47. *Anomalous Dispersion of Elastic Surface Waves. II.*[1]

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1. *Introduction.*

In a previous paper[2] we showed that even should the velocity of bodily waves in the surface layer be higher than that in the subjacent medium, it is possible for Rayleigh-type waves of certain ranges of wave-length to be transmitted, the propagational velocity of the waves then diminishing with increase in wave-length. Under the condition of the problem particularly selected in the last case, namely, the condition that $\rho'/\rho=1/2$, $\mu'/\mu=1$, the dispersion curve beginning discontinuously with ordinate $V/\sqrt{\mu/\rho}=1{\cdot}000$ and abscissa $L/H=4{\cdot}650$, existed only in the range, $L/H=4{\cdot}650\sim\infty$, tending to assume the asymptotic value $0{\cdot}9194$ at $L/H\rightarrow\infty$. If any part of the dispersion curve were within the range $L/H=4{\cdot}650\sim 0$, the value of $V/\sqrt{\mu/\rho}$ would be greater than 1, in which case the mathematical conditions prevent the waves from assuming a surface type or even a permanent type.

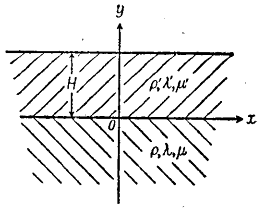

Fig. 1.

Although in the condition just mentioned, the dispersion curve incidentally meets the line $V/\sqrt{\mu/\rho}=1$, with the result that the same curve ends at an intermediate value of L/H, it is however possible to expect another case in which the curve does not meet the line $V/\sqrt{\mu/\rho}=1$, the end of the curve probably intersecting with the abscissa $L/H=0$. With a view to ascertaining the general feature of the problem for widely different cases of the dispersion curves, the calculation is now extended to the cases $\mu'/\mu=0{\cdot}8$, $0{\cdot}58$, $0{\cdot}50$, $0{\cdot}44$, $0{\cdot}40$ ($\lambda=\mu$, $\lambda'=\mu'$), ρ'/ρ being kept at $0{\cdot}5$; at the same time the nature of the velocity equation that we obtained in a very simplified form a few years ago, is now rediscussed from a more analytical point of view.

1) Preliminary report published in the *Proc. Imp. Acad.*, **14** (1938), 246∼249.

2) K. SEZAWA, "Anomalous Dispersion of Elastic Surface Waves," *Bull. Earthq. Res. Inst.*, **16** (1938), 225∼233.

2. *Velocity equation of dispersive Rayleigh-waves.*

Let the thickness of the surface layer be H and the wave-length L, and let also densities and elastic constants of the layer and the subjacent medium be ρ', λ', μ'; ρ, λ, μ respectively. Then, the equation that is necessary for obtaining the velocity of transmission of the dispersive Rayleigh-waves of frequency p is expressed by[3)]

$$\begin{aligned}\frac{4r's'}{f^2}\Big(2-\frac{k'^2}{f^2}\Big)\eta-\frac{r's'}{f^2}\Big\{4\vartheta+\Big(2-\frac{k'^2}{f^2}\Big)^2\zeta\Big\}\cosh r'H\cos s'H \\ +\frac{r'}{f}\varphi\Big\{\frac{4rs'^2}{f^3}+\frac{s}{f}\Big(2-\frac{k'^2}{f^2}\Big)^2\Big\}\cosh r'H\sin s'H \\ +\frac{s'}{f}\varphi\Big\{-\frac{4sr'^2}{f^3}+\frac{r}{f}\Big(2-\frac{k'^2}{f^2}\Big)^2\Big\}\sinh r'H\cos s'H \\ +\Big\{-\frac{4r'^2s'^2}{f^4}\zeta+\Big(2-\frac{k'^2}{f^2}\Big)^2\vartheta\Big\}\sinh r'H\sin s'H=0\,,\end{aligned}\tag{1}$$

where

$$\left.\begin{aligned}&\varphi=\frac{\mu'k^2k'^2}{\mu f^4},\quad \zeta=\frac{4rs}{f^2}\Big(\frac{\mu'}{\mu}-1\Big)^2-\alpha^2,\quad \eta=\frac{2rs}{f^2}\Big(\frac{\mu'}{\mu}-1\Big)\beta-\alpha\gamma,\\ &\vartheta=\frac{rs}{f^2}\beta^2-\gamma^2,\quad \alpha=\frac{2\mu'}{\mu}-\Big(2-\frac{k^2}{f^2}\Big),\quad \beta=\frac{\mu'}{\mu}\Big(2-\frac{k'^2}{f^2}\Big)-2\,,\\ &\gamma=\frac{\mu'}{\mu}\Big(2-\frac{k'^2}{f^2}\Big)-\Big(2-\frac{k^2}{f^2}\Big),\quad f=\frac{2\pi}{L},\\ &r^2=f^2-h^2,\quad s^2=f^2-k^2,\quad r'^2=f^2-h'^2,\quad s'^2=k'^2-f^2,\\ &h^2=\rho p^2/(\lambda+2\mu),\quad h'^2=\rho'p^2/(\lambda'+2\mu'),\quad k^2=\rho p^2/\mu,\quad k'^2=\rho'p^2/\mu'.\end{aligned}\right\}\tag{2}$$

