37. *Damping of Periodic Visco-Elastic Waves with Increase in Focal Distance.*

By Katsutada SEZAWA and Kiyoshi KANAI,

Earthquake Research Institute.

(Read May 17, 1938.—Received June 20, 1938.)

1. *Introduction.*

It is evident that the problem of the decay of visco-elastic waves has been receiving the attention of investigators. In the solution that we previously obtained,[1, 2] particularly, with a view to ascertaining the manner of deformation of a given initial pulse, the elementary part of the solution was of the kind to show the condition of the damping of waves with time increase. The solutions of the problem obtained since then by other authors were also of the kind just mentioned, at any rate with regard to the condition of the damping of elementary waves. On the other hand, notwithstanding that the behaviour in transmission of a periodic visco-elastic wave whose amplitude at its source is constant, is also important, no one has so far discussed it, excepting such a special case as that of waves transmitted along a bar in the form of flexural waves, the expression for which we were the first to formulate.[3] We shall now solve the general cases of bodily waves as well as those of surface waves, by means of which, such parts of the previous solutions as were treated approximately can now be represented in their exact forms with little complexity. Although in the previous paper, in seeking the solution for the transmission of Rayleigh-waves we assumed that $\lambda/\mu=\lambda'/\mu'$, in the present paper it is shown that it is possible to obtain solutions for waves of the same type for any ratios of λ/μ, λ'/μ', μ/μ'.[4]

---

1) K. SEZAWA, "On the Decay of Waves in Visco-Elastic Solid Bodies", *Bull. Earthq. Res. Inst.*, **3** (1927), 43~50.

2) *Ditto*, "On the Diffusion of Tremors on the Surface of a Semi-infinite Solid Body", *Bull. Earthq. Res. Inst.*, **5** (1928), 71~83.

3) *Ibid.* 1), 50~53; and some other papers.

4) J. H. C. THOMPSON, citing our result, *loc. cit.* 1), discussed several conditions of λ'/μ', *Phil. Trans. Roy. Soc.*, **231** (1933), 347~350. Our previous assumption on this condition was made merely for the sake of mathematical simplicity. K. IIDA also dealt with a problem similar to Thompson's from his experimental result for soil, etc., *Bull. Earthq. Res. Inst.*, **16** (1938), 391~406.

Jeffreys[5] considered two kinds of damping in waves, namely, firmo-viscosity (similar to Voigt's viscous condition) and elastico-viscosity (similar to Maxwell's relaxation condition). Gutenberg and Schlechtweg[6] also studied Jeffreys's case. Here, however, we shall take the former condition of the two, namely, the firmo-viscous state of a solid body, calling it a visco-elastic body.

We shall first discuss the Rayleigh-type waves and Love-type waves, and then the plane and radial bodily waves, all of which are transmitted through visco-elastic bodies.

2. *Rayleigh-waves.*

Let λ, μ be Lamé's elastic constants and λ', μ' the dilatational and tangential coefficients of solid viscosities. The equations of motion of a visco-elastic body are then expressed by

$$\left.\begin{aligned} &\left\{(\lambda+2\mu)+(\lambda'+2\mu')\frac{\partial}{\partial t}\right\}\left(\frac{\partial^2\Delta}{\partial x^2}+\frac{\partial^2\Delta}{\partial y^2}\right)=\rho\frac{\partial^2\Delta}{\partial t^2}, \\ &\left\{\mu+\mu'\frac{\partial}{\partial t}\right\}\left(\frac{\partial^2\varpi}{\partial x^2}+\frac{\partial^2\varpi}{\partial y^2}\right)=\rho\frac{\partial^2\varpi}{\partial t^2}, \end{aligned}\right\} \tag{1}$$

where

$$\Delta=\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}, \qquad 2\varpi=\frac{\partial v_2}{\partial x}-\frac{\partial u_2}{\partial y}. \tag{2}$$

The solutions of (1) are

$$\left.\begin{aligned} \Delta&=Ae^{-ry+i(pt-fx)}, \\ 2\varpi&=Be^{-sy+i(pt-fx)}, \end{aligned}\right\} \tag{3}$$

in which

$$\left.\begin{aligned} &r^2=f^2-h^2, \quad s^2=f^2-k^2, \\ &h^2=\frac{\rho p^2}{(\lambda+2\mu)+ip(\lambda'+2\mu')}, \quad k^2=\frac{\rho p^2}{\mu+ip\mu'}. \end{aligned}\right\} \tag{4}$$

The solutions for u_1, v_1 and u_2, v_2 corresponding to Δ and ϖ are

$$\left.\begin{aligned} &u_1=\frac{if}{h^2}Ae^{-ry+i(pt-fx)}, \quad v_1=\frac{r}{h^2}Ae^{-ry+i(pt-fx)}, \\ &u_2=-\frac{s}{k^2}Be^{-sy+i(pt-fx)}, \quad v_2=\frac{if}{k^2}Be^{-sy+i(pt-fx)}. \end{aligned}\right\} \tag{5}$$

5) H. JEFFREYS, *M.N.R.A.S.*, **77** (1917), 449~456; *The Earth* (1929), 263.
6) B. GUTENBERG and H. SCHLECHTWEG, *Phys. ZS.*, **31** (1930), 745~752.

