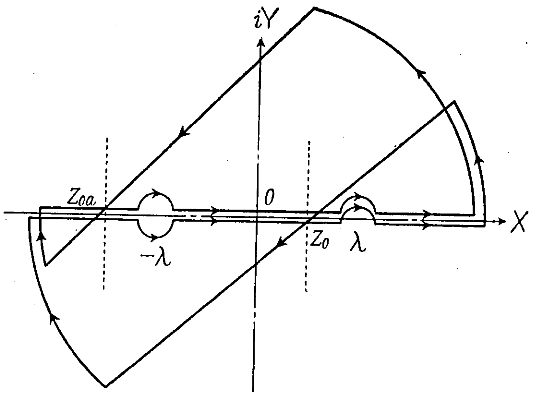

Fig. 1.

The inclined part in the contour is so chosen that the same line, passing through the saddle point of the exponent in the integrand, traverses the line of steepest descent of the same exponent. To determine the saddle point of (9), for example, we write

$$\Phi(Z)=iZ^2Bt+iZ(At-x+a). \qquad [Z=X+iY] \tag{11}$$

This function is stationary when

$$\Phi'(Z)=0, \tag{12}$$

from which we determine the coordinates of the saddle point. In the present case, we shall specially write

$$\Phi(Z)=R+iI, \tag{13}$$

in which

$$\left.\begin{aligned} R&=-2BtXY-(At-x+a)Y, \\ I&=Bt(X^2-Y^2)+(At-x+a)X, \end{aligned}\right\} \tag{14}$$

and perform the operations

$$\frac{\partial R}{\partial X}=0, \qquad \frac{\partial R}{\partial Y}=0, \tag{15}$$

from which we get the coordinates of the saddle point, such that

$$X=\frac{-(At-x+a)}{2Bt}, \qquad Y=0. \tag{16}$$

The value of R at this saddle point is

$$R=0. \tag{17}$$

If we seek the locus of points at which the value of R is the same as that at the saddle point, we find a pair of straight lines, defined by

$$X=-\frac{(At-x+a)}{2Bt}, \qquad Y=0. \tag{18}$$

The valleys are thus nearly in the first and third quadrants and the hills nearly in the second and fourth quadrants. The line of steepest descent is that which bisects the angle between the two lines shown in (18). Since the important part of the integration along such a line is only that from the part near the saddle point, we shall expand $\Phi(Z)$ in the form

$$\Phi(Z)=\Phi(Z_0)+\frac{1}{2}(Z-Z_0)^2F''(Z_0)+\cdots, \tag{19}$$

remembering that $\Phi'(Z_0)=0$, taking the expanded terms to only the second order. If, for simplicity, we write

$$Z-Z_0=qe^{i\alpha}, \tag{20}$$

since $\Phi''(Z_0)=2iBt$, we get

$$2\int_0^\infty e^{\Phi(Z)}dZ=2e^{-\frac{i(At-x+a)^2}{4Bt}}\int_0^\infty e^{iBtq^2}dq e^{i\alpha} \qquad \left(\alpha=\frac{\pi}{4}\right)$$

$$=-i\frac{\sqrt{\pi}}{\sqrt{Bt}}e^{-\frac{i(At-x+a)^2}{4Bt}}. \tag{21}$$

In the same way, it is possible to get the integration of a similar kind for the case that corresponds to (10).

The contribution from the above integration to expression (8) is now written

$$v_1=\frac{1}{4\sqrt{\pi Bt}}e^{-\frac{i(At-x+a)^2}{4Bt}}\left\{\frac{-e^{ia\lambda}}{\lambda-\frac{At-x+a}{2Bt}}+\frac{e^{-ia\lambda}}{\lambda+\frac{At-x+a}{2Bt}}\right\}$$

$$+\frac{1}{4\sqrt{\pi Bt}}e^{-\frac{i(At-x)^2}{4Bt}}\left\{\frac{1}{\lambda-\frac{At-x}{2Bt}}-\frac{1}{\lambda+\frac{At-x}{2Bt}}\right\}. \tag{22}$$

There is no contribution from parts of circles of infinite radius. Since the parts of these circles range from 0° to 45° and also from 180° to 225°, the factor of the type $e^{\alpha Z^2}$ of each integrand in (9), (10) tends to zero of the order of $e^{-\alpha' R^2}$. Although the factor of the type $e^{\beta Z}$ in the same integrand diverges to infinity of the order $e^{\beta' R}$, the resultant of both factors causes the integrand to vanish.

Although the nature of the singular points $Z=-\lambda$, λ is very simple, owing to the position of that part of the contour that corresponds to the line of steepest descent, the contribution from such points to the integral is somewhat complex.

We shall take twenty different conditions of the contour, namely, conditions relating to the position of the line of steepest descent relative to the singular points, that is, the poles. The coordinate of the saddle point for integral (9) is shown in (16), namely,

$$X=\frac{-(At-x+a)}{2Bt}, \qquad Y=0, \tag{23}$$

and that for integral (10) is given by

$$X=\frac{-(At-x)}{2Bt}, \qquad Y=0. \tag{24}$$

We shall call the saddle points shown in (23), (24), Z_{0a} and Z_0, respectively.

(A) Let us assume that B is positive.

(i) When the positions of the saddle points and poles are arranged, as showe in Fig. 2a, we have the conditions

$$-\lambda > -\left(\frac{At-x+a}{2Bt}\right), \quad \lambda < -\left(\frac{At-x}{2Bt}\right), \tag{25a}$$

$$\frac{At-x+a}{2Bt} > 0 > \frac{At-x}{2Bt}, \tag{26a}$$

in which case the contribution to the integral comes from both poles. In that case, the contribution from the poles to the integral v_2 is

$$v_2 = -\frac{1}{4}\left[-2e^{-i\lambda t(A-B\lambda)+i\lambda x} - 2e^{i\lambda t(A+B\lambda)-i\lambda x}\right]$$

$$= e^{i\lambda^2 Bt}\cos\lambda(x-At). \tag{27a}$$

(ii) When the saddle points and poles are arranged as shown in Fig. 2b, we have the conditions

$$-\lambda > -\left(\frac{At-x+a}{2Bt}\right), \quad \lambda > -\left(\frac{At-x}{2Bt}\right), \tag{25b}$$

