3. *A Contribution to the Tidal Theory of the Origin of the Solar Planets.*

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1. *Introduction.*

There seems to be no theory that perfectly explains the origin of the solar planets. Although the tidal theory of T. C. Chamberlin and F. R. Moulton also involves certain difficulties, since, as Jeffreys[1] has remarked, there are still some excellent advantages in that theory, we shall now direct our attention solely to the particular theory under consideration and confirm it dynamically, so far as it is possible to do so. According to the tidal theory of Chamberlin and Moulton, if an external massive star were to encounter the primitive sun with a moderate relative velocity, the deformation of the sun then under tidal force would be so large that planets are in some way are ejected from it. Even if the velocity of encounter were either quicker or slower than the moderate one just mentioned, it is improbable that it would result in a large deformation of the primitive sun.

Although a treatment of the deformation of the primitive sun in the event that the position of the external star relative to the sun changes with time is greatly to be desired, since the problem is extremely difficult, we shall restrict ourselves to the discussion of the simple case in which the intensity of the tidal force of an external star at a given distance from the sun changes with time; although even under this condition, the nature of the problem would not differ much from the one just mentioned. It is also assumed, for simplicity, that the sun is incompressible, particularly in that condition of continuity of the medium. Although, as a matter of fact, the sun is gaseous, as the result of pressure under gravitational forces, the hydrodynamical condition below a certain depth from the outer surface of the sun could be incompressible.

1) H. JEFFREYS, *The Earth*. (Cambridge, 1929).

2. *Gravitational potential and hydrodynamical condition for small oscillation.*

Let r, θ be the coordinates of a point with respect to the centre of a spherical sun, and D the distance of an external star in the direction $\theta=0$, from the centre of the same sun; then the gravitational potential of the point r, θ is expressed by

$$V=-2\pi\gamma\int_0^\infty dr_1\int_{-1}^{1}\frac{\rho r_1^2\,d\cos\theta_1}{\sqrt{r^2-2rr_1\cos(\theta-\theta_1)+r_1^2}}, \tag{1}$$

where γ is the gravitational constant. It is known that

$$\left.\begin{aligned}\frac{1}{\sqrt{r^2-2rr_1\cos(\theta-\theta_1)+r_1^2}}&=\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r_1}{r}\right)^n P_n\{\cos(\theta-\theta_1)\}, \quad (r>r_1)\\ &=\frac{1}{r_1}\sum_{n=0}^{\infty}\left(\frac{r}{r_1}\right)^n P_n\{\cos(\theta-\theta_1)\}. \quad (r<r_1)\end{aligned}\right\} \tag{2}$$

It is possible to assume that in a slightly deformed condition, the sun's radius a becomes $a+\delta$, so that the thickness δ is ideally constant on the spherical surface. The gravitational potential within the spherical body then assumes the form

$$\begin{aligned}\Omega=&-2\pi\gamma\int_0^r dr_1\int_{-1}^{1}\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r_1}{r}\right)^n P_n\{\cos(\theta-\theta_1)\}\rho_1(r)r_1^2\,d\cos\theta_1\\ &-2\pi\gamma\int_r^a dr_1\int_{-1}^{1}\frac{1}{r_1}\sum_{n=0}^{\infty}\left(\frac{r}{r_1}\right)^n P_n\{\cos(\theta-\theta_1)\}\rho_1(r)r_1^2\,d\cos\theta_1\\ &-2\pi\gamma\int_a^{a+\delta} dr_1\int_{-1}^{1}\frac{1}{r_1}\sum_{n=0}^{\infty}\left(\frac{r}{r_1}\right)^n P_n\{\cos(\theta-\theta_1)\}\rho_1' r_1^2\,d\cos\theta_1\\ &-\gamma M\frac{1}{D}\sum_{n=0}^{\infty}\left(\frac{r}{D}\right)^n P_n(\cos\theta)\,.\end{aligned} \tag{3}$$