The boundary conditions are

$$\left.\begin{aligned}&\lambda\Delta+2\mu\frac{\partial v}{\partial y}+\frac{\partial}{\partial t}\left(\lambda'\Delta+2\mu'\frac{\partial v}{\partial y}\right)=0,\\&\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\mu'\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0,\end{aligned}\right\}\tag{6}$$

in which $u=u_1+u_2$, $v=v_1+v_2$.

Substituting (5) in (6), we get

$$\left\{\left(\frac{k}{f}\right)^2-2\right\}^2-\frac{4rs}{f^2}=0,\tag{7}$$

which is of the same form as the usual velocity equation for Rayleigh-waves, the condition being that the symbols k, r, s shall be written in the forms given in (4).

We write

$$\frac{k^2}{f^2}=\frac{\rho p^2}{f^2\mu\left(1+\frac{ip\mu'}{\mu}\right)}=a+ib,\tag{8}$$

whence

$$\frac{r^2}{f^2}=1-(a+ib)(P+iQ),\qquad \frac{s^2}{f^2}=1-(a+ib),\tag{9}$$

where

$$P=\frac{\left(\frac{\lambda}{\mu}+2\right)+\left(\frac{p\mu'}{\mu}\right)^2\left(\frac{\lambda'}{\mu'}+2\right)}{\left(\frac{\lambda}{\mu}+2\right)^2+\left(\frac{p\mu'}{\mu}\right)^2\left(\frac{\lambda'}{\mu'}+2\right)^2},\quad Q=\frac{\frac{p\mu'}{\mu}\left(\frac{\lambda}{\mu}-\frac{\lambda'}{\mu'}\right)}{\left(\frac{\lambda}{\mu}+2\right)^2+\left(\frac{p\mu'}{\mu}\right)^2\left(\frac{\lambda'}{\mu'}+2\right)^2},\tag{10}$$

from which the velocity equation (7) is transformed into the two expressions

$$\left.\begin{aligned}&(a-2)^2-b^2-4\left\{(1-aP+bQ)^2+(aQ+bP)^2\right\}^{\frac{1}{4}}\left\{(1-a)^2+b^2\right\}^{\frac{1}{4}}\\&\qquad\cdot\cos\frac{1}{2}\left(\tan^{-1}\frac{aQ+bP}{1-aP+bQ}+\tan^{-1}\frac{b}{1-a}\right)=0,\end{aligned}\right\}\tag{11}$$

$$b(a-2)+2\left\{(1-aP+bQ)^2+(aQ+bP)^2\right\}^{\frac{1}{4}}\left\{(1-a)^2+b^2\right\}^{\frac{1}{4}}$$
$$\cdot \sin\frac{1}{2}\left(\tan^{-1}\frac{aQ+bP}{1-aP+bQ}+\tan^{-1}\frac{b}{1-a}\right)=0.$$

From (8), (10) and (11) it is possible to get the velocity and damping coefficient (with increase in focal distance) of the periodic visco-elastic Rayleigh-waves for any ratios of λ/μ, λ'/μ', $p\mu'/\mu$.

Since, generally speaking, the f-value that defines the wave length assumes the form

$$f=f_1+if_2, \tag{12}$$

it follows that the actual wave length is $2\pi/f_1$ and that the damping coefficient of waves with increase in focal distance is f_2. The coefficients r, s showing the decay in amplitude with depth, are no longer aperiodic, but of the types $r=r_1+ir_2$, $s=s_1+is_2$, which indicate oscillatory decrease in wave amplitudes with depth. It should however be borne in mind that, in this case, the integral of the wave energy through the whole depth is not infinite.