$$\frac{At-x+a}{2Bt} > 0 > \frac{At-x}{2Bt}, \tag{26b}$$

the contribution to v_2 being then

$$v_2 = \frac{1}{2}e^{i\lambda(x-At)+i\lambda^2 Bt}. \tag{27b}$$

(iii) When the saddle points and poles are arranged, as shown in Fig. 2c, the conditions are

$$-\lambda < -\left(\frac{At-x+a}{2Bt}\right), \quad \lambda < -\left(\frac{At-x}{2Bt}\right), \tag{25c}$$

$$\frac{At-x+a}{2Bt} > 0 > \frac{At-x}{2Bt}, \tag{26c}$$

the contribution to v_2 being

$$v_2 = \frac{1}{2}e^{-i\lambda(x-At)+i\lambda^2 Bt}. \tag{27c}$$

(iv) When the saddle points and poles are arranged, as shown in Fig. 2d, we have

$$-\lambda < -\left(\frac{At-x+a}{2Bt}\right), \quad \lambda > -\left(\frac{At-x}{2Bt}\right), \tag{25d}$$

$$\frac{At-x+a}{2Bt} > 0 > \frac{At-x}{2Bt}, \tag{26d}$$

in which case the contributions from both poles cancel each other, so that

$$v_2 = 0. \tag{27d}$$

(v) When the saddle points and poles are arranged as shown in Fig. 2e, we have

$$-\lambda > -\left(\frac{At-x}{2Bt}\right); \quad \frac{At-x}{2Bt} > 0, \qquad (25e),\ (26e)$$

from which

$$v_2 = 0. \tag{27e}$$

(vi) When the saddle points and poles are arranged as shown in Fig. 2f, the conditions are

$$-\lambda > -\left(\frac{At-x+a}{2Bt}\right), \quad -\lambda < -\left(\frac{At-x}{2Bt}\right); \quad \frac{At-x}{2Bt} > 0, \qquad (25f),\ (26f)$$

so that

$$v_2 = \frac{1}{2} e^{i\lambda(x-At)+i\lambda^2 Bt}. \tag{27f}$$

(vii) When the saddle points and poles are arranged as shown in Fig. 2g, the conditions are

$$-\lambda < -\left(\frac{At-x+a}{2Bt}\right); \quad \frac{At-x}{2Bt} > 0, \qquad (25g),\ (26g)$$

the contribution to v_2 being

$$v_2 = 0. \tag{27g}$$

(viii), (ix) From their quantitative natures, the arrangements of the saddle points and poles, as shown in Figs. 2h and 2i, are impossible.

(x) When the saddle points and poles are arranged as shown in Fig. 2j, the conditions are

$$\lambda > -\left(\frac{At-x}{2Bt}\right); \quad \frac{At-x+a}{2Bt} < 0, \qquad (25j),\ (26j)$$

the contribution to v_2 being zero, that is

$$v_2=0. \tag{27j}$$

(xi) When the saddle points and poles are arranged as shown in Fig. 2k, we have the conditions

$$\lambda>-\left(\frac{At-x+a}{2Bt}\right), \quad \lambda<-\left(\frac{At-x}{2Bt}\right); \quad \frac{At-x+a}{2Bt}<0, \qquad (25k),\ (26k)$$

so that

$$v_2=\frac{1}{2}e^{-i\lambda(x-At)+i\lambda^2Bt}. \tag{27k}$$

(xii), (xiii), (xiv) From the nature of the problem, the arrangements of the saddle points and poles as shown in Figs. 2l, 2m, 2n are impossible.

(xv) When the saddle points and poles are arranged as shown in Fig. 2o, we have the conditions

$$\lambda<-\left(\frac{At-x+a}{2Bt}\right); \quad \frac{At-x+a}{2Bt}<0, \qquad (25o),\ (26o),$$

from which we get

$$v_2=0. \tag{27o}$$

(xvi), (xvii), (xviii), (xix), (xx) For the same reason as that already shown, the arrangements of saddle points and poles as shown in Figs. 2p, 2q, 2r, 2s, 2t are impossible.

(B) If B be negative, we write $B=-B'$.

(i) When the saddle points and poles are arranged as shown in Fig. 2s, we have the conditions

$$\frac{At-x+a}{2B't}>\lambda, \quad \frac{At-x}{2B't}<-\lambda, \tag{25s'}$$

$$\frac{At-x+a}{2B't}>0>\frac{At-x}{2B't}, \tag{26s'}$$

the contribution to v_2 being

$$v_2=-e^{-i\lambda^2B't}\cos\lambda(x-At). \tag{27s'}$$

(ii) When the saddle points and poles are arranged as shown in Fig. 2r, the conditions are

$$\frac{At-x+a}{2B't}>\lambda, \quad \frac{At-x}{2B't}>-\lambda, \tag{25r'}$$

$$\frac{At-x+a}{2B't}>0>\frac{At-x}{2B't}, \tag{26r'}$$

from which we have

$$v_2=-\frac{1}{2}e^{i\lambda(At-x)-i\lambda^2B't}. \tag{27r'}$$

(iii) When the saddle points and poles are arranged as shown in Fig. 2n, the conditions are

$$\frac{At-x+a}{2B't}<\lambda, \quad \frac{At-x}{2B't}<-\lambda, \tag{25n'}$$

$$\frac{At-x+a}{2B't}>0>\frac{At-x}{2B't}, \tag{26n'}$$

from which we get

$$v_2=-\frac{1}{2}e^{-i\lambda(At-x)-i\lambda^2B't}. \tag{27n'}$$

(iv) When the saddle points and poles are arranged as shown in Fig. 2m, we have

$$\frac{At-x+a}{2B't}<\lambda, \quad \frac{At-x}{2B't}>-\lambda, \tag{25m'}$$

$$\frac{At-x+a}{2B't}>0>\frac{At-x}{2B't}, \tag{26m'}$$

from which we get

$$v_2=0. \tag{27m'}$$

(v) When the saddle points and poles are arranged as shown in Fig. 2h, we have

$$\frac{At-x}{2B't}>-\lambda; \quad \frac{At-x+a}{2B't}<0, \qquad \text{(25h')}, \text{(26h')}$$

the contribution to v_2 being

$$v_2=0. \tag{27h'}$$

(vi) When the saddle points and poles are arranged as shown in Fig. 2j, the conditions are

$$\frac{At-x+a}{2B't}>-\lambda, \quad \frac{At-x}{2B't}<-\lambda; \quad \frac{At-x+a}{2B't}<0, \qquad (25t'), (26t')$$

from which we have

$$v_2=-\frac{1}{2}e^{-i\lambda(At-x)-i\lambda^2B't}. \qquad (27t')$$

(vii), (viii), (ix) From the nature of things, the arrangements of the saddle points and poles as shown in Figs. 2g, d, c are impossible.

