1. *The Range of Possible Existence of Stoneley-waves, and Some Related Problems.*

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1. *Introduction.*

Many years ago Stoneley[1] found that it is possible for Rayleigh-type waves to be transmitted along the surface of separation of two solids. Although Stoneley concluded that the waves under consideration only exist in that case wherein the densities and elasticities of the media are likely to satisfy Wiechert's condition, namely, $\rho'/\rho = \lambda'/\lambda = \mu'/\mu$, no accurate condition for the range of possible existence of the same waves has yet been determined. On the other hand, some of our problems[2),3)] already discussed were more or less related to Stoneley-waves, from which we are now in a position to ascertain the condition for the waves to exist.

2. *Velocity equation.*

Let ρ, λ, μ; ρ', λ', μ' be the densities and the elastic constants in both media respectively, and $2\pi/f$, $2\pi/p$ the length and the period of Stoneley-waves. Our previous investigation[4)] showed that the velocity equation of the waves assumes the form

$$\begin{aligned}&\mu'^2(f^2-\sqrt{f^2-h^2}\sqrt{f^2-k^2})\{(2f^2-k'^2)^2-4f^2\sqrt{f^2-h'^2}\sqrt{f^2-k'^2}\}\\&-\mu\mu'k^2k'^2(\sqrt{f^2-h^2}\sqrt{f^2-k'^2}+\sqrt{f^2-k^2}\sqrt{f^2-h'^2})\\&+2\mu\mu'f^2(2\sqrt{f^2-h^2}\sqrt{f^2-k^2}+k^2-2f^2)(2f^2-k'^2-2\sqrt{f^2-h'^2}\sqrt{f^2-k'^2})\\&+\mu^2(f^2-\sqrt{f^2-h'^2}\sqrt{f^2-k'^2})\{(2f^2-k^2)^2-4f^2\sqrt{f^2-h^2}\sqrt{f^2-k^2}\}=0. \quad (1)\end{aligned}$$

1) R. STONELEY, "Elastic Waves at the Surface of Separation of Two Solids", *Proc. Roy. Soc., London*, **106** (1924), 416~428.

2) K. SEZAWA and K. KANAI, "The Formation of Boundary Waves at the Surface of Discontinuity within the Earth's Crust. I", *Bull. Earthq. Res. Inst.*, **16** (1938), 504~526.

3) K. SEZAWA and K. KANAI, "Anomalous Dispersion of Elastic Surface Waves, II", *Bull. Earthq. Res. Inst.*, **16** (1938), 683~689.

4) *loc. cit.* 2), 506.

The velocity equation originally obtained by Stoneley[5] is of the type

$$c^4\{(\rho_1-\rho_2)^2-(\rho_1A_2+\rho_2A_1)(\rho_1B_2+\rho_2B_1)\}$$
$$+2Kc^2\{\rho_1A_2B_2-\rho_2A_1B_1-\rho_1+\rho_2\}+K^2(A_1B_1-1)(A_2B_2-1)=0. \quad (2)$$

If the symbols used by Stoneley are replaced by the corresponding ones in the present case, such that

$$\left.\begin{array}{l} c^2\equiv\dfrac{p^2}{f^2},\quad \alpha_1^2=\dfrac{\lambda_1+2\mu_1}{\rho_1}\equiv\dfrac{1}{p^2h^2},\quad \beta_1^2=\dfrac{\mu_1}{\rho_1}\equiv\dfrac{1}{p^2k^2},\quad \rho_1\equiv\rho,\ \lambda_1\equiv\lambda,\ \mu_1\equiv\mu,\\ \alpha_2^2=\dfrac{\lambda_2+2\mu_2}{\rho_2}\equiv\dfrac{1}{p^2h'^2},\quad \beta_2^2=\dfrac{\mu_2}{\rho_2}\equiv\dfrac{1}{p^2k'^2},\quad \rho_2\equiv\rho',\ \lambda_2\equiv\lambda',\ \mu_2\equiv\mu',\\ A_1=\left(1-\dfrac{c^2}{\alpha_1^2}\right)^{\frac{1}{2}},\ A_2=\left(1-\dfrac{c^2}{\alpha_2^2}\right)^{\frac{1}{2}},\ B_1=\left(1-\dfrac{c^2}{\beta_1^2}\right)^{\frac{1}{2}},\ B_2=\left(1-\dfrac{c^2}{\beta_2^2}\right)^{\frac{1}{2}},\\ K=2(\rho_1\beta_1^2-\rho_2\beta_2^2), \end{array}\right\} \quad (3)$$