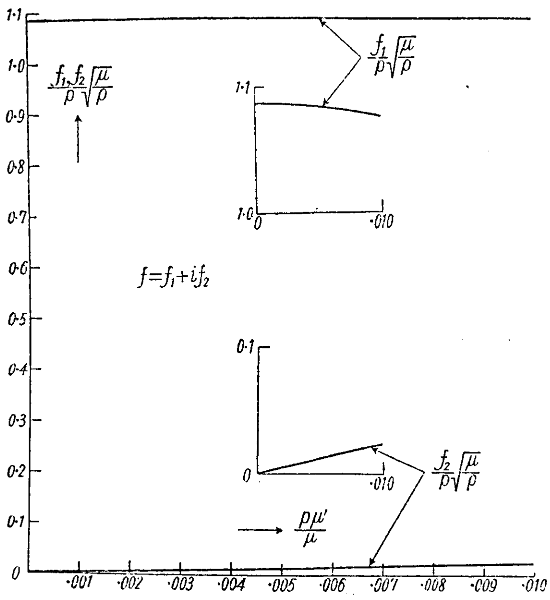

Fig. 1.

The results of calculating the various ratios of $p\mu'/\mu$ in the case $\lambda/\mu=1$, $\lambda'/\mu'=100\,(\lambda'\neq\mu')$ are shown in the formulae below and in Fig. 1.

$$\frac{p\mu'}{\mu}=0; \qquad f=1{\cdot}08766p\sqrt{\frac{\rho}{\mu}},$$

$$\frac{p\mu'}{\mu}=\frac{1}{1000}; \qquad f=(1{\cdot}0875-i0{\cdot}002987)p\sqrt{\frac{\rho}{\mu}},$$

$$\frac{p\mu'}{\mu}=\frac{1}{200}; \qquad f=(1{\cdot}0843-i0{\cdot}01367)p\sqrt{\frac{\rho}{\mu}},$$

$$\frac{p\mu'}{\mu}=\frac{1}{100}; \qquad f=(1{\cdot}0764-i0{\cdot}02411)p\sqrt{\frac{\rho}{\mu}}.$$

Since $(f_1/p)\sqrt{\mu/\rho}$ and $(f_2/p)\sqrt{\mu/\rho}$ correspond to the first and second terms of

$$\frac{f}{p}\sqrt{\frac{\mu}{\rho}}\left\{\equiv(f_1+if_2)\frac{1}{p}\sqrt{\frac{\mu}{\rho}}\right\} \tag{13}$$

in the solution of the velocity equation, curves A and B are proportional to the coefficient for wave length and to the coefficient for damping with increase in focal distance respectively.

It will be seen from Fig. 1 that whereas the increase in the damping coefficient with increase in $p\mu'/\mu$ is very marked, the increase in wave length (that is, decrease of f) with increase in the same $p\mu'/\mu$ is rather gradual. A similar condition will probably hold for any ratio of λ/μ as well as for any ratio of λ'/μ'. At all events, the respective expressions for the dilatational and distortional components assume the forms

$$\left.\begin{aligned}\Delta&=Ae^{-(r_1+ir_2)y+i(pt-f_1x)-f_2x},\\ 2\varpi&=Be^{-(s_1+is_2)y+i(pt-f_1x)-f_2x},\end{aligned}\right\} \tag{14}$$

f_1, f_2, r_1, r_2, s_1, s_2 being real. These expressions show that even should the damping of pure dilatational waves differ from that of pure distortional waves, the damping coefficients of both components (dilatational and distortional) in Rayleigh-waves are the same.

Although every feature of damping in Rayleigh-waves is involved implicitly in the curves in Fig. 1, since it is not an easy matter to compare the nature of the damping in Rayleigh-waves with that in bodily waves, we shall now show the expressions for the Rayleigh-waves in a special case. The condition of the case is such that

$$\frac{\lambda+2\mu}{\mu}=\frac{p(\lambda'+2\mu')}{p\mu'}\equiv n. \tag{15}$$

Solving the problem as in the general case, we have

$$f=\sqrt{\frac{\rho p^2}{q\mu\sqrt{1+\left(\frac{p\mu'}{\mu}\right)^2}}}\left\{\cos\frac{1}{2}\tan^{-1}\frac{p\mu'}{\mu}-i\sin\frac{1}{2}\tan^{-1}\frac{p\mu'}{\mu}\right\}, \tag{16}$$

$$\left.\begin{aligned} r &= \sqrt{\frac{\left(1-\frac{q\mu}{\lambda+2\mu}\right)\rho p^2}{q\mu\sqrt{1+\left(\frac{p\mu'}{\mu}\right)^2}}}\left\{\cos\frac{1}{2}\tan^{-1}\frac{p\mu'}{\mu}-i\sin\frac{1}{2}\tan^{-1}\frac{p\mu'}{\mu}\right\}, \\ s &= \sqrt{\frac{(1-q)\rho p^2}{q\mu\sqrt{1+\left(\frac{p\mu'}{\mu}\right)^2}}}\left\{\cos\frac{1}{2}\tan^{-1}\frac{p\mu'}{\mu}-i\sin\frac{1}{2}\tan^{-1}\frac{p\mu'}{\mu}\right\}, \end{aligned}\right\} \tag{17}$$