(x) When the saddle points and poles are arranged as shown in Fig. 2l, the conditions are

$$\frac{At-x+a}{2B't}<\lambda; \quad \frac{At-x}{2B't}>0, \qquad (25l'), (26l')$$

from which we get

$$v_2=0. \qquad (27l')$$

(xi) When the saddle points and poles are arranged as shown in Fig. 2q, the conditions are

$$\frac{At-x+a}{2B't}>\lambda, \quad \frac{At-x}{2B't}<\lambda; \quad \frac{At-x}{2B't}>0, \qquad (25q'), (26q')$$

from which

$$v_2=-\frac{1}{2}e^{i\lambda(At-x)-i\lambda^2B't}. \qquad (27q')$$

(xii), (xiii), (xiv), (xv), (xvi), (xvii), (xviii) No such arrangements of saddle points and poles, as shown in Figs. 2j, b, k, o, e, f, a, can possibly exist.

(xix) When the saddle points and poles are arranged as shown in Fig. 2p, the conditions are

$$\frac{At-x}{2B't}>\lambda; \quad \frac{At-x}{2B't}>0, \qquad (25p'), (26p')$$

from which we get

$$v_2=0. \qquad (27p')$$

(xx) When, finally, the saddle points and poles are arranged as shown in Fig. 2i, we have

$$\frac{At-x+a}{2B't}<-\lambda; \quad \frac{At-x+a}{2B't}<0, \qquad (25i'), (26i')$$

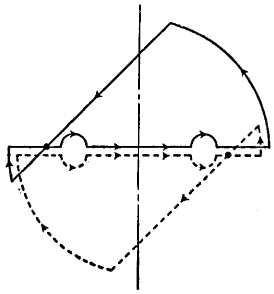

Fig. 2a.

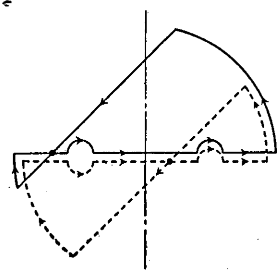

Fig. 2b.

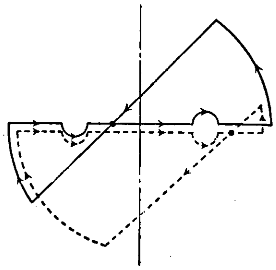

Fig. 2c.

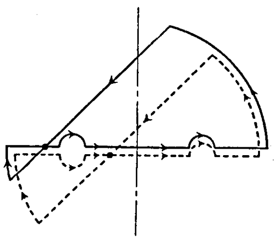

Fig. 2d.

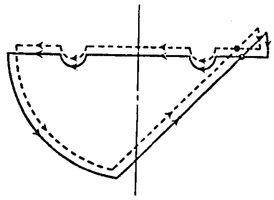

Fig. 2e.

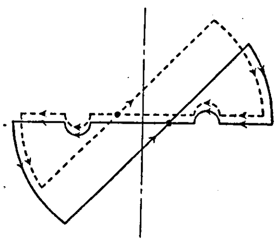

Fig. 2f.

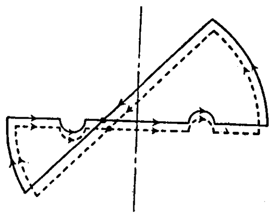

Fig. 2g.

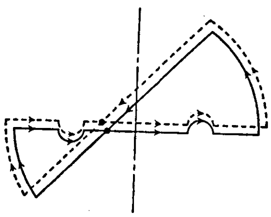

Fig. 2h.

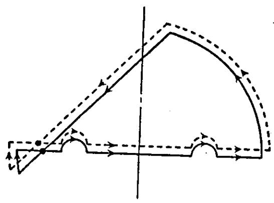

Fig. 2i.

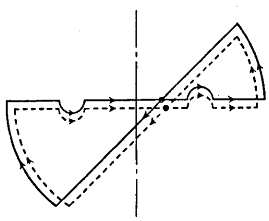

Fig. 2j.

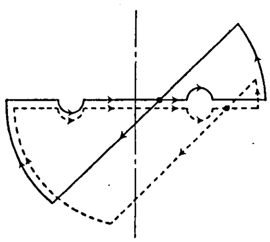

Fig. 2k.

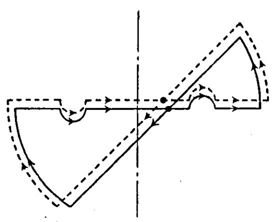

Fig. 2l.

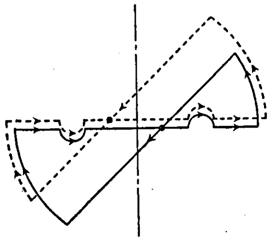

Fig. 2m.

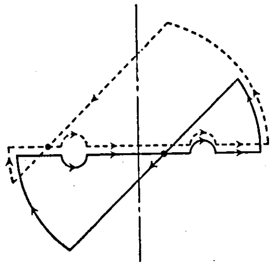

Fig. 2n.

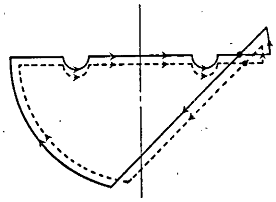

Fig. 2o.

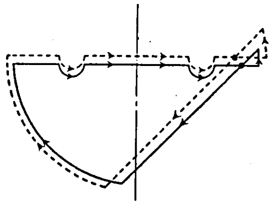

Fig. 2p.

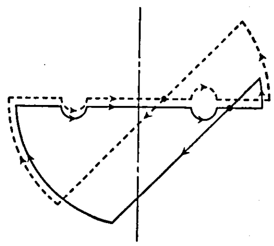

Fig. 2q.