the form of the expression in (2) reduces to that in (1).

Using the symbols from another one[6] of our papers, the velocity equation becomes

$$\frac{rs}{f^2}\left\{\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-2\right\}^2-\left\{\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-\left(2-\frac{k^2}{f^2}\right)\right\}^2-\frac{4rsr's'}{f^4}\left(\frac{\mu'}{\mu}-1\right)^2$$
$$+\frac{r's'}{f^2}\left\{\frac{2\mu'}{\mu}-\left(2-\frac{k^2}{f^2}\right)\right\}^2+\frac{\mu'k^2k'^2}{\mu f^4}\left(\frac{rs'}{f^2}+\frac{r's}{f^2}\right)=0, \quad (4)$$

where

$$\left.\begin{array}{l} r^2=f^2-h^2,\ \ s^2=f^2-k^2,\ \ r'^2=f^2-h'^2,\ \ s'^2=f^2-k'^2,\\ h^2=\rho p^2/(\lambda+2\mu),\ \ k^2=\rho p^2/\mu,\ \ h'^2=\rho' p^2/(\lambda'+2\mu'),\ \ k'^2=\rho' p^2/\mu'. \end{array}\right\} \quad (5)$$

If we multiply (4) by $-\mu^2f^6$, the form of the expression in (4) reduces to that in (1). The expression in (4), as a matter of fact, represents the velocity equation of particular Rayleigh-waves transmitted along a stratified surface layer.

3. *The determination of the range of possible existence of the waves.*

Since the decreases in amplitudes in the respective media with distance from the boundary surface are given by

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5) *loc. cit.* 1).

6) *loc. cit.* 3). In the previous paper, for obtaining the expression in (4), we changed the expressions of types *cos*s'H, *sin*s'H into those of types *cosh*s'H, *sinh*s'H.

$$e^{-ry},\quad e^{-sy},\quad e^{r'y},\quad e^{s'y},$$

where

$$r^2 = f^2-h^2,\quad s^2 = f^2-k^2,\quad r'^2 = f^2-h'^2,\quad s'^2 = f^2-k'^2,$$

if any one among f^2-h^2, f^2-k^2, $f^2-h'^2$, $f^2-k'^2$ be negative, the condition of the boundary waves fails, whence, the critical condition of the possible existence of Stoneley-waves is determined by the condition that either one among f^2-h^2, f^2-k^2, $f^2-h'^2$, $f^2-k'^2$ changes its sign from positive to negative. Since, from the nature of things, $h^2<k^2$, $h'^2<k'^2$, the condition under consideration is given by $f^2 = k^2$ in the case $k^2>k'^2$, and by $f^2 = k'^2$ in the case $k'^2>k^2$.

We also calculated the velocity of transmission of the waves in the critical conditions just mentioned. The limiting cases for Poisson's ratio are $\sigma = 1/4$ and $\sigma = 1/2$, that is to say, $\lambda/\mu = \lambda'/\mu' = 1$, and $\lambda/\mu = \lambda'/\mu' = \infty$, both of them having been calculated.

The calculation of the critical conditions in two limiting cases of Poisson's ratio are shown in Tables I, II, and plotted in Figs. 1, 2.