When $fH\to 0$, that is, $L/H\to\infty$, the equation (1) reduces to

$$\frac{r's'k'^4}{f^6}\Big\{\Big(2-\frac{k^2}{f^2}\Big)^2-\frac{4rs}{f^2}\Big\}=0\,,\tag{3}$$

which is of the same form as the velocity equation of Rayleigh-waves for a semi-infinite body. This shows that even should the velocity of bodily waves in the surface layer be much higher than that in the subjacent medium, the behaviour of Rayleigh-waves of great length in passing through the stratified body does not differ from that of the

3) *loc. cit.*

same waves in the case of a semi-infinite body which is of the material as the subjacent medium just mentioned.

When $fH\rightarrow\infty$, that is $L/H\rightarrow 0$, the equation (1) reduces to a pair of factors, as follows,

$$\left\{\left(2-\frac{k'^2}{f^2}\right)^2-\frac{4r's'}{f^2}\right\}\left[\frac{rs}{f^2}\left\{\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-2\right\}^2-\left\{\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-\left(2-\frac{k^2}{f^2}\right)\right\}^2\right.$$

$$\left.-\frac{4r's'rs}{f^4}\left(\frac{\mu'}{\mu}-1\right)^2+\frac{r's'}{f^2}\left\{\frac{2\mu'}{\mu}-\left(2-\frac{k^2}{f^2}\right)\right\}^2+\frac{\mu'k^2k'^2sr'}{\mu f^6}+\frac{\mu'k^2k'^2rs'}{\mu f^6}\right]=0\,. \quad (4)$$

The first factor represents the velocity equation of Rayleigh-waves that are transmitted through a semi-infinite body composed of the same material as that in the surface layer, and the second one the velocity equation of the waves that are transmitted along the boundary between the surface and the subjacent media. The latter kind of waves corresponds to the one that Stoneley[4] already found, the type of expression in (4) being formulated by us for other purposes.[5] As a matter of fact, the decomposition of the velocity equation was shown by Love[6] for the case of incompressible materials, and also by Lee[7] for a more general case in Love's type of expression. Although it appears possible for two kinds of waves to exist for any value of ρ'/ρ, or μ'/μ, Stoneley wave only exists in the case wherein the densities and elasticities of the media are likely to satisfy Wiechert's condition, namely, $\rho'/\rho=\mu'/\mu=\lambda'/\lambda$.

One of the important conditions for the present kind of waves that the same waves should be of surface or boundary type, would be $V/\sqrt{\mu/\rho}<1$; otherwise, the amplitude distribution of the waves in the subjacent medium would be sinusoidal, from which the total energy of the waves, integrated through the whole depth, would assume an infinitely large value even should the amplitude at any depth in the subjacent medium be very small.

In the present problem, so far as the nature of the waves differed from that of surface or boundary waves, the waves under consideration would not be of permanent type at the same time, whence follows the importance with respect to two different conditions of the relation $V/\sqrt{\mu/\rho}<1$.

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4) R. STONELEY, "Elastic Waves at the Surface of Separation of Two Solids", *Proc. Roy. Soc.*, **106** (1924), 416~428.

5) K. SEZAWA and K. KANAI, *Bull. Earthq. Res. Inst.*, **16** (1938), 504~526.

6) A. E. H. LOVE, *Some Problems of Geodynamics*, **1911**, 168.

7) A. W. LEE, *M. N. R. A. S. Geophys. Suppl.*, **3** (1934), 252.

3. *Numerical examples and interpretation of their results.*

As mentioned in the introductory part, we calculated the cases (I) $\mu'/\mu=1$, (II) $\mu'/\mu=0{\cdot}8$, (III) $\mu'/\mu=0{\cdot}58$, (IV) $\mu'/\mu=0{\cdot}5$, (V) $\mu'/\mu=0{\cdot}44$, (VI) $\mu'/\mu=0{\cdot}40$; $\rho'/\rho=0{\cdot}5$, $\lambda=\mu$, $\lambda'=\mu'$ being kept constant. The results of the calculation are shown in the annexed tables.

Table I. $\mu'/\mu=1$.

