39. *The Effect of Cooling on a Plastic Earth under Gravitational Forces. II.*

By Katsutada SEZAWA and Kiyoshi KANAI,

Earthquake Research Institute.

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1. *Preliminary notes.*

In our previous paper[1] we ascertained that the horizontal compression exceeding the vertical in the surface crust of the Earth is the result of the usual gravitational forces, the effect of temperature fall of the substratum being negligible compared with the action of gravity. Whereas in the previous paper we discussed the case in which a layer of uniform thickness is cooled uniformly, in the present paper we shall deal with the case in which the cooling is of any distribution with depth in the earth's crust.

Although we shall deal with the problem as in the case of plasticity as well as in that of elasticity, since our aim is to find the maximum shear stress and not the normal stress, the condition that the materials forming the earth's crust shall be compressible, even in the statical state, is particularly important. If, on the other hand, the Earth is incompressible in the statical state, the shear stresses at any part of the Earth are zero, so that there is no plastic stress; the manner of treatment, whether as for plasticity or for elasticity, being therefore immaterial.

As already mentioned in the previous paper, the equation of equilibrium in the case of a pure plasticity problem is

$$\rho\frac{\partial V}{\partial r}+\frac{\partial \widehat{rr}}{\partial r}+\frac{2}{r}(\widehat{rr}-\widehat{\theta\theta})-\beta\frac{\partial \theta}{\partial r}=0\,, \tag{1}$$

with a plastic condition

$$\widehat{rr}-\widehat{\theta\theta}=\pm 2k\,.$$

Although the solutions of the above two equations in the case of uniform ρ and k, for example, are merely written

1) K. SEZAWA and K. KANAI, "The Effect of Cooling on a Plastic Earth under Gravitational Forces", *Bull. Earthq. Res. Inst.*, 16 (1938), 244~255.

$$\left.\begin{aligned}\widehat{rr}&=4k\log\frac{a}{r}-\frac{\eta}{2}(a^2-r^2)+\beta\{\theta_{r=a}-\theta\},\\ \widehat{\theta\theta}&=4k\log\frac{a}{r}-\frac{\eta}{2}(a^2-r^2)+\beta\{\theta_{r=a}-\theta\}-2k,\end{aligned}\right\}\tag{2}$$

since these normal stresses are not so significant as the plastic stress $(\widehat{rr}-\widehat{\theta\theta})/2$, namely k, it is possible to conclude that were the Earth purely plastic, the effect of cooling has no place in the problem. All plasticity problems in the present paper will be so treated as in the manner of elasticity.

It should be borne in mind that we are to ascertain the stress state of a solid Earth with reference to the state of a liquid Earth, the gravitational conditions for both cases being the same.

2. *General theory, the problem being treated in the manner of elasticity.*

Let us assume that both density and temperature fall vary with the radius of the Earth. The equation of equilibrium is written

$$\rho\frac{\partial V}{\partial r}+(\lambda+2\mu)\left(\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}-\frac{2u}{r^2}\right)-\beta\frac{\partial\theta}{\partial r}=0,\tag{3}$$

where $\rho\partial V/\partial r$ is the rate of change of bodily force with radius, λ, μ are elastic constants, and $\beta=(\lambda+2\mu/3)q$; $\lambda+2\mu/3$, q being the modulus of compression and the expansion coefficient respectively.