where $q=0{\cdot}8453$ for $\lambda=\mu$ and $q=0{\cdot}9126$ for $\lambda/\mu=\infty$. The expressions for the dilatational and distortional components of Rayleigh-waves are the same as those in (14). These results are virtually the same as those that we obtained in the previous paper.[7] Comparing (16) with the damping coefficients of bodily waves, the expressions for which will appear presently, it will be seen that under the condition (15), Rayleigh-waves, bodily distortional waves, and bodily dilatational waves have successively smaller damping coefficients with reference to increase in focal distance. If, on the other hand, the damping coefficient as referred to increase in time, that is to say, g in e^{-gt}, were considered, it would be found that the coefficients under consideration for the present particular case are the same for the three kinds of waves. Speaking generally, it seems that the damping coefficient of Rayleigh-waves as referred to time increase is intermediate between the coefficient of dilatational waves and that of distortional waves.[8] As an example, we shall take the case, $\lambda/\mu=1$, $\lambda'/\mu'=100$ (which differs from the condition (15), and arrange the damping coefficients of the three kinds of waves as follows.

Table I. Damping coefficient $f(\div p\sqrt{\rho/\mu})$ as referred to increase in focal distance.

$p\mu'/\mu$	0	1/1000	1/200	1/100
Dilatational waves	0	0·00954	0·0476	0·0954
Rayleigh-waves	0	0·002987	0·01367	0·02411
Distortional waves	0	0·00050	0·0025	0·0050

Table II. Damping coefficient $g(\div p)$ as referred to time increase.

$p\mu'/\mu$	0	1/1000	1/200	1/100
Dilatational waves	0	0·0165	0·0805	0·165
Rayleigh-waves	0	0·002745	0·01256	0·02218
Distortional waves	0	0·00050	0·0025	0·0050

7) *loc. cit.* 1).
8) *loc. cit. poste.* § 4.

The condition that the damping coefficient of Rayleigh-waves as referred to increase in focal distance is intermediate between those of both bodily waves, is not very important. Since, nevertheless, Rayleigh-waves are in the condition of dilatational waves coupled with distortional waves, it is likely that the damping coefficient of Rayleigh-waves as referred to increase in time is generally intermediate between the damping coefficients of the two kinds of bodily waves.

3. *Love-waves.*

The equations of motion in this case are written

$$\left.\begin{aligned}\left(\mu_1+\mu_1'\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial z^2}\right)&=\rho_1\frac{\partial^2 v_1}{\partial t^2},\\ \left(\mu_2+\mu_2'\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 v_2}{\partial x^2}+\frac{\partial^2 v_2}{\partial z^2}\right)&=\rho_2\frac{\partial^2 v_2}{\partial t^2},\end{aligned}\right\} \tag{18}$$

where μ_1, μ_1', ρ_1, v_1 refer to the surface layer and μ_2, μ_2', ρ_2, v_2 to the subjacent medium. The solutions of (18) are

$$\left.\begin{aligned}v_1&=(A\cos s_1z+B\sin s_1z)e^{i(pt-fx)},\\ v_2&=Ce^{s_2z}e^{i(pt-fx)},\end{aligned}\right\} \tag{19}$$

where

$$s_1^2=\frac{\rho_1p^2}{\mu_1+i\mu_1'}-f^2,\quad s_2^2=f^2-\frac{\rho_2p^2}{\mu_2+i\mu_2'}. \tag{20}$$

Substituting (19) in the boundary conditions

$$\left.\begin{aligned}z=H;&\qquad \left(\mu_1+\mu_1'\frac{\partial}{\partial t}\right)\frac{\partial v_1}{\partial z}=0,\\ z=0;&\qquad v_1=v_2,\quad \left(\mu_1+\mu_1'\frac{\partial}{\partial t}\right)\frac{\partial v_1}{\partial z}=\left(\mu_2+\mu_2'\frac{\partial}{\partial t}\right)\frac{\partial v_2}{\partial z}=0,\end{aligned}\right\} \tag{21}$$