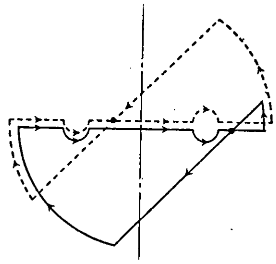

Fig. 2r.

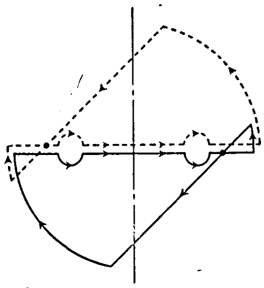

Fig. 2s.

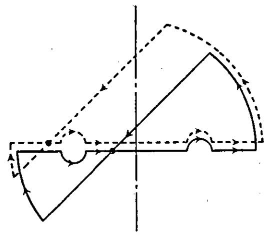

Fig. 2t.

from which the contribution to v_2 becomes

$$v_2=0. \tag{27i'}$$

We shall now summarize the results shown in (25)'s, (26)'s, and (27)'s as follows, $|B|$ being the absolute value of B. The upper and lower signs correspond to cases for B-positive and B-negative respectively.

$\{a+t(A-2|B|\lambda)\}>x>t(A+2|B|\lambda)$;

$$v_2=\pm\frac{1}{2}\left[e^{i\lambda\{x-t(A-B\lambda)\}}+e^{-i\lambda\{x-t(A+B\lambda)\}}\right],$$

$\{a+t(A-2|B|\lambda))\}>x>At, \quad t(A+2|B|\lambda)>x;$

$$v_2=\pm\frac{1}{2}e^{i\lambda\{x-t(A-B\lambda)\}},$$

$(a+At)>x>\{a+t(A-2|B|\lambda)\}, \quad x>t(A+2|B|\lambda);$

$$v_2=\pm\frac{1}{2}e^{-i\lambda\{x-t(A+B\lambda)\}},$$

$(a+At)>x>\{a+t(A-2|B|\lambda)\}, \quad t(A+2|B|\lambda)>x>At; \qquad v_2=0,$

$\{a+t(A+2|B|\lambda))\}>x>t(A+2|B|\lambda), \quad x>(a+At);$

$$v_2=\pm\frac{1}{2}e^{-i\lambda\{x-t(A+B\lambda)\}},$$

$t(A+2|B|\lambda)>x>(a+At); \qquad v_2=0,$

$x>\{a+t(A+2|B|\lambda)\}; \qquad v_2=0,$

$\{a+t(A-2|B|\lambda))\}>x>t(A-2|B|\lambda), \quad At>x;$

$$v_2=\pm\frac{1}{2}e^{i\lambda\{x-t(A-B\lambda)\}},$$

$t(A-2|B|\lambda)>x; \qquad v_2=0,$

$At>x>\{a+t(A-2|B|\lambda)\}; \qquad v_2=0. \qquad (28)$

As already remarked, v_1 is the contribution from the saddle points and v_2 that from the poles. Strictly speaking, v_1 is distributed through the whole range of x for any t, whereas v_2 extends through only a narrow range of x for every t. If we examine the natures of v_1 and v_2 more closely, there are some features to be noticed.

For v_1, the wave amplitudes are maximum at two phase conditions, namely, the condition of the phase $x-At=0$ and that of the phase $x-At-a=0$. With increases in the absolute values of phases $(x-At)$ and $(x-At-a)$, amplitude v_1 decreases. Independent of these conditions, v_1 also decreases with increase in t, in consequence of which v_1 tends to decay with time.

The conditions of phases $(x-At)$, $(x-At+a)$ above given are rather approximate. Mathematically, there are four phases, such that $(A+2B\lambda)t-x$, $(A+2B\lambda)t-x+a$, $(A-2B\lambda)t-x$, $(A-2B\lambda)t-x+a$. Although the phases $(A+2B\lambda)t-x$, $(A+2B\lambda)t-x+a$ represent the group wave that corresponds to wave length $2\pi/\lambda$, the forms of the remaining phases, namely, $(A-2B\lambda)t-x$, $(A-2B\lambda)t-x+a$ are very odd.

Another feature of v_1 to be remarked is that, with increase in the absolute values of phases $x-At$, $x-At-a$, the form of v_1 becomes increasingly undulatory, the wave lengths of the respective undulatory parts augmenting with lapse of time.

The amplitude of v_2 does not change during the transmission of waves through a medium. The disturbed part of the waves also does not change much, except in certain special cases. Although the velocities of the leading and trailing parts of the waves cannot be expressed in simple forms, it is however possible to conclude that, whereas the velocity of the leading part of the disturbance never exceeds $A+2|B|\lambda$, the trailing part is never less than $A-2|B|\lambda$. Since $A+2|B|\lambda$ and $A-2|B|\lambda$ are group velocities of waves of length for B-positive and B-negative respectively, it is likely that the usual criterion that a wave group shall be tansmitted with group velocity, in the present particular problem, is available only for the velocity of the leading part for the case of B-positive and to the velocity of the trailing part for the case of B-negative.

In the case of B-positive, the group velocity is higher than the phase velocity, while in the case of B-negative, the condition is exactly reversed. On the other hand, the energy of the waves is transmitted with a velocity that corresponds to the group velocity of the waves. It holds, then, that the leading part, at least, of the waves for the case of B-positive, should be transmitted with group velocity. It is likely that a similar condition exists for the trailing part of the waves for B-negative.

The values of the phase velocities are indicated by the exponent in every expression of v_2 in (28). It will be seen that although, generally, the phase is propagated with the corresponding phase velocity, in some special cases the velocity of transmission of such phase differs from what it ought to be.