The term $\rho(\partial V/\partial r)$ in (3) is determined from the conditions

$$\rho\frac{\partial V}{\partial r}=-\frac{\gamma M}{r^2}\rho,\quad M=\frac{4}{3}\pi\int_0^r\rho dr^3.\tag{4}$$

Let us assume that the displacement u in (3) is of the form

$$u=u_1+u_2,\tag{5}$$

where u_1, u_2 satisfy the equations

$$\left.\begin{aligned}(\lambda+2\mu)\left(\frac{\partial^2 u_1}{\partial r^2}+\frac{2}{r}\frac{\partial u_1}{\partial r}-\frac{2u_1}{r^2}\right)-\beta\frac{\partial\theta}{\partial r}&=0,\\ (\lambda+2\mu)\left(\frac{\partial^2 u_2}{\partial r^2}+\frac{2}{r}\frac{\partial u_2}{\partial r}-\frac{2u_2}{r^2}\right)+\rho\frac{\partial V}{\partial r}&=0.\end{aligned}\right\}\tag{6}$$

The superposition of u_1 and u_2 obviously satisfies the equation (3). Since it is now possible to get the solutions u_1, u_2 separately, treatment of the problem is greatly simplified.

3. *The determination of u_1 in the case of irregular distribution of temperature fall.*

Let us assume that the distribution of temperature (rise) is

$$\theta = f(r) = \alpha \sum_{n=0}^{\infty} A_n r^n. \tag{7}$$

To solve the first equation of (6), we write

$$u_1 = \sum_{n=0}^{\infty} a_n r^n. \tag{8}$$

From (5), (6), (7), (8) we find the relation

$$a_n = \frac{\alpha A_{n-1}}{n+2}, \quad (n = 2,\ 3, \cdots \infty) \tag{9}$$

so that the expressions for displacement and the stresses are

$$u_1 = B' r + \frac{\beta \alpha}{\lambda + 2\mu} \sum_{n=2}^{\infty} \frac{A^{n-1}}{n+2} r^n, \tag{10}$$

$$\widehat{rr}_1 = (3\lambda + 2\mu) B' + \frac{\beta\alpha}{\lambda + 2\mu} \sum_{n=2}^{\infty} \left[\frac{r^{n-1} A_{n-1}}{n+2} \Big\{ 2\lambda + n(\lambda + 2\mu) \Big\} - A_n r^n (\lambda + 2\mu) \right], \tag{11}$$

$$\widehat{\theta\theta}_1 = \widehat{\phi\phi}_1 = (3\lambda + 2\mu) B'$$
$$+ \frac{\beta\alpha}{\lambda + 2\mu} \sum_{n=2}^{\infty} \left[\frac{r^{n-1} A_{n-1}}{n+2} \Big\{ 2(\lambda + \mu) + n\lambda \Big\} - A_n r^n (\lambda + 2\mu) \right]. \tag{12}$$

Since the free surface has no normal stress, that is,

$$r = a; \qquad \widehat{rr}_1 = 0, \tag{13}$$

we find that

$$(3\lambda + 2\mu) B' = -\frac{\beta\alpha}{\lambda + 2\mu} \sum_{n=2}^{\infty} \left[\frac{a^{n-1} A_{n-1}}{n+2} \Big\{ 2\mu n + \lambda(n+2) \Big\} - A_n a^n (\lambda + 2\mu) \right]. \tag{14}$$

The plastic stress is

$$\frac{\widehat{rr}_1 - \widehat{\theta\theta}_1}{2} = \frac{\beta\alpha\mu}{\lambda + 2\mu} \sum_{n=2}^{\infty} \frac{(n-1) r^{n-1} A_{n-1}}{n+2}. \tag{15}$$

From the expression (15) it will be seen that, were the rigidity of the earth's crust very small, that is, $\mu\to 0$, the plastic stress would then be zero for any distribution of temperature in the crust.