From Figs. 1, 2 it will be seen that it is possible for Stoneley-waves to exist within a relatively narrow range of μ'/μ for every ratio

Table I. $\lambda = \mu$, $\lambda' = \mu'$.

ρ'/ρ	0		0·02		0·1		0·3	
μ'/μ	0·3435	0	0·3492	0·01709	0·3771	0·0888	0·458	0·2855
$(p/f)\sqrt{\rho/\mu}$	1	0·9196	1	0·9244	1	0·9420	1	0·9753

ρ'/ρ	0·5		0·8		1·0		1·25		2	
μ'/μ	0·563	0·4927	0·8013	0·7996	1	1	1·248	1·251	1·777	2·030
$(p/f)\sqrt{\rho/\mu}$	1	0·9926	1	0·9998	1	1	0·9995	1	0·9425	1

ρ'/ρ	10/3		10		50		∞	
μ'/μ	2·184	3·503	2·653	11·265	2·864	58·53	2·912	∞
$(p/f)\sqrt{\rho/\mu}$	0·809	1	0·515	1	0·2394	1	0	1

Table II. $\lambda/\mu = \infty$, $\lambda'/\mu' = \infty$.

ρ'/ρ	0		0·1		0·2		0·5	
μ'/μ	0·3090	0	0·3493	0·09362	0·3924	0·1911	0·5523	0·4957
$(p/f)\sqrt{\rho/\mu}$	1	0·9553	1	0·9673	1	0·9773	1	0·9958

ρ'/ρ	0·7		1		10/7		2	
μ'/μ	0·7064	0·6990	1	1	1·415	1·430	1·811	2·017
$(p/f)\sqrt{\rho/\mu}$	1	0·9995	1	1	0·9956	1	0·9518	1

ρ'/ρ	5		10		∞	
μ'/μ	2·547	5·231	2·863	10·68	3·236	∞
$(p/f)\sqrt{\rho/\mu}$	0·7140	1	0·5350	1	0	1

of ρ'/ρ. This feature is remarkable, particularly, when the ratio of ρ'/ρ is nearly unity, at which condition Stoneley-waves could not exist unless Wiechert's condition, $\rho'/\rho = \mu'/\mu$, be almost satisfied. On the other hand, when the ratio of ρ'/ρ is relatively small, say, less than 0·5, or relatively large, say, greater than 2, the range of μ'/μ in which Stoneley-waves exist is relatively wide, Wiechert's condition being then rather unimportant. As a matter of fact, the greater the ratio of ρ'/ρ,

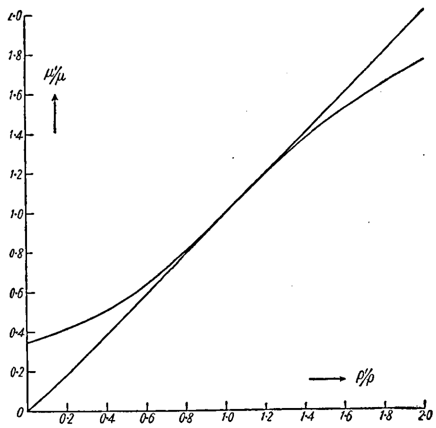

Fig. 1. Range of existence of Stoneley-waves ($\lambda = \mu$, $\lambda' = \mu'$), as shown by the narrow parts between two curves.

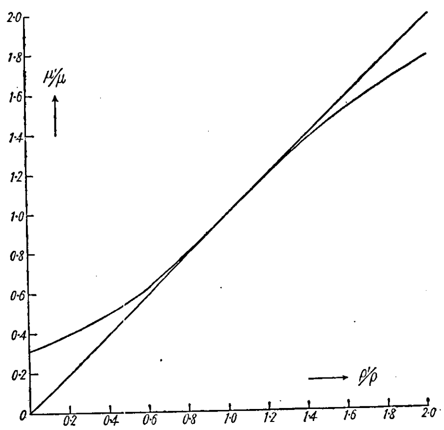

Fig. 2. Range of existence of Stoneley-waves ($\lambda = \infty$, $\lambda' = \infty$), as shown by the narrow parts between two curves.