Now, since the conditions of the waves v_2 as given by the expressions in (28) are fairly complex, the nature of the problem, in its details, is somewhat difficult to interpret. On the other hand, the quantities $At+a+2|B|t\lambda,\cdots$ in the same expressions can be arranged in the order of their values, as in the following five possible conditions.

$$\left.\begin{array}{ll}
\text{(I)} & (At+a+2|B|t\lambda)>(At+a)>(At+a-2|B|t\lambda)>(At \\
& \quad +2|B|t\lambda)>At>(At-2|B|t\lambda), \\
\text{(II)} & (At+a+2|B|t\lambda)>(At+a)>(At+2|B|t\lambda)>(At \\
& \quad +a-2|B|t\lambda)>At>(At-2|B|t\lambda), \\
\text{(III)} & (At+a+2|B|t\lambda)>(At+2|B|t\lambda)>(At+a)>(At \\
& \quad +a-2|B|t\lambda)>At>(At-2|B|t\lambda),
\end{array}\right\} \qquad (30)$$

$$\left.\begin{array}{ll}\text{(IV)} & (At+a+2|B|t\lambda)>(At+2|B|t\lambda)>(At \\ & \quad +a)>At>(At+a-2|B|t\lambda)>(At-2|B|t\lambda), \\ \text{(V)} & (At+a+2|B|t\lambda)>(At+a)>(At+2|B|t\lambda)>At> \\ & \quad (At+a-2|B|t\lambda)>(At-2|B|t\lambda),\end{array}\right\}$$

between the respective limits of which x may be intercepted. If we write ten cases

$$\left.\begin{array}{llll}\text{(i)} & v_2=e^{i\lambda^2Bt}\cos\lambda(x-At), & \text{(vi)} & v_2=0, \\ \text{(ii)} & v_2=\frac{1}{2}e^{i\lambda^2Bt+i\lambda(x-At)}, & \text{(vii)} & v_2=0, \\ \text{(iii)} & v_2=\frac{1}{2}e^{i\lambda^2Bt-i\lambda(x-At)}, & \text{(viii)} & v_2=\frac{1}{2}e^{i\lambda^2Bt+i\lambda(x-At)}, \\ \text{(iv)} & v_2=0, & \text{(ix)} & v_2=0, \\ \text{(v)} & v_2=\frac{1}{2}e^{i\lambda^2Bt-i\lambda(x-At)}, & \text{(x)} & v_2=0,\end{array}\right\} \quad (30)$$

for wave amplitudes, these cases can possibly exist for x lying within their respective ranges between the limits defined by the conditions as shown in (29). They are arranged as follows. The meaning of $>$(v)$>$, for example, is that the value of v_2 at x between $>x>$ should assume the value (v).

$$\begin{array}{ll}\text{(I)} & 0>(At+a+2|B|t\lambda)>\text{(v)}>(At+a)>\text{(iii)}>(At+a-2|B|t\lambda) \\ & \quad >\text{(i)}>(At+2|B|t\lambda)>\text{(ii)}>At>\text{(viii)}>(At-2|B|t\lambda)>0, \\ \text{(II)} & 0>(At+a+2|B|t\lambda)>\text{(v)}>(At+a)>\text{(iii)}>(At+2|B|t\lambda) \\ & \quad >0>(At+a-2|B|t\lambda)>\text{(ii)}>At>\text{(viii)}>(At-2|B|t\lambda)>0, \\ \text{(III)} & 0>(At+a+2|B|t\lambda)>\text{(v)}>(At+2|B|t\lambda)>0>(At+a) \\ & \quad >0>(At+a-2|B|t\lambda)>\text{(ii)}>At>\text{(viii)}>(At-2|B|t\lambda)>0, \\ \text{(IV)} & 0>(At+a+2|B|t\lambda)>\text{(v)}>(At+2|B|t\lambda)>0>(At+a) \\ & \quad >0>At>0>(At+a-2|B|t\lambda)>\text{(viii)}>(At-2|B|t\lambda)>0, \\ \text{(V)} & 0>(At+a+2|B|t\lambda)>\text{(v)}>(At+a)>\text{(iii)}>(At+2|B|t\lambda) \\ & \quad >0>At>0>(At+a-2|B|t\lambda)>\text{(viii)}>(At-2|B|t\lambda)>0.\end{array} \quad (31)$$

It will be seen that, in every condition, no wave exists for $x>At+a+2|B|t\lambda$ and $x<At-2|B|t\lambda$, —the same result as shown previously. For conditons II, III, IV, V, no wave exists for an intermediate phase, and the wave train is split into two packets. For condition I, not only does no split of the wave packet occur, but also the amplitude of the waves at the middle part of the wave train is doubled.

Although it may appear that the above feature is of interest merely from the mathematical point of view, the same feature has, as a matter of fact, an important bearing on the nature of the group velocity of the disturbance. Comparing the respective conditions in (31), we find that the centroid of the whole of the wave trains is transmitted with velocity A. A represents the group velocity of the infinitely long waves, not the group velocity peculiar to the wave length $2\pi/\lambda$.

It is now possible to conclude that, notwithstanding that the periodic disturbance has a group velocity that is proper to the wave length $2\pi/\lambda$, the centroid of the train of the disturbances, that is, the centroid of the wave packet, should rather be transmitted with a group velocity that is peculiar to the periodic waves of infinite length, that is, $L=2\pi/0=\infty$. It is not a matter of importance whether or not the group velocity under consideration is higher than that for any wave of finite length.

It should be borne in mind that a must be finite, otherwise the integral expressions do not apply to the present problem.

3. *A single pulse that obeys dispersion formula* (*a*).

The problem of periodic waves of the type shown in Section 2, which however obeys dispersion formula (a), has already been shown in a previous paper,[6] the result of which was that the leading or trailing part of the wave train is transmitted with group velocity A. We shall now show the case of a single pulse obeying also the law in (a).

In the case of a symmetrical pulse, we put

$$c=A+\frac{B}{f}, \quad F(\sigma)=e^{-\frac{\sigma^2}{a^2}} \tag{32}$$

in the integral

$$v=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ifct}df\int_{-\infty}^{\infty} F(\sigma)e^{if(\sigma-x)}d\sigma, \tag{33}$$

which then reduces to

$$v=e^{-\left(\frac{At-x}{a}\right)^2+iBt}. \tag{34}$$

In the case of an anti-symmetrical pulse, we shall write

$$F(\sigma)=\sigma e^{-\frac{\sigma^2}{a^2}}. \tag{35}$$

6) *loc. cit.*, 3).

Then, from (33) we get

$$v=(x-At)e^{-\left(\frac{At-x}{a}\right)^2+iBt}. \tag{36}$$

Taking the respective real parts, we have

$$v=e^{-\left(\frac{At-x}{a}\right)^2}\cos Bt, \qquad v=(x-At)e^{-\left(\frac{At-x}{a}\right)^2}\cos Bt \tag{37}$$

in place of (34), (36), respectively.