If the deformation, particularly, is of the type of zonal harmonics, we put

$$\rho_1'=cP_n(\cos\theta_1), \tag{4}$$

where c is a constant determined from other conditions. If the actual displacement of the free suface $r=a$ be ξ, then we have

$$\rho_1'\delta=\rho_1(a)\xi\,. \tag{5}$$

Substituting (4), (5) in (3), and using the relations

$$\left.\begin{aligned}&\int_{-1}^{1} P_m(\cos\theta_1)P_n^s(\cos\theta_1)\,d\cos\theta_1=0, \qquad (m\neq n)\\ &\int_{-1}^{1}\left\{P_n(\cos\theta_1)\right\}^2 d\cos\theta_1=\frac{2}{2n+1},\end{aligned}\right\} \tag{6}$$

$$P_n\{\cos(\theta-\theta_1)\}=P_n(\cos\theta)P_n(\cos\theta_1)+2\sum_{s=1}^{s=n}\left\{\frac{(n-s)!}{(n+s)!}P_n^s(\cos\theta)P_n^s(\cos\theta_1)\right\}, \tag{7}$$

we get

$$\Omega=-4\pi\gamma\left[\frac{\rho r^2}{3}+\frac{\rho(a^2-r^2)}{2}+\frac{r^n cP_n(\cos\theta)\delta a^{1-n}}{2n+1}\right]-\gamma M\frac{1}{D}\sum_{n=0}^{\infty}\left(\frac{r}{D}\right)^n P_n(\cos\theta), \tag{8}$$

ρ being assumed to be constant.

The value of M is assumed to change, as in the type

$$M=M_0e^{i\sigma t}. \tag{9}$$

We shall now deal with the hydrodynamical condition. The equations of motion of the fluid mass in the sun in rectangular coordinates are as follows[2)]

$$\left.\begin{aligned}&-\frac{\partial^2\phi}{\partial t\partial x}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}=-\frac{\partial\Omega}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x},\\ &-\frac{\partial^2\phi}{\partial t\partial y}+u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial y}=-\frac{\partial\Omega}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y},\\ &-\frac{\partial^2\phi}{\partial t\partial z}+u\frac{\partial u}{\partial z}+v\frac{\partial v}{\partial z}+w\frac{\partial w}{\partial z}=-\frac{\partial\Omega}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z},\end{aligned}\right\} \tag{10}$$

where $u=-\partial\phi/\partial x$, $v=-\partial\phi/\partial y$, $w=-\partial\phi/\partial z$ are components of velocity at the point x, y, z in x-, y-, z-directions and p is the pressure at the same point. The integral of the differentials in (10) is

$$\frac{p}{\rho}=\frac{\partial\phi}{\partial t}-\Omega-\frac{1}{2}(u^2+v^2+w^2)+F(t). \tag{11}$$

Assuming that the velocity of the medium is small, and neglecting the squares of such a small quantity as that just given, we have

$$\frac{p}{\rho}=\frac{\partial\phi}{\partial t}-\Omega+\text{const}. \tag{12}$$

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2) H. Lamb, *Hydrodynamics.*

Since, on the other hand, the material of which the sun is composed is assumed to be incompressible, equation

$$\nabla^2\phi=0 \tag{13}$$

holds, from which we get

$$\phi=\sum_1^{\infty}A_n\frac{r^n}{a^n}P_n(\cos\theta)e^{i\sigma t}. \tag{14}$$

The kinematic condition on the free surface of the sun is

$$\frac{\partial\xi}{\partial t}=-\frac{\partial\phi}{\partial r}, \tag{15}$$

for which we write

$$\xi=fP_n(\cos\theta)e^{i\sigma t} \tag{16}$$

on that surface. On the free surface of the sun, furthermore, there is no vibratory pressure, so that from (12) we obtain the condition

$$\frac{\partial\phi}{\partial t}-\mu\frac{\partial\xi}{\partial t}=\text{variable part of } \Omega_{r=a}, \tag{17}$$

where μ is a small damping coefficient that is made zero after the calculation is performed.

We shall now solve the problem with these equations. Putting (14), (16) in (15), we get

$$A_n=-\frac{i\sigma a}{n}f. \tag{18}$$

The variable part of Ω, when small quantities of the first order are taken, is such that

$$\left[-4\pi\gamma\left\{\frac{2\rho a\xi}{3}-\rho a\xi+\frac{a\rho\xi}{2n+1}\right\}-\gamma M_0\frac{1}{D}\left(\frac{a}{D}\right)^n\right]P_n(\cos\theta). \tag{19}$$

Substituting this and (14), (18) in (17), we finally get

$$\xi=e^{i\sigma t}\sum_{n=1}^{\infty}\frac{-n\gamma M_0\dfrac{a^{n-1}}{D^{n+1}}P_n(\cos\theta)}{\sigma^2-\dfrac{in\mu}{a}\sigma-\dfrac{8n(n-1)\pi\gamma\rho}{3(2n+1)}}, \tag{20}$$