$V/\sqrt{\mu/\rho}$	1·000	0·980	0·950	0·930	0·9194
L/H	4·650	8·240	18·91	58·50	∞

Table II. $\mu'/\mu=0{\cdot}8$.

$V/\sqrt{\mu/\rho}$	1·000	0·9798	0·9644	0·9194
L/H	3·217	6·042	8·760	∞

Table III. $\mu'/\mu=0{\cdot}58$.

$V/\sqrt{\mu/\rho}$	0·9902	0·9798	0·9487	0·9194
L/H	0	2·565	7·604	∞

Table IV. Case IV. $\mu'/\mu=0{\cdot}5$.

$V/\sqrt{\mu/\rho}$	0·9194	0.9327		0·9534		0·9487		0·9194
L/H	0	1·359	10·13	2·303	3·736	1·946	4·862	∞

Table IV(S). Case IV(S). $\mu'/\mu=0{\cdot}5$.

$V/\sqrt{\mu/\rho}$	0·9951	0·9960	0·9970	0·9975	0·9995	0·9999	1·000
L/H	0	0·6000	0·7950	0·8748	1·1575	1·267	1·282

Table V. $\mu'/\mu=0{\cdot}44$.

$V/\sqrt{\mu/\rho}$	0 8625	0·9000	0·9165	0·9247		0·9298		0·9194
L/H	0	1·751	2·112	2·363	11·256	2·574	7·868	∞

Table VI. $\mu'/\mu=0{\cdot}4$.

$V/\sqrt{\mu/\rho}$	0·8223	0·8944	0·9165	0·9195		0·9194
L/H	0	2·235	2·881	3·046	12·738	∞

The same results are also shown in Fig. 2. The dispersion curve in case I is reproduced from our preceding result.

It will be seen that, as in case I, the dispersion curve in case II beginning discontinuously at such an abscissa of L/H as not zero with ordinate $V/\sqrt{\mu/\rho}=1{\cdot}000$, tends to touch the asymptotic line $V/\sqrt{\mu/\rho}=0{\cdot}9194$ at $L/H\rightarrow\infty$, the curve thus lying within a limited range of abscissa. In cases III, IV, V, VI, on the other hand, the respective dispersion curves extend through the whole range of L/H.

There are two dispersion curves in case IV, where Wiechert's condition, namely $\rho'/\rho=\mu'/\mu$, is satisfied. In this case the dispersion curve passing through the ordinate corresponding to Stoneley wave is indicated by IV(S). The question may arise whether or not dispersion curves analogous to IV(S) can exist in the respective remaining cases, namely, I, II, III, V, VI. The answer is relatively simple. It was confirmed that in cases I, II, III, no point, excepting those shown in the sketch, on the line $V/\sqrt{\mu/\rho}=1$, can represent real propagational velocity. Since, on the other hand, the curve analogous to IV(S) should be so inclined as to meet the line $V/\sqrt{\mu/\rho}=1$, it is scarcely possible for the curve in question to exist. In case V, there is no point analogous to IV(S) on the line, $V/\sqrt{\mu/\rho}=0{\cdot}938$, beyond which ordinate the value of s' becomes imaginary, that is to say, the amplitude distribution in the surface layer becomes of sinusoidal type. For the same reason as that given in cases I, II, III, it is impossible for the second dispersion curves to exist. It is also obvious that in case VI no second dispersion curve exists. At all events, ordinate of every dispersion curve tends to assume the asymptotic value $V/\sqrt{\mu/\rho}=0{\cdot}9194$ at $L/H\rightarrow\infty$.

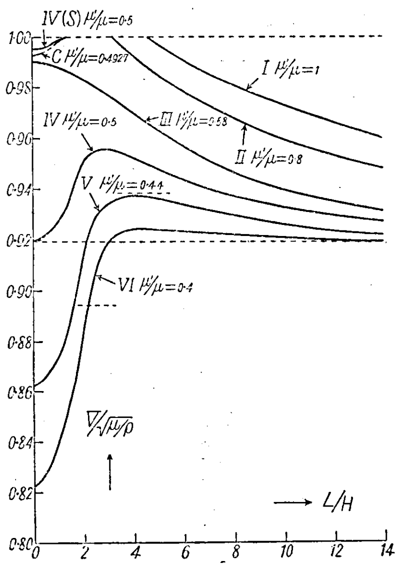

Fig. 2. Dispersion curves.