we get

$$\tan s_1H=\frac{(\mu_2+ip\mu_2')s_2}{(\mu_1+ip\mu_1')s_1}. \tag{22}$$

Writing

$$\rho_1p^2/\mu_1f^2=a+ib, \tag{23}$$

the expression (22) transforms into two equations

$$\left.\begin{aligned}
&\frac{\tan\left\{pH\sqrt{\frac{\rho_1}{\mu_1}}\zeta\sqrt{\frac{1+\cos\theta}{2}}\right\}\operatorname{sech}^2\left\{pH\sqrt{\frac{\rho_1}{\mu_1}}\zeta\sqrt{\frac{1-\cos\theta}{2}}\right\}}{1+\tan^2\left\{pH\sqrt{\frac{\rho_1}{\mu_1}}\zeta\sqrt{\frac{1+\cos\theta}{2}}\right\}\tanh^2\left\{pH\sqrt{\frac{\rho_1}{\mu_1}}\zeta\sqrt{\frac{1-\cos\theta}{2}}\right\}}\\
&\quad-\frac{\mu_2}{\mu_1}\sqrt{\frac{\frac{1}{2}(1+\cos\varphi)\sqrt{\eta^2+\vartheta^2}}{\left\{(a-1)+\frac{p\mu_1'}{\mu_1}\left(-b+\frac{p\mu_1'}{\mu_1}\right)\right\}^2+\left\{b+\frac{p\mu_1'}{\mu_1}(a-2)\right\}^2}}=0,\\
&\frac{\sec^2\left\{pH\sqrt{\frac{\rho_1}{\mu_1}}\zeta\sqrt{\frac{1+\cos\theta}{2}}\right\}\tanh\left\{pH\sqrt{\frac{\rho_1}{\mu_1}}\zeta\sqrt{\frac{1-\cos\theta}{2}}\right\}}{1+\tan^2\left\{pH\sqrt{\frac{\rho_1}{\mu_1}}\zeta\sqrt{\frac{1+\cos\theta}{2}}\right\}\tanh^2\left\{pH\sqrt{\frac{\rho_1}{\mu_1}}\zeta\sqrt{\frac{1-\cos\theta}{2}}\right\}}\\
&\quad-\frac{\mu_2}{\mu_1}\sqrt{\frac{\frac{1}{2}(1-\cos\varphi)\sqrt{\eta^2+\vartheta^2}}{\left\{(a-1)+\frac{p\mu_1'}{\mu_1}\left(-b+\frac{p\mu_1'}{\mu_1}\right)\right\}^2+\left\{b+\frac{p\mu_1'}{\mu_1}(a-2)\right\}^2}}=0,
\end{aligned}\right\}\quad(24)$$

where

$$\zeta=\sqrt{\frac{\left\{(a^2+b^2)-a\left(1+\frac{p^2\mu_1'^2}{\mu_1^2}\right)\right\}^2+\left\{\frac{p\mu_1'}{\mu_1}(a^2+b^2)-b\left(1+\frac{p^2\mu_1'^2}{\mu_1^2}\right)\right\}^2}{(a^2+b^2)\left(1+\frac{p^2\mu_1'^2}{\mu_1^2}\right)}},$$

$$\cos\theta=\frac{(a^2+b^2)-a\left(1+\frac{p^2\mu_1'^2}{\mu_1^2}\right)}{\sqrt{\left\{(a^2+b^2)-a\left(1+\frac{p^2\mu_1'^2}{\mu_1^2}\right)\right\}^2+\left\{\frac{p\mu_1'}{\mu_1}(a^2+b^2)-b\left(1+\frac{p^2\mu_1'^2}{\mu_1^2}\right)\right\}^2}},$$

$$\begin{aligned}
\eta=&\left\{1-\frac{p^2\mu_2'^2}{\mu_2^2}+\frac{\rho_2\mu_1}{\rho_1\mu_2}\left(b\frac{p\mu_2'}{\mu_2}-a\right)\right\}\left\{(a-1)+\frac{p\mu_1'}{\mu_1}\left(-b+\frac{\mu_1'}{\mu_1}\right)\right\}\\
&+\left\{2\frac{p\mu_2'}{\mu_2}-\frac{\rho_2\mu_1}{\rho_1\mu_2}\left(b+a\frac{p\mu_2'}{\mu_2}\right)\right\}\left\{b+\frac{p\mu_1'}{\mu_1}(a-2)\right\},
\end{aligned}$$

$$\begin{aligned}
\vartheta=&-\left\{1-\frac{p^2\mu_2'^2}{\mu_2^2}+\frac{\rho_2\mu_1}{\rho_1\mu_2}\left(b\frac{p\mu_2'}{\mu_2}-a\right)\right\}\left\{b+\frac{p\mu_1'}{\mu_1}(a-2)\right\}\\
&+\left\{2\frac{p\mu_2'}{\mu_2}-\frac{\rho_2\mu_1}{\rho_1\mu_2}\left(b+a\frac{p\mu_2'}{\mu_2}\right)\right\}\left\{(a-1)+\frac{p\mu_1'}{\mu_1}\left(-b+\frac{p\mu_1'}{\mu_1}\right)\right\},
\end{aligned}$$

$$\cos\varphi=\frac{\eta}{\sqrt{\eta^2+\vartheta^2}},\qquad(25)$$

every quantity in (24) being real. If a, b were determined from (24), the value f_1 defining the actual wave length, and damping coefficient f_2 for increase in focal distance x, is expressed by

$$f(=f_1+if_2)=\frac{p\sqrt{\frac{\rho_1}{\mu_1}}\left\{\sqrt{\frac{1}{2}\left(1+\frac{a}{\sqrt{a^2+b^2}}\right)}-i\sqrt{\frac{1}{2}\left(1-\frac{a}{\sqrt{a^2+b^2}}\right)}\right\}}{(a^2+b^2)^{\frac{1}{4}}}. \quad (26)$$