ρ being constant. Writing

$$\frac{8n(n-1)\pi\gamma\rho}{3(2n+1)}\equiv\alpha, \qquad \frac{n\mu}{a}\equiv\beta, \tag{21}$$

then, the denominator of each term in (20) is

$$\left[\frac{1}{\sigma-\frac{1}{2}(i\beta+\sqrt{4\alpha-\beta^2})}-\frac{1}{\sigma-\frac{1}{2}(i\beta-\sqrt{4\alpha-\beta^2})}\right]\frac{1}{\sqrt{4\alpha-\beta^2}}. \tag{22}$$

The expression in (20) shows that the oscillatory change in the intensity of the external star excites an infinite number of modes in the free oscillation of the sun.

When the disturbance is periodic with such a frequency as to make the denominator of (20) or (22) vanish, then a resonance condition of the vibratory oscillation of the sun occurs. However, the disturbance now under consideration is not periodic; it is rather of the type of a single pulse of certain duration, for which reason we shall now use Fourier's double integral that will fit any type of disturbance.

3. *The solution for the case of arbitrary disturbance.*

Let the intensity of the disturbance M in the external star be of the form

$$f(t), \tag{23}$$

that is to say,

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\sigma\int_{-\infty}^{\infty}f(T)e^{i\sigma(t-T)}dT, \tag{23'}$$

then

$$\xi=\sum_{n=1}^{\infty}\frac{-P_n(\cos\theta)}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\sigma-\frac{1}{2}(i\beta+\sqrt{4\alpha-\beta^2})}\right.$$

$$\left.-\frac{1}{\sigma-\frac{1}{2}(i\beta-\sqrt{4\alpha-\beta^2})}\right]\frac{Cd\sigma}{\sqrt{4\alpha-\beta^2}}\int_{-\infty}^{\infty}f(T)e^{i\sigma(t-T)}dT, \tag{24}$$

$$C=n\gamma M_0a^{n-1}/D^{n+1}. \tag{25}$$

As an example, we shall specially assume that the disturbance is of the type

$$f(t)=e^{-\frac{t^2}{c^2}}, \tag{26}$$

then, a single term in the summation of (24) becomes

$$\xi=\frac{CcP_n(\cos\theta)}{2\sqrt{\pi}\sqrt{4\alpha-\beta^2}}\left[\int_{-\infty}^{\infty}\frac{e^{-\frac{c^2\sigma^2}{4}+i\sigma t}d\sigma}{-\sigma+\frac{1}{2}(\sqrt{4\alpha-\beta^2}+i\beta)}+\int_{-\infty}^{\infty}\frac{e^{-\frac{c^2\sigma^2}{4}+i\sigma t}d\sigma}{\sigma+\frac{1}{2}(\sqrt{4\alpha-\beta^2}-i\beta)}\right]. \tag{27}$$

To integrate (27) for $t>0$, we shall consider an integral of the type

$$\frac{CcP_n(\cos\theta)}{2\sqrt{\pi}\sqrt{4\alpha-\beta^2}}\left[\int\frac{e^{-\frac{c^2Z^2}{4}+iZt}dZ}{-Z+\frac{1}{2}(\sqrt{4\alpha-\beta^2}+i\beta)}+\int\frac{e^{-\frac{c^2Z^2}{4}+iZt}dZ}{Z+\frac{1}{2}(\sqrt{4\alpha-\beta^2}-i\beta)}\right], \tag{28}$$

taken around the contour shown in the sketch, the singular points of the first and the second integrals lying in the first and second quadrants respectively. The part of the integral that was taken along the real axis corresponds to the expression in (27). When OA and OD are infinitely long, the integrals along parts AB and DC vanish. The integrations around both poles E, F,

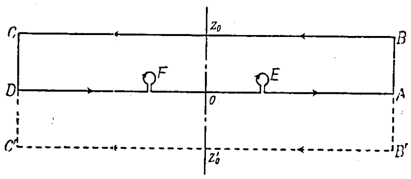

Fig. 1.

that is, $Z=\pm\frac{1}{2}\sqrt{4\alpha-\beta^2}+\frac{i\beta}{2}$ are very simple, the results being

$$\frac{2\sqrt{\pi}CcP_n(\cos\theta)}{\sqrt{4\alpha-\beta^2}}\sin\left\{\frac{\sqrt{4\alpha-\beta^2}}{2}\left(\frac{c^2\beta}{4}-t\right)\right\}e^{-\frac{1}{2}\left\{\frac{c^2}{4}(2\alpha-\beta^2)+t\beta\right\}}. \tag{29}$$