From these expressions, it will be seen that the respective centroids of the symmetrical waves and anti-symmetrical waves are transmitted with group velocity A, without any feature showing diffusion of wave forms with time lapse. The factor $\cos Bt$ in each of the equations (37), however, represents the condition that the amplitude of the whole wave changes sinusoidally with period $2\pi/B$, showing that the sense of amplitude of the pulse changes periodically during its transmission, and also that the amplitude at any part of the pulse wave disappears periodically. This implies the condition that the whole energy of the wave changes from potential to kinentic and then from kinetic to potential. Interpreted physically, since B in (32) gives the phase velocity $B\lambda/2\pi$ for very long waves of length λ, $2\pi/B$ represents the period of the same long waves, with the result that the amplitude of the pulse appears and disappears with a period corresponding to that of the long waves just mentioned.

At all events, since the centroid of the wave energy is defined by the phase relation $x-At$ without regard to the kind of energy, whether potential or kinetic, it holds that in this case, the centroid of the wave energy under consideration is transmitted with the group velocity of the wave system. Since, furthermore, the position of the maximum amplitude (or maximum displacement velocity) relative to the centroid of the wave packet remains constant, it is indiscernible in the present case whether it is the centroid of the wave or the phase of the maximum amplitude that is transmitted with group velocity.

4. *A single pulse that obeys dispersion formula (b).*

The case of a single pulse obeying the law of dispersion in (b) is not so simple as that shown in the preceding section.

In the case of a symmetrical pulse, we write

$$c=A+Bf, \qquad F(\sigma)=e^{-\frac{\sigma^2}{a^2}} \tag{38), (39}$$

in the integral

$$v=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ifct}df\int_{-\infty}^{\infty}F(\sigma)e^{if(\sigma-x)}d\sigma. \tag{40}$$

Substituting (38), (39) in (40), we get

$$v=\frac{a}{2\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\frac{a^2f^2}{4}+if\{(A+Bf)t-x\}}df. \tag{41}$$

To integrate (41), we shall consider an integral of the type

$$\int e^{-\frac{a^2Z^2}{4}+iZ\{(A+BZ)t-x\}}dZ \tag{42}$$

taken round the contour shown in the sketch. When the radius of each circular arc is very large, the integrals along such arcs vanish, particularly, when the arcs are restricted within the ranges $\theta=0\sim45°$ and $180°\sim225°$, the reason of which is that if these arcs be of the ranges just given, the factor $exp.\{-a^2Z^2/4+iBZ^2t\}$ of the integrand in (42) will always tend to zero of the order e^{-aR^2} for $R\to\infty$. Although the factor $exp.i(At-x)Z$ of the same integrand may diverge to infinity of the order e^{bR}, the resultant of both factors still causes the integrand to vanish. There is no singular point within the contour.

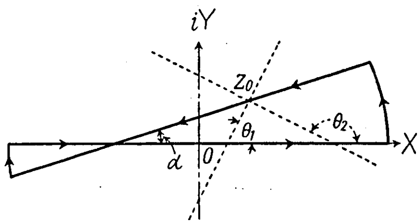

Fig. 3.

The inclined part in the contour is so chosen that the same line passes through the saddle point of the exponent in the integrand just given, and traverses along the line of steepest descent of the same exponent. To determine the saddle point, we write, as in Section 2,

$$\Phi(Z)=-\frac{a^2Z^2}{4}+iZ\Big\{(A+BZ)t-x\Big\}, \tag{43}$$

the stationary point of which is given by

$$\Phi'(Z)=0, \tag{44}$$

its coordinate being therefore

$$Z_0=\frac{i(At-x)}{\frac{a^2}{2}-2iBt}, \tag{45}$$

which is the saddle point already given. The same coordinates may also be obtained alternatively. Write

$$\Phi(Z)=R+iI \tag{46}$$

where

$$\left.\begin{aligned} R&=-\frac{a^2}{4}(X^2-Y^2)-2XYBt-Y(At-x), \\ I&=Bt(X^2-Y^2)-\frac{a^2}{2}XY+X(At-x). \end{aligned}\right\} \tag{47}$$

Performing the operations

$$\frac{\partial R}{\partial X}=0, \quad \frac{\partial R}{\partial Y}=0, \tag{48}$$

we get the coordinates of $Z_0=X+iY$ as follows

$$X=\frac{8Bt(x-At)}{a^4+16B^2t^2}, \quad Y=\frac{-2a^2(x-At)}{a^4+16B^2t^2}. \tag{49}$$

The value of R at the saddle point is

$$R=\frac{-a^2(x-At)^2}{a^4+16B^2t^2}. \tag{50}$$

We shall now equate the general expression of R as shown in (47) to the particular value of R above given, when we get a quadratic equation in X and Y, the expression of which can, however, be decomposed into the pair of equations

$$aX\sqrt{1+\frac{16B^2t^2}{a^4}}-aY\sqrt{1+\frac{16B^2t^2}{a^4}}\left\{-\frac{4Bt}{a^2}+\sqrt{1+\frac{16B^2t^2}{a^4}}\right\} +2\left(\frac{At}{a}-\frac{x}{a}\right)=0, \tag{51}$$

$$aX\sqrt{1+\frac{16B^2t^2}{a^4}}\left\{-\frac{4Bt}{a^2}+\sqrt{1+\frac{16B^2t^2}{a^4}}\right\}+aY\sqrt{1+\frac{16B^2t^2}{a^4}} -2\left(\frac{At}{a}-\frac{x}{a}\right)\left\{-\frac{4Bt}{a^2}+\sqrt{1+\frac{16B^2t^2}{a^4}}\right\}=0, \tag{52}$$

both being equations for straight lines. These straight lines represent the loci of points at which the value of R is equal to that at the saddle point. The inclinations of the lines (51), (52) to the X-axis are given by

$$\tan\theta_1=\frac{1}{\sqrt{1+\frac{16B^2t^2}{a^4}-\frac{4Bt}{a^2}}},\quad \tan\theta_2=-\left\{\sqrt{1+\frac{16B^2t^2}{a^4}}-\frac{4Bt}{a^2}\right\}, \tag{53}$$

respectively. Since $\tan\theta_1\tan\theta_2=-1$, both lines intersect at right angles. The valleys of R exist at places between θ_1 and θ_2. Thus, the inclination α of the line of steepest descent is represented by

$$a=\theta_1-\frac{\pi}{4}=\tan^{-1}\left\{\frac{1}{\sqrt{1+\frac{16B^2t^2}{a^4}-\frac{4Bt}{a^2}}}\right\}-\frac{\pi}{4}. \tag{54}$$

The inclination of α is between 0° and 45°. Because, for $Bt/a^2=0$, θ_1 is 45° and for $Bt/a^2=\infty$, θ_1 is 90°; it follows from (53) that θ_1 shall always be between 45° and 90°, so that a is between 0° and 45°.