the greater the ratio of μ'/μ for the real existence of Stoneley-waves, in consequence of which the condition of reality for Stoneley-waves would be alike, even were the parameter ρ'/ρ replaced by μ'/μ, as will be seen in Figs. 1, 2. It should be borne in mind that the two respective curves in Figs. 1, 2, corresponding to the criticals just mentioned, intersect at the point with abscissa $\rho'/\rho = 1$ and ordinate $\mu'/\mu = 1$.

Stoneley-waves are not dispersive, that is to say, the velocity of transmission of these waves is independent of the length of the waves.

and is constant for given ρ', λ', μ', ρ, λ, μ. As has been shown, for certain conditions of ρ'/ρ, μ'/μ, Stoneley-waves can not exist. In such a critical condition, the velocity of transmission of Stoneley-waves is the same as that of distortional waves in a layer in which the velocity of the bodily waves is less than that in the other. It is likely that in the condition intermediate between the criticals, the velocity of transmission of Stoneley-waves is less than either one of the velocities of distortional waves in both media. This will be shown in the next section.

4. *On Stoneley-waves transmitted along the lower boundary of the surface layer of the earth.*

In our previous papers, we remarked on Stoneley-waves transmitted along the lower boundary of the surface layer, in evaluating the velocity of such waves for a particular case. The calculation will be now extended to more widely different conditions of ρ'/ρ, μ'/μ, and the thickness of the surface layer.

Let us assume that $\rho'/\rho = 0{\cdot}5$, and take the cases (I) $\mu'/\mu = 0{\cdot}563$, (II) $\mu'/\mu = 0{\cdot}530$, (III) $\mu'/\mu = 0{\cdot}5$, (IV) $\mu'/\mu = 0{\cdot}495$, (V) $\mu'/\mu = 0{\cdot}4927$, (VI) $\mu'/\mu = 0{\cdot}470$; the relations $\lambda = \mu$, $\lambda' = \mu'$ being retained. The velocity equation for any case is generally

$$\left.\begin{aligned}&\frac{4r's'}{f^2}\Big(2-\frac{k'^2}{f^2}\Big)\eta-\frac{r's'}{f^2}\Big\{4\vartheta+\Big(2-\frac{k'^2}{f^2}\Big)^2\zeta\Big\}\cosh r'H\cosh s'H\\&+\frac{r'}{f}\varphi\Big\{\frac{s}{f}\Big(2-\frac{k'^2}{f^2}\Big)^2-\frac{4rs'^2}{f^3}\Big\}\cosh r'H\sinh s'H\\&+\frac{s'}{f}\varphi\Big\{\frac{r}{f}\Big(2-\frac{k'^2}{f^2}\Big)^2-\frac{4sr'^2}{f^3}\Big\}\sinh r'H\cosh s'H\\&+\Big\{\Big(2-\frac{k'^2}{f^2}\Big)^2\vartheta+\frac{4r'^2s'^2}{f^4}\zeta\Big\}\sinh r'H\sinh s'H=0,\end{aligned}\right\}\qquad(6)$$

where

$$\left.\begin{aligned}&\varphi=\frac{\mu'k^2k'^2}{\mu f^4},\quad \zeta=\frac{4rs}{f^2}\Big(\frac{\mu'}{\mu}-1\Big)^2-\alpha^2,\quad \eta=\frac{2rs}{f^2}\Big(\frac{\mu'}{\mu}-1\Big)\beta-\alpha\gamma,\\&\vartheta=\frac{rs}{f^2}\beta^2-\gamma^2,\quad \alpha=\frac{2\mu'}{\mu}-\Big(2-\frac{k^2}{f^2}\Big),\quad \beta=\frac{\mu'}{\mu}\Big(2-\frac{k'^2}{f^2}\Big)-2,\\&\gamma=\frac{\mu'}{\mu}\Big(2-\frac{k'^2}{f^2}\Big)-\Big(2-\frac{k^2}{f^2}\Big),\quad f=\frac{2\pi}{L}.\end{aligned}\right\}\qquad(7)$$