Since in the present case there are a number of quantities, such as $p\mu_1'/\mu_1$, μ_2'/μ_1', μ_2/μ_1, ρ_2/ρ_1, $pH\sqrt{\rho_1/\mu_1}$, for determining the coefficient, $f(=f_1+if_2)$, the numerical treatment of the problem is extremely complex. We shall therefore complete this section by merely concluding that Love-waves probably damp in much the same way as do Rayleigh-waves.

4. *Plane dilatational waves and plane distortional waves.*

The problem here is very simple. The equation of motion and its solution for dilatational waves are

$$\left\{(\lambda+2\mu)+(\lambda'+2\mu')\frac{\partial}{\partial t}\right\}\frac{\partial^2 v}{\partial x^2}=\rho\frac{\partial^2 v}{\partial t^2}, \quad (27)$$

$$v=A\,exp.\left[ipt-x\frac{\sqrt{\rho}\,p}{\{(\lambda+2\mu)^2+p^2(\lambda'+2\mu')^2\}^{\frac{1}{4}}}\cdot\left\{i\cos\frac{1}{2}\tan^{-1}\frac{p(\lambda'+2\mu')}{\lambda+2\mu}+\sin\frac{1}{2}\tan^{-1}\frac{p(\lambda'+2\mu')}{\lambda+2\mu}\right\}\right], \quad (28)$$

and those for distortional waves are

$$\left(\mu+\mu'\frac{\partial}{\partial t}\right)\frac{\partial^2 v}{\partial x^2}=\rho\frac{\partial^2 v}{\partial t^2}, \quad (29)$$

$$v=A\,exp.\left[ipt-x\frac{\sqrt{\rho}\,p}{\{\mu^2+p^2\mu'^2\}^{\frac{1}{4}}}\left\{i\cos\frac{1}{2}\tan^{-1}\frac{p\mu'}{\mu}+\sin\frac{1}{2}\tan^{-1}\frac{p\mu'}{\mu}\right\}\right]. \quad (30)$$

The second coefficient of x in every expression of v in (28), (30) corresponds to the damping coefficient under consideration. Particularly, when the solid viscosities are not large, the approximate values of damping coefficients as referred to increase in focal distance for both kinds of waves are

$$p^2\sqrt{\frac{\rho}{\lambda+2\mu}}\,\frac{(\lambda'+2\mu')}{2(\lambda+2\mu)}, \qquad p^2\sqrt{\frac{\rho}{\mu}}\,\frac{\mu'}{2\mu} \quad (31)$$

respectively, from which the numerical values for bodily waves shown in Tables I, II were calculated. It appears, at all events, that whereas the damping coefficients referred to distance, even approximately, are as those shown by (31), the damping coefficients referred to time are nearly

$$\frac{p^2(\lambda'+2\mu')}{2(\lambda+2\mu)}, \quad \frac{p^2\mu'}{2\mu}. \tag{32}$$

5. *Spherical dilatational waves and spherical distortional waves.*

The general solutions of spherical waves transmitted in a pure elastic body have already been obtained.[9] Here we shall modify the solutions by writing

$$h^2=\frac{\rho p^2}{(\lambda+2\mu)+ip(\lambda'+2\mu')}, \quad k^2=\frac{\rho p^2}{\mu+ip\mu'}, \tag{33}$$

that is to say

$$h=\left(\frac{\rho p^2}{\mu}\right)^{\frac{1}{2}}\left\{\left(\frac{\lambda}{\mu}+2\right)^2+\left(\frac{p\mu'}{\mu}\right)^2\left(\frac{\lambda'}{\mu'}+2\right)^2\right\}^{-\frac{1}{4}}e^{-\frac{i\theta_1}{2}},$$

$$k=\left(\frac{\rho p^2}{\mu}\right)^{\frac{1}{2}}\left\{1+\left(\frac{p\mu'}{\mu}\right)^2\right\}^{-\frac{1}{4}}e^{-\frac{i\theta_2}{2}}, \tag{33'}$$

where

$$\theta_1=\tan^{-1}\frac{\frac{p\mu'}{\mu}\left(\frac{\lambda'}{\mu'}+2\right)}{\frac{\lambda}{\mu}+2}, \qquad \theta_2=\tan^{-1}\frac{p\mu'}{\mu}. \tag{34}$$