The contour of the integration is now drawn as shown in Fig. 3. The inclined part passes through the line of steepest descent. The parts of the integration for the large circles contribute nothing, because a is between 0° and 45°.

Taking the important part near the saddle point, it is possible to work out the definite integral belonging to the present problem.

We finally get the solution for the displacement, the result being

$$v=\frac{a}{(a^4+16B^2t^2)^{\frac{1}{4}}}\,exp.\left[-\frac{(At-x)^2(a^2+4iBt)}{4(a^4+16B^2t^2)}+i\alpha\right]. \tag{55}$$

From this result it will be seen that the part of the largest amplitude is transmitted with the group velocity of infinitely long periodic waves. The general wave form, nevertheless, appears and disappears with gradually varying periods. After a very long time, the periodic change of the wave form disappears. At the same time, the general amplitudes of the waves decay with time lapse.

It is now possible to conclude that, in the present case, the centroid of the wave energy, that is the centroid of the energy of the wave assembly, is transmitted with a group velocity of infinitely long waves, regardless of whether or not the same group velocity is higher than the group velocity of any wave of finite length.

5. *A single pulse that obeys dispersion formula* (*b*). (*continued*)

This problem is very similar to that in the preceding section, with the exception that, here, the initial wave form is anti-symmetrical and of the type

$$\Phi(\sigma)=\sigma e^{-\frac{\sigma^2}{a^2}}, \tag{56}$$

the dispersion formula being again of the form

$$c=A+Bf. \tag{57}$$

Proceeding in the same way as in the preceding section, we finally get

$$v=\frac{-a^3(At-x)}{(a^4+16B^2t^2)^{\frac{3}{4}}}\,exp.\left[-\frac{(At-x)^2(a^2+4iBt)}{4(a^4+16B^2t^2)}+i\alpha+\tan^{-1}\frac{4Bt}{a^2}\right]. \tag{58}$$

In this case, too, the general feature of the problem is the same as that in the preceding section.

6. *The case of a trapezoid-formed pulse that obeys dispersion formula* (*b*).

In this case we put

$$\left.\begin{aligned}\Phi(\sigma)&=-K\sigma, \quad [-b<x<-a]\\ &=0, \quad [-b>x,\ -a<x]\end{aligned}\right\} \tag{59}$$

the dispersion formula being $c=A+Bf$; the displacement velocity of the waves

$$\frac{\partial v}{\partial t}=\frac{i}{2\pi}\int_{-\infty}^{\infty}fce^{ifct}df\int_{-\infty}^{\infty}F(\sigma)e^{if(\sigma-x)}d\sigma \tag{60}$$

changes to

$$\left.\begin{aligned}\frac{\partial v}{\partial t}&=\frac{-i}{2\pi}\int_{-\infty}^{\infty}f(A+Bf)e^{ift(A+Bf)-ifx}df\int_{-b}^{-a}K\sigma e^{if\sigma}d\sigma, \quad [-b<x<-a]\\ &=0. \quad [-b>x, \quad -a<x]\end{aligned}\right\} \tag{61}$$

Since

$$\int_{-b}^{-a}\sigma e^{if\sigma}d\sigma=\frac{-ae^{-ifa}}{if}+\frac{be^{-ifb}}{if}+\frac{e^{-ifa}}{f^2}-\frac{e^{-ifb}}{f^2}, \tag{62}$$

we have

$$\frac{\partial v}{\partial t}=\frac{K}{2\pi}\int_{-\infty}^{\infty}\left\{e^{-ifa}(Aa-iB)+e^{-ifb}(-Ab+iB)+iA\left(-\frac{e^{-ifa}}{f}+\frac{e^{-ifb}}{f}\right)\right.$$
$$\left.+B(afe^{-ifa}-bfe^{-ifb})\right\}df. \tag{63}$$

We shall now use the method of the line of steepest descent. The two kinds of exponents $\Phi(Z)$'s in the functions of the integrand in (63) have saddle points of the forms

$$Z_0=-\left(\frac{At-x-a}{2Bt}\right),\quad Z_0=-\left(\frac{At-x-b}{2Bt}\right) \qquad (64),\ (64')$$

respectively, which arise from the relations

$$\Phi(Z)=R+iI, \qquad (65)$$

$$\frac{\partial R}{\partial X}=0,\quad \frac{\partial R}{\partial Y}=0. \qquad (66)$$

The boundaries between the valleys and hills are

$$X=-\left(\frac{At-x-a}{2Bt}\right),\quad Y=0 \qquad (67)$$

for the first case and

$$X=-\left(\frac{At-x-b}{2Bt}\right),\quad Y=0 \qquad (67')$$

for the second case. The value of R on these boundaries, including the saddle points, is

$$R=0. \qquad (68)$$

The line of steepest descent is therefore inclined at 45° to the X-axis. The contour for integration is shown in Fig. 4. From the nature of the integrand in (63), there is a pole at the origin of coordinates. Parts of large circles contribute nothing to the integration, owing to the condition that these circular arcs are between 0° and 45° and also between 180° and 225°.

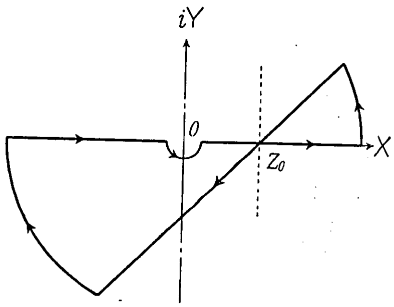

Fig. 4.

In agreement with the position of the saddle points relative to that of the poles, contribution from the same poles to the integration may or may not exist.