Using equation (6), we calculated the velocities of transmission for cases (I)$\sim$(VI), the result being shown in Fig. 3. Althogh it is possible to get the velocity of transmission of Rayleigh-waves also from (6), since the dispersion curves for such waves are, as a matter of fact, outside the area in Fig. 3, and since discussion of Rayleigh-waves is not our present object, calculation of these Rayleigh-waves has been omitted.

When $fH \to 0$, that is, $L/H \to \infty$, the equation (6) reduces to a pair of factors, one of which represents the velocity equation of Rayleigh-waves that are transmitted through a semi-infinite body, the same as that in the surface layer, and the other one the velocity equation of the waves that are transmitted through the boundary between the surface and the subjacent media, namely, Stoneley-waves.[7]

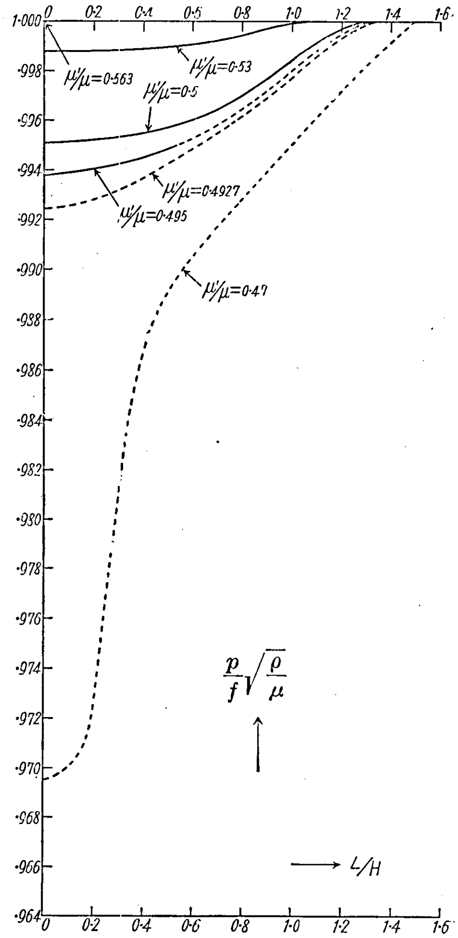

Fig. 3. Dispersion curves of Stoneley-waves transmitted along the lower boundary of a surface layer. $\rho'/\rho = 1/2$, $\lambda = \mu$, $\lambda' = \mu'$.

Since the waves in the layer, of which H/L is not infinity, are more or less of surface-wave type, even should the wave energy be accumulated near the lower boundary of the layer, the ordinates in Fig.3, excepting those at the abscissa $L/H = 0$, do not represent actual Stoneley-waves. The ordinates at zero abscissa only correspond to Stoneley-waves.

For the case in which L/H is not zero, the amplitudes of the waves in the surface layer either decrease exponentially or are distributed sinusoidally with distance from the boundary under consideration. In the present case, if $\mu'/\mu > 0{\cdot}5$, the amplitudes in the surface layer de-

7) The decomposition of the equation (6) in such a special case was performed by *Love*, *Stoneley* (*M. N. R. A. S. G. S.*, **3** (1934), 232), *Lee*, and the authors in different manners, respectively.