The expressions of spherical dilatational waves at a relatively large distance then assume the forms

$$\left.\begin{aligned}
u_1&=A_{mn}\sqrt{\frac{2}{\pi}}\left\{\frac{1}{r^2}\vartheta^{-\frac{5}{2}}e^{i\left(\frac{n+1}{2}\pi+\frac{5}{4}\theta_1\right)}\right.\\
&\qquad\left.-\frac{1}{r}\vartheta^{-\frac{3}{2}}e^{i\left(\frac{n}{2}\pi+\frac{3}{4}\theta_1\right)}\right\}e^{-r(h_1+ih_2)}P_n^m(\cos\theta)\cos m\phi e^{ipt},\\
v_1&=-A_{mn}\sqrt{\frac{2}{\pi}}\frac{1}{r^2}\vartheta^{-\frac{5}{2}}e^{-r(h_1+ih_2)+i\left(\frac{n+1}{2}\pi+\frac{5}{4}\theta_1\right)}\frac{dP_n^m(\cos\theta)}{d\theta}\cos m\phi e^{ipt},\\
w_1&=mA_{mn}\sqrt{\frac{2}{\pi}}\frac{1}{r^2}\vartheta^{-\frac{5}{2}}e^{-r(h_1+ih_2)+i\left(\frac{n+1}{2}\pi+\frac{5}{4}\theta_1\right)}\frac{P_n^m(\cos\theta)}{\sin\theta}\sin m\phi e^{ipt}.
\end{aligned}\right\} \tag{35}$$

9) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **6** (1929), 12; K. SEZAWA and K. KANAI, *ibid.*, **10** (1932), 299.

The distortional waves of the second and the first kinds have respectively the forms

$$\left.\begin{aligned} u_2&=0,\\ v_2&=\frac{mB_{mn}}{n(n+1)}\sqrt{\frac{2}{\pi}\frac{1}{r}}\zeta^{-\frac{1}{2}}e^{-r(k_1+ik_2)+i\left(\frac{n+1}{2}\pi+\frac{1}{4}\theta_2\right)}\frac{P_n^m(\cos\theta)}{\sin\theta}\cos m\phi e^{ipt},\\ w_2&=-\frac{B_{mn}}{n(n+1)}\sqrt{\frac{2}{\pi}\frac{1}{r}}\zeta^{-\frac{1}{2}}e^{-r(k_1+ik_2)+i\left(\frac{n+1}{2}\pi+\frac{1}{4}\theta_2\right)}\frac{dP_n^m(\cos\theta)}{d\theta}\sin m\phi e^{ipt}, \end{aligned}\right\} \quad (36)$$

$$\left.\begin{aligned} u_3&=-\frac{n(n+1)C_{mn}}{m}\sqrt{\frac{2}{\pi}\frac{1}{r^2}}\zeta^{-\frac{5}{2}}e^{-r(k_1+ik_2)+i\left(\frac{n+1}{2}\pi+\frac{5}{4}\theta_2\right)}P_n^m(\cos\theta)\cos m\phi e^{ipt},\\ v_3&=\frac{C_{mn}}{m}\sqrt{\frac{2}{\pi}\frac{1}{r}}\zeta^{-\frac{3}{2}}e^{-r(k_1+ik_2)+i\left(\frac{n+2}{2}\pi+\frac{3}{4}\theta_2\right)}\frac{dP_n^m(\cos\theta)}{d\theta}\cos m\phi e^{ipt},\\ w_3&=C_{mn}\sqrt{\frac{2}{\pi}\frac{1}{r}}\zeta^{-\frac{3}{2}}e^{-r(k_1+ik_2)+i\left(\frac{n}{2}\pi+\frac{3}{4}\theta_2\right)}\frac{P_n^m(\cos\theta)}{\sin\theta}\sin m\phi e^{ipt}. \end{aligned}\right\} \quad (37)$$

The symbols ϑ, ζ, h_1, h_2, k_1, k_2 in (35), (36), (37) are such that

$$\left.\begin{aligned} &\vartheta=\left(\frac{\rho p^2}{\mu}\right)^{\frac{1}{2}}\left\{\left(\frac{\lambda}{\mu}+2\right)^2+\left(\frac{p\mu'}{\mu}\right)^2\left(\frac{\lambda'}{\mu'}+2\right)^2\right\}^{-\frac{1}{4}},\quad \zeta=\left(\frac{\rho p^2}{\mu}\right)^{\frac{1}{2}}\left\{1+\left(\frac{p\mu'}{\mu}\right)^2\right\}^{-\frac{1}{4}},\\ &h_1=\vartheta\sqrt{\frac{1-\cos\theta_1}{2}},\qquad h_2=\vartheta\sqrt{\frac{1+\cos\theta_1}{2}},\\ &k_1=\zeta\sqrt{\frac{1-\cos\theta_2}{2}},\qquad k_2=\zeta\sqrt{\frac{1+\cos\theta_2}{2}}. \end{aligned}\right\} \quad (38)$$

The damping coefficients here are the same as those in the case of plane bodily waves, at any rate in the case of relatively large focal distances from the origin of the waves.