The part BC in the contour is so chosen that the same line passes through the point $Z=Z_0=2it/c^2$, satisfying $\partial F(Z)/\partial Z=(\partial/\partial Z)(-c^2Z^2/4+iZt)=0$, and parallel to the X-axis. At the point Z_0 the value of $\exp.(-c^2Z^2/4+iZt)$ becomes stationary, that is to say, it becomes the saddle point (due to Debye). The line BC is thus the curve of the steepest descent, which runs from one valley to the other in passing through the saddle point just mentioned. Since the important part of the integration along such a line is merely that from the point near the saddle point, we shall expand $F(Z)=-c^2Z^2/4+iZt$ in the form

$$F(Z)=F(Z_0)+\frac{1}{2}(Z-Z_0)^2F''(Z_0)+\cdots\cdots, \tag{30}$$

remembering that $F'(Z_0)=0$, and taking the expanded terms to the second order only. If we write

$$Z-Z_0=q, \tag{31}$$

since $F''(Z_0)=-c^2/2$, the sum of both integrals in (28) along BC becomes

$$\frac{Cc}{\sqrt{\pi}}\frac{P_n(\cos\theta)e^{-\frac{t^2}{c^2}}}{a\left(1-\frac{2\beta t}{ac^2}+\frac{4t^2}{ac^4}\right)}\int_0^\infty e^{-\frac{c^2q^2}{4}}dq=\frac{Ce^{-\frac{t^2}{c^2}}P_n(\cos\theta)}{a\left(1-\frac{2\beta t}{ac^2}+\frac{4t^2}{ac^4}\right)}. \tag{32}$$

Since the integral (28) taken around the contour and the pole vanishes, it is possible to find the definite integral along the real axis, namely (27). The result of the calculation will be given later.

When t is negative, the corresponding saddle point lies on the negative side of the Y-axis with ordinate $Z=Z_0'=2it/c^2$ $(t<0)$. In this case $2t/c^2<0$, the contour then being $AB'C'D$. Since in this case, no singular point exists within the contour, there is no principal value of the definite integral. The integrals along AB' and $C'D$ vanish for the same reason as in the case $t>0$. The integral along $B'C'$, namely, that along the curve of the steepest descent, can be obtained also in the same way as in the case of $t>0$, the result being

$$\frac{Ce^{-\frac{t^2}{c^2}}P_n(\cos\theta)}{a\left(1+\frac{2\beta t}{ac^2}+\frac{4t^2}{ac^4}\right)}, \tag{33}$$

from which it is possible to obtain the value of the definite integral shown in (27).

Since the actual displacement of the free surface of the sun is expressed by the summation given in (24), we get

$$\xi=-\sum_{n=1}^{\infty}\frac{Ce^{-\frac{t^2}{c^2}}P_n(\cos\theta)}{a\left(1+\frac{2\beta t}{ac^2}+\frac{4t^2}{ac^4}\right)}, \qquad (t<0) \tag{34}$$

$$\xi=-\sum_{n=1}^{\infty}\frac{Ce^{-\frac{t^2}{c^2}}P_n(\cos\theta)}{a\left(1-\frac{2\beta t}{ac^2}+\frac{4t^2}{ac^4}\right)}$$
$$-\sum_{n=1}^{\infty}\left\{\frac{2Cc\sqrt{\pi}P_n(\cos\theta)}{\sqrt{4a-\beta^2}}e^{-\frac{1}{2}\left\{\frac{c^2}{4}(2a-\beta^2)+t\beta\right\}}\sin\left\{\frac{\sqrt{4a-\beta^2}}{2}\left(\frac{c^2\beta}{4}-t\right)\right\}\right\}, \qquad (t>0) \tag{35}$$

where

$$C=\frac{n\gamma M_0 a^{n-1}}{D^{n+1}}, \qquad a=\frac{8n(n-1)\pi\gamma\rho}{3(2n+1)}, \qquad \beta=\frac{n\mu}{a}. \tag{36}$$

It will be seen from (34), (35) that the motion of the sun consists of the aggregate of two types of movements. The amplitudes of the first type, which are not oscillatory, vary symmetrically with respect to time $t=0$. The amplitudes of the second type, on the other hand, are oscillatory, and exist for $t>0$ only. The same amplitudes, furthermore, decrease with increase in t.