4. *The determination of u_2 in the case of irregular distribution of mass density.*

Let the distribution of mass density be of the form

$$\rho = F(r) = \varepsilon \sum_{n=0}^{\infty} B_n r^n, \tag{16}$$

then, by means of (4),

$$\rho \frac{\partial V}{\partial r} = -4\pi\gamma \sum_{n=0}^{\infty} \frac{1}{n+3} B_n^2 r^{2n+1}. \tag{17}$$

Writing

$$u_2 = \sum_{n=0}^{\infty} b_n r^n, \tag{18}$$

and substituting it in the second equation of (6), we get

$$b_{2n+3} = \frac{1}{2(n+1)(2n+5)} \frac{4\pi\gamma\varepsilon^2}{n+3} B_n^2 \frac{1}{\lambda+2\mu}. \quad (n=0,\ 1,\ 2,\cdots\infty) \tag{19}$$

The expressions for displacement and the stresses are

$$u_2 = B''r + \sum_{n=0}^{\infty} \frac{r^{2n+3} 4\pi\gamma\varepsilon^2 B_n^2}{2(\lambda+2\mu)(n+1)(n+3)(2n+5)}, \tag{20}$$

$$\widehat{rr}_2 = (\lambda+2\mu)B'' + 4\pi\gamma\varepsilon^2 \sum_{n=0}^{\infty} \frac{r^{2n+2} B_n^2}{2(n+1)(n+3)(2n+5)} \left\{ \frac{\lambda(2n+5)+2\mu(2n+3)}{\lambda+2\mu} \right\}, \tag{21}$$

$$\widehat{\theta\theta}_2 = \widehat{\phi\phi}_2 = (\lambda+2\mu)B'' + 4\pi\gamma\varepsilon^2 \sum_{n=0}^{\infty} \frac{r^{2n+2} B_n^2}{2(n+1)(n+3)(2n+5)} \frac{\lambda(2n+5)+2\mu}{\lambda+2\mu}. \tag{22}$$

The value of B'' can be determined from the condition

$$r = a; \qquad \widehat{rr}_2 = 0, \tag{23}$$

the result being

$$(3\lambda+2\mu)B'' = -4\pi\gamma\varepsilon^2 \sum_{n=0}^{\infty} \frac{a^{2n+2} B_n^2}{2(n+1)(n+3)(2n+5)} \frac{\{\lambda(2n+5)+2\mu(2n+3)\}}{\lambda+2\mu}. \tag{24}$$

The plastic stress is

$$\frac{\widehat{rr}_2-\widehat{\theta\theta}_2}{2}=\frac{4\pi\gamma\varepsilon^2\mu}{\lambda+2\mu}\sum_{n=0}^{\infty}\frac{r^{2n+2}B_n^2}{(n+3)(2n+5)}, \tag{25}$$

which shows that if the rigidity of the crust is very small, the plastic stress assumes zero value, as in the preceding case.

5. *General solution for the case in which distribution of temperature and that of mass density are both given.*

When the distribution of temperature as well as that of density are both specified in such forms as (7) and (16), the solutions for displacement and the stresses are obtained by merely algebraically superposing the corresponding expressions in two cases, namely,

$$u=u_1+u_2,\quad \widehat{rr}=\widehat{rr}_1+\widehat{rr}_2,\cdots\cdots\cdots\cdots,\frac{\widehat{rr}-\widehat{\theta\theta}}{2}=\frac{\widehat{rr}_1-\widehat{\theta\theta}_1}{2}+\frac{\widehat{rr}_2-\widehat{\theta\theta}_2}{2}. \tag{26}$$

The resultants u, $rr,\cdots$, $(\widehat{rr}-\widehat{\theta\theta})/2$ obviously satisfy the equation (3) and the boundary condition

$$r=a;\qquad \widehat{rr}=0. \tag{27}$$

The ratio of plastic stress due to temperature distribution to that due to gravitational forces is obtained from (15) and (25) in the form

$$\frac{\beta\alpha}{4\pi\gamma\varepsilon^2}\left\{\sum_{n=2}^{\infty}\frac{(n-1)r^{n-1}A_{n-1}}{n+2}\Big/\sum_{n=0}^{\infty}\frac{r^{2n+2}B_n^2}{(n+3)(2n+5)}\right\}, \tag{28}$$

which shows that, provided the values of $\beta\alpha$ and $\gamma\varepsilon^2$ were specified in addition to the distributions of temperature and density, the ratio under consideration would invariably be the same for any value of μ — a satisfactory condition for concluding that the smallness of μ never increases that part of the plastic stress that is due to temperature change in the total plastic stress.