The results of calculation are as follows

$$\frac{\partial v}{\partial t}=\frac{\partial v_1}{\partial t}+\frac{\partial v_2}{\partial t}, \qquad (69)$$

where

$$\frac{\partial v_1}{\partial t}=\frac{AK}{2\sqrt{\pi}}e^{\frac{-i(At-x-a)^2}{4Bt}}\left\{\frac{a}{\sqrt{Bt}}-\frac{i\sqrt{B}}{A\sqrt{t}}+\frac{a(a+x-At)}{2A\sqrt{Bt^3}}-\frac{2i\sqrt{Bt}}{a+x-At}\right\}$$

$$+\frac{AK}{2\sqrt{\pi}}e^{\frac{-i(At-x-b)^2}{4Bt}}\left\{-\frac{b}{\sqrt{Bt}}+\frac{i\sqrt{B}}{A\sqrt{t}}-\frac{b(b+x-At)}{2A\sqrt{Bt^3}}+\frac{2i\sqrt{Bt}}{b+x-At}\right\}, \qquad (70)$$

and

$$\left.\begin{aligned}\frac{\partial v_2}{\partial t}&=-2\pi AK, \qquad [At-a>x>At-b]\\ &=0. \qquad [At-a<x,\quad At-b>x]\end{aligned}\right\} \qquad (71)$$

Although the initial condition is given for the wave form, for mathematical simplicity, the calculation is performed for displacement velocity of the waves.

In this case, although the main part of the waves is transmitted with a group velocity of the longest possible waves, the additional part extends outside the main part, and its phase condition is undulatory. This additional part decays with time lapse. It is now possible to conclude that in this case, too, the apparent centroid of the wave energy, that is, the centroid of the energy of wave packet, is transmitted with a group velocity of the longest sinusoidal waves.

6. *Summary and concluding remarks.*

From a few mathematical examples, we have ascertained that, even should the wave form be irregular, the energy of the waves is transmitted with a special group velocity. If the sinusoidal components composing the irregular waves have different group velocities, it is likely that the centroid of the wave energy, that is, the centroid of the energy of the wave packet, will be transmitted with the group velocity that corresponds to the waves of infinite length involved in the irregular waves.

If the group velocity differs with difference in length of the sinusoidal waves, even should the disturbance be of sine form of finite extent, the centroid of the same disturbance is still transmitted with group velocity of sinusoidal waves of infinite length.

It is not a matter of importance whether or not the group velocity of the longest wave is higher than that of any shorter wave.

It is also likely that in the case of irregular wave form, the velocity of change in the form of phase waves concerns the velocity of phase waves of infinite length.

Since the cases that we have here discussed are likely to correspond to that of relatively long waves and that of relatively short waves, both for Rayleigh- or Love-waves, the results in the present paper could be adapted to a rather wide range of dispersion conditions of waves of irregular form.

The condition that the wave packet of irregular form shall be transmitted with group velocity of infinitely long sinusoidal waves, holds in the problems discussed by Coulomb.[7] Notwithstanding that, in Coulomb's case, the dispersion formula is of the type $c=A-Bf^2$, he arrived at the same conclusion as that in this paper.

It is then likely that, if the dispersion formula is of the form,

$$c=A\pm Bf^n,$$

the centroid of a packet of irregular waves is transmitted with velocity A, that is, the group velocity of the longest possible waves composing the wave packet. It is not important whether or not the group velocity of the longest waves last given is higher than the group velocities of other sinusoidal waves also composing the irregular disturbance.

In the analysis of such dispersive seismic waves as the L-waves and M-waves, it is usually assumed that nearly periodic oscillations are of sine curve, so that the waves are transmitted with a group velocity peculiar to the period of those oscillations. Even in the case of only one or two oscillations existing, the condition for transmission is also assumed to be the same. In the case of a period varying in successive oscillations, the above criterion applies to every one of the successive oscillations. One of us, also, once had such an opinion.[8] From the present investigation, on the other hand, if the wave form deviated slightly from a sine curve, the wave packet would be transmitted with the group velocity of the longest waves composing the disturbance. It follows then that a good deal of error will invariably result in the usual analysis of dispersive waves.

Another way of estimating the velocity of long waves is to examine the approximate beginning of the oscillation that corresponds to the same waves. As has been remarked, although the velócity of transmission of the leading part is not always the group velocity of the waves, in taking it as such, however, the error arising from reading the beginning of the motion in estimating group velocity, will be far less than that resulting from assuming the apparent periodic oscillation to be a sine curve.

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7) M. J. COULOMB, *loc. cit.*, 2).
8) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **8** (1935), 245.

Macelwane[9] obtained a table of the travel times of seismic waves, including the L- and M-phases. According to his table, the values for the L- and M-phases are rather regular, irrespective of wave lengths. He probably took the wave part of the largest amplitude or, at least, the beginning of the motion that coresponds to the long waves. There is a reason for his values for L- and M-phases being arranged with fair regularity.

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18. 不規則なる波形をなす分散彈性波の群圍速度について

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地震研究所 金井清

無限につづく周期的波動の場合の群速度については今更議論をする必要がないが，波形が不規則な場合や，周期的であるにしても短い範圍の波群の場合は，未だよくしらべられてないやうである．

レーレー波やラブ波の波長が比較的に短い場合や比較的に長い場合は計算が割合に簡單に取扱へるから，それ等の場合をしらべて見ると甚だ面白い性質があることがわかつた．波形が不規則な場合には，波の重心にあたる部分が，無限大の波長の正弦波に相當する群速度を以て傳播移動することがわかつたのである．波形が正弦形であつても，その波が一定の範圍にしか涉つてをらぬ場合には，その正弦波の波長に相當する群速度で重心が移動するのでなく，やはり無限大波長の正弦波に相當する群速度で移動するらしい．之等の性質は，無限大波長の正弦波に相當する群速度が有限長の正弦波に相當する群速度よりも高くても低くてもあり得るのである．

不規則な波形のときの位相波の變形速度も無限長波の位相波の速度に關係がある．

地震記象から表面波の群速度を決定する場合に態々波長を考慮に入れて問題を取扱ふ事があるけれども，以上の事柄に照して必しも正しいとはいひ得ないのである．マケルウエンは表面波迄も入れた震波走時表を作つたが，表面波は案外一定の速度で進む傾向があるやうである．之は同氏が簡單に表面波の最大振幅のみや，又は表面波の出始めのみを取つたものであるとすれば，只今の理論に照して見て，表面波が反て一定速度になるべきであるといふ結論に達するのである．

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9) J. B. MACELWANE's table, 1933.