The amplitude of the movement of the first type is maximum at $t=0$, the value of the same maximum being constant for any rapidity c of the disturbance $\exp.(-t^2/c^2)$. The maximum amplitude of the movement of the second type varies with change in the value of c, whence it appears that the problem due to Chamberlin and Moulton concerns the second type of the motion now under consideration. This condition will be discussed in the next section.

4. *The change in the maximum amplitude with change in the rapidity of the disturbance.*

Consider the second term in the expression in (35). Although for a given c, an infinite number of modes of oscillatory movements with different amplitudes and phases are excited, since the treatment is very complex, we shall consider only a single term in the same expression, namely,

$$\frac{2Cc\sqrt{\pi}P_n(\cos\theta)}{\sqrt{4\alpha-\beta^2}}e^{-\frac{1}{2}\left\{\frac{c^2}{4}(2\alpha-\beta^2)+t\beta\right\}}\sin\frac{\sqrt{4\alpha-\beta^2}}{2}\left(\frac{c^2\beta}{4}-t\right). \tag{37}$$

Assuming that the term $c^2\beta/4$ in $\sin\frac{\sqrt{4\alpha-\beta^2}}{2}\left(\frac{c^2\beta}{4}-t\right)$ is negligible, the value of (37) is maximum when

$$1-\frac{c^2}{4}\left(2\alpha-\beta^2\right)=0\,, \tag{38}$$

that is to say, when

$$c^2=\frac{4}{2\alpha-\beta^2}$$

$$=\frac{4}{\frac{16n(n-1)\pi\gamma\rho}{3(2n+1)}-\frac{n^2\mu^2}{a^2}}. \tag{39}$$

Since the resonance circular frequency σ_0 for the n-th mode is

$$\sigma_0^2\approx\frac{8n(n-1)\pi\gamma\rho}{3(2n+1)} \tag{40}$$

approximately, the relation of the maximum amplitude is written

$$\frac{c^2\sigma_0^2}{2}=1. \tag{41}$$

Writing $\sigma_0=2\pi/\tau_0$, (41) transforms to

$$c=\tau_0/\sqrt{2}\pi. \tag{42}$$

It follows then that if the external star encounters the sun with moderate rapidity, that is, a rapidity comparable to the natural period of one mode of the free oscillations of the sun, the movement of the sun corresponding to that mode becomes maximum, although not infinitely large. If the rapidity of the encounter were greater or less than that specified in (42), the movement of the oscillation of the sun corresponding to the mode in question would be rather less than that given above.

Since, on the other hand, the expression (35) or (37) shows that the higher the mode of the excited oscillation, the smaller is the amplitudes of oscillation, the lowest mode, namely, that corresponding to $n=2$, forms an important part of the problem. Thus, (42) may be replaced by

$$c=\sqrt{\frac{30}{16\pi\gamma\rho}}, \tag{43}$$

showing that the rapidity with which the amplitude of the sun is maximum, depends on the density of the same sun and the gravitational constant.

5. *Concluding remarks.*

From mathematical investigation it was found that if an external star encounters the primitive sun, a free vibration of the same sun is possibly excited, while for a moderate rapidity of the encounter, the amplitude of the vibration becomes maximum, which result well agrees with the conclusion of Chamberlin and Moulton. It should however be borne in mind that the amplitude in question is not infinitely large, but merely assumes its maximum value, as a result of which it is also possible for the velocity of the encounter to be of a certain range of rapidity.

In conclusion, we wish to express our thanks to Professor Nagaoka for his kind advices and suggestions.

3. 太陽惑星の潮汐論的起原に關する一問題

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太陽惑星の起因に關して Chamberlin 其他の人の潮汐論的説明があり，近來多くの人に支持されてゐるやうであるが，之を流體力學的に解いて見たのである．外部の大質量の星が原太陽の近くに飛び込んで來る代りに一定の距離にあるその星の質量が種々の速さで一定の値に增加し次に減少するとして取扱つて見た．この增加減少の速度が原太陽の自己振動の周期とある關係をなすときに原太陽の振動變位が極大となり，速度が上記の周期よりも速くても遲くても原太陽の振動變位は反て小さくなることがわかつた．但し誤解してならぬ事であるが，極大といふのは，單に他の場合に比して大なる事であり，無限大の振幅になるといふ意味ではないのである．