If the ratio of μ'/μ be somewhat less than 0·495, the amplitude distribution, of the waves similar to IV(S), within the surface layer becomes of sinusoidal type for a relatively large wave-length, say $L/H>0{\cdot}5$. In the limiting case, where the ratio of μ'/μ is 0·4927, the amplitude distribution, of the waves also similar to IV(S), in the surface layer is sinusoidal for any wave-length. Such a limiting case is shown by curve C in Fig. 2, its ordinate at $L/H=0$ being 0·9926. Although it is possible to consider dispersion curves, of the waves similar to IV(S), below the curve C for such cases that the elastic constants satisfy the condition $\mu'/\mu<0{\cdot}4927$, since the amplitude distribution within the surface layer in any one of such cases is sinusoidal, it is improbable for the waves under consideration to be of boundary type. Thus, Stoneley waves are possible to exist only for the range between $\mu'/\mu=0{\cdot}4926\sim0{\cdot}563$, the velocities corresponding to these limits then being $0{\cdot}9926\sim1$.

The dispersion curve in case IV has an odd form. Notwithstanding that the velocity of longitudinal waves or transverse waves in the surface layer is the same as that in the subjacent medium, the velocity of Rayleigh-waves is considerably greater than that of usual Rayleigh-waves, namely $V=0{\cdot}9194\sqrt{\mu/\rho}=0{\cdot}9194\sqrt{\mu'/\rho'}$, excepting for the condition $L/H=0$ and $L/H=\infty$. The dispersion curve in case V resembles that in case IV; after reaching a maximum it is slowly depressed, and approaches the asymptote $V/\sqrt{\mu/\rho}=0{\cdot}9194$. In the dispersion curve in case VI, no appreciable maximum of ordinate exists for intermediate value of L/H. It appears that for a smaller value of μ'/μ, the dispersion curve is similar to that in the case of the normal dispersion of Rayleigh-waves.

The values of μ'/μ, for which the ordinate of the dispersion curve is exactly $V/\sqrt{\mu/\rho}=1$ at $L/H=0$, is $\mu'/\mu=0{\cdot}592$. It follows then that so long as $\mu'/\mu>0{\cdot}592$ in the case $\rho'/\rho=1/2(\lambda=\mu,\ \lambda'=\mu')$, there should exist a certain range of L/H within which no surface wave (permanent type) can exist.

4. *Concluding remarks.*

From mathematical calculations we found that in certain elasticity and density conditions, the dispersion of Rayleigh-waves is anomalous. When the stratification nearly satisfies Wiechert's condition, the velocity of Rayleigh-waves for an intermediate ratio of L/H is considerably higher than that of the usual Rayleigh-waves. In this case there is a second dispersion curve that passes through the point corresponding to the velocity of Stoneley waves. When the stratification is nearly in the usual condition rather than Wiechert's condition, the dispersion curve tends to assume the normal dispersion type. Should, on the other hand, the condition of stratification be the reverse of that in the usual meaning even beyond Wiechert's condition, the dispersion curve becomes quite anomalous and, beyond a certain critical condition, there should exist a range of L/H within which no surface wave (permanent type) can exist. Particularly, in the present problem, the condition that prevents the existence of surface waves also prevents the waves from being of permanent type.

The condition of stratification just given, that is to say, the condition that the increase of density is more rapid than that of elasticity, both with increase in depth, is more likely to be actual than merely ideal in the surface crust of the Earth. The fact that the vibration of Rayleigh-waves in some case is limited to certain periods, may result from the condition of the waves possibly existing for a certain range of wave-lengths. The evidence that the group velocity of surface waves is greater than their phase velocity, is also likely to be connected with the problem in the present paper.

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47. 彈性表面波の異常分散(第2報)

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前回の研究に於て，地殼の深い所程その物質の密度が大きい場合に，分散性レーリー波が異常分散になり得る事を説明して置いた．今回はそれに關係のある問題を一層廣い範圍に渉つてしらべて見たのである．

地殼の層がウイーヘルトの條件を滿足する場合には，中間位の波長に限つて普通のレーリー波から想像できる速度よりも高い傳播速度があり得るのである．しかしウイーヘルトの條件よりも普通の層狀に近い場合，即ち表面程固體波の速度が遲い場合には，普通の場合の型の分散曲線が出るものである．又普通の層狀とは逆になり，而もウイーヘルトの條件を越してをる狀態になると，波長が長い程速度が遲くなるといふ場合があり得る．その場合には，ある範圍の波長については實在の波の存在し得ぬものである．

ウイーヘルトの條件を滿足するやうな場合に近い狀態では固體と固體との間の境界を傳播するレーリー型の波が存在することもわかつた．

只今のやうな波の出る場合は，よく考へるとむしろ大いに可能な狀態である．又，地震波から見ても，表面波が或る範圍の波長にしか渉つてをらぬ場合や，波の群速度がその位相速度よりも速い場合などが只今の問題と大いに關係のあることもわかつた．