6. *Concluding remarks.*

By treating the behaviour of transmission of a periodic visco-elastic wave whose amplitude at its source is constant, it was possible for us to get accurate expressions for the damping condition of that wave for any ratio in every one of λ/μ. λ'/μ', $p\mu'/\mu$. The cases discussed in this paper cover Rayleigh-waves, Love-waves, and plane and radial bodily waves (dilatational as well as distortional).

In the case of Rayleigh-waves, even should the damping of pure dilatational waves in the medium of transmission differ from that of pure distortional waves in the same medium, the damping coefficients of both (dilatational and distortional) the components in Rayleigh-waves are the same. It also appears that the coefficients r, s showing the decay in amplitude with depth, are never aperiodic but are of the types $r=r_1+ir_2$, $s=s_1+is_2$, which point to oscillatory decrease in wave amplitudes with depth. The integral of the wave energy throughout the whole depth is, in this case, finite. Since Rayleigh-waves are in the condition of dilatational waves coupled with distortional waves, it is probable that the damping coefficient of Rayleigh-waves as referred to time increase is usually intermediate between the damping coefficients of the two kinds of bodily waves.

Although, in the case of Love-waves, there are a number of quantities, such as $p\mu_1'/\mu$, μ_2'/μ_1', μ_2/μ_1, ρ_2/ρ_1, $pH\sqrt{\rho_1/\mu_1}$ for determining the wave length and damping coefficient, the problem being then extremely complex, it is likely that a condition quite similar to that of Rayleigh-waves holds.

The problems in plane dilatational waves and plane distortional waves are very simple. When, in particular, the solid viscosities are not large, the approximate values of the damping coefficients as referred to increase in focal distance for both kinds of waves are $p^2(\lambda'+2\mu')\sqrt{\rho/(\lambda+2\mu)}/2(\lambda+2\mu)$, $p^2\mu'\sqrt{\rho\mu}/2\mu$ respectively, whereas the damping coefficients as referred to time increase are nearly $p^2(\lambda'+2\mu')/2(\lambda+2\mu)$, $p^2\mu'/2\mu$ for the respective kinds of waves.

The damping coefficients in the case of spherical bodily waves are somewhat complex, but when the focal distance is relatively large, the coefficients under consideration assume the same values as those in the case of plane bodily waves.

37. 周期的粘彈性波の走波距離についての減衰

地震研究所 {妹澤克惟 / 金井清}

粘彈性波の減衰といふ問題は絶えず研究家の注意を惹いて來たやうである．我々が以前に試みた研究では，與へられた初變形が時間的に如何に變化するかを見るために，解の基礎部分を時間的に減衰する型にして置いたのである．其後の他の研究家の試みた解法もやはり同様な型を用ひたやうである．然るに一方では原點の振動振幅が一定で走波距離について振幅が減少する問題も大切であるのに拘らず，棒に傳播する減衰屈曲波（我々が最初に形を作つた）の場合以外は誰も之に就て議論する人がなかつたのである．そこで一般の彈性波について（周體波にも，表面波にも），この方針で問題を解くことにした．之によつて表面波の場合に便宜上用ひた粘性の特定條件をも取除くことができたのである．

レーレー波の場合にはそれを構成する縦波の粘性と横波の粘性とが違つてをつても，レーレー波の波動方向に從つての縦波振動の減衰係數と横波振動の減衰係數とが同じになる．之を時間的の減衰係數に換算すると，レーレー波の減衰係數は縦波の減衰係數と横波の減衰係數との中間の値を取る．之はレーレー波は縦波と横波の聯成する振動であるからである．尙，レーレー波の地中に於ける振幅は，粘性のないときのやうに無周期的でなく，周期的減幅性を表はす．しかし振動勢力をすべての深さを通して積分しても無限大にならぬから好都合である．

ラブ波の場合は問題が複雑であるけれども，レーレー波の場合と似た性質のあることが確められた．

平面周體波については，縦波でも横波でもその減衰性が極めて簡單な形で表される．球面周體波の場合でも震原距離が少しく大きくなると平面周體波と同じ減衰係數を取るものである．