crease exponentially for any wave-length in the dispersion curves. If, on the other hand, $\mu'/\mu<0{\cdot}5$, the amplitude distribution is exponential for shorter waves and sinusoidal for longer waves. For example, in the case, $\mu'/\mu=0{\cdot}495$, the distribution is periodic for $L/H>0{\cdot}540$, namely $(p/f)\sqrt{\rho/\mu}>0{\cdot}9950$, and exponential for $L/H<0{\cdot}540$. When $\mu'/\mu=0{\cdot}4927$, the distribution is sinuisoidal for any wave length, excepting the condition $L/H=0$. If the ratio is $\mu'/\mu<0{\cdot}4927$, the waves in the layer are distributed sinusoidally for any wave length, including the case $L/H=0$. In Fig. 3, the parts of dispersion curves, showing the condition that the amplitudes of the waves in the surface layer, vary exponentially and the parts, showing the condition that the same amplitudes are distributed sinusoidally, are indicated by full lines and broken lines, respectively.

From the above consideration it is possible to conclude that Stoneley-waves do exist for the range $0{\cdot}563>\mu'/\mu>0{\cdot}4927$ under the condition that $L/H=0$, the corresponding velocities of transmission ranging between $\sqrt{\mu/\rho}$ and $0{\cdot}9926\sqrt{\mu/\rho}$. This agrees with the result shown in Table I.

The value of μ'/μ for which the ordinate of the dispersion curve is exactly $(\rho/f)\sqrt{\rho/\mu}=1$ at abscissa $L/H=0$ is $\mu'/\mu=0{\cdot}563$. Since in this case, however, no point in the dispersion curve, excepting this ordinate, really exists, Stoneley-waves are the only waves that can be transmitted through the lower boundary of the surface layer.

5. *Concluding remarks.*

In the present paper, we determined the range of ρ'/ρ and μ'/μ, within which Stoneley-waves could possibly exist, and furthermore investigated the nature of such waves as resemble Stoneley-waves that are propagated through the lower boundary of the surface layer of the earth. It has been ascertained that, whereas for such a ratio as $\rho'/\rho\sim1$, the range of μ'/μ within which Stoneley-waves exist is quite narrow, for a very small ratio or a very large ratio of ρ'/ρ, the range under consideration is relatively wide. The nature of the waves transmitted through the lower boundary of the surface layer would resemble that of Stoneley-waves, provided the ratio of ρ'/ρ is not far from that of μ'/μ. If, on the other hand, the ratio of ρ'/ρ greatly differs from that of μ'/μ, the amplitudes of the waves in the surface layer are distributed sinusoidally, so that the waves cannot be of the Stoneley type. Although waves that are similar to Stoneley-waves in the surface layer show a certain dispersive nature (normal), the real Stoneley-waves are not dispersive, their velocity being constant if ρ', λ', μ', ρ, λ, μ were given.

In conclusion, we wish to express our thanks to Mr. Unoki, through whose kind aid the calculations were greatly facilitated.

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1. ストンレー波の實在範圍と之に關聯する問題

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10 年以上前に Stoneley が 2 層の境界に傳播する Rayleigh 型波の存在する事を提唱し，最近には我々が之を應用して一二の問題を解いた．しかし，Stoneley 波が存在するには 2 層の彈性及密度に對して制限があるのであつて，その制限の範圍をこの論文で確めたのである．この論文では更に地表の表面層の下の面に沿うて傳播する Stoneley 型類似の波の性質をも研究して置いたのである．

研究によると，2 層の密度が略々近いときには 2 層の彈性も非常に近い場合しか Stoneloy 波が存在しないが，密度の差が著しいときには 2 層の彈性の差の相當廣い範圍にわたつて Stoneley 波が存在し得ることがわかつた．

表面層の下面に沿うて傳播する波は，Wiechert の條件を滿足するやうな密度と彈性の關係になつてをるときに限り，Stoneley 波に類似するけれども，然らざるときには波の性質殊に表面層中での波動分布がStoneley 波とは全然異なることがわかつた．

表面層の下面に傳はる波は Stoneley 波に類似する場合でも分散性があるけれども，純粹の Stoneley 波では速度が一定であることも確められた．