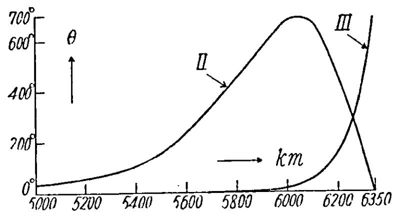

Fig. 1. Distributions of temperature fall.

6. *Numerical examples in the case of uniform mass density.*

(i) We shall first consider the case in which a part of the earth's crust, corresponding to a substratum of it, is cooled in a distribution such as that shown by curve II in Fig. 1, in which case the expression for θ is empirically of the form

$$\theta \, in \, C^0 = f(r) = -14.10^4\left\{\left(\frac{r}{6350}\right)^{32}\left(1-\frac{r}{6150}\right)+\frac{200}{6150}\left(\frac{r}{6350}\right)^{64}\right\}, \tag{29}$$

r being in km. The maximum temperature change occurs at a depth 300 km from the earth's surface and assumes the value $-700°$C. The temperature changes at depths 900 km and 1300 km from the earth's surface are only about $-100°$C and $-30°$C respectively. Let us assume that

$$\left.\begin{aligned}&\beta=\left(\lambda+\frac{2}{3}\mu\right)q=(10^{12}.3).(0{\cdot}08.10^{-4})=24.10^6,\\ &\mu/\lambda=1, \quad \rho=5{\cdot}52, \quad \gamma=648.10^{-10}.\end{aligned}\right\} \tag{30}$$

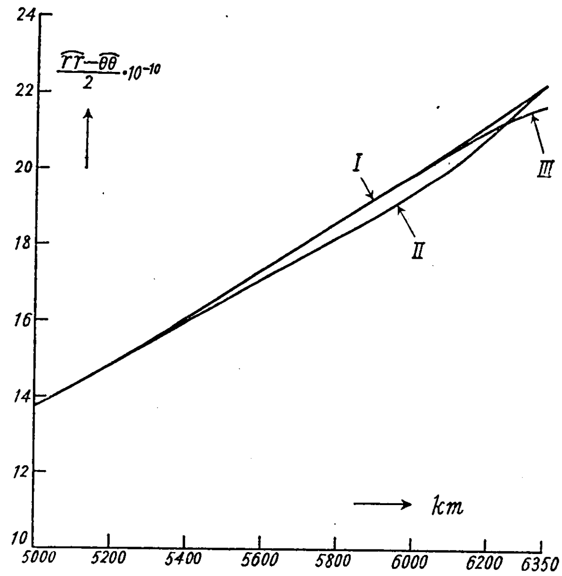

Fig. 2. Distributions of plastic stress.

Then, since from (29)

$$a=-14.10^4, \quad A_{32}=\frac{1}{6350^{32}}, \quad A_{33}=\frac{1}{6350^{32}.6150}, \quad A_{64}=\frac{200}{6350^{64}.6150}, \tag{31}$$

we get the plastic stress and radial stress due to both temperature change and the gravitational forces

$$\widehat{rr}-\widehat{\theta\theta}=10^{10}\Bigg[1{\cdot}103r'^2 + 224\left(\frac{r'}{6{\cdot}35}\right)^{32}\left\{-0{\cdot}9146+0{\cdot}9160\frac{r'}{6{\cdot}15}-0{\cdot}03104\left(\frac{r'}{6{\cdot}35}\right)^{32}\right\}\Bigg], \tag{32}$$

$$\widehat{rr}=10^{10}\Big[-122{\cdot}2+3{\cdot}033r'^2-0{\cdot}1684$$

$$+336\left(\frac{r'}{6{\cdot}35}\right)^{32}\left\{0{\cdot}038095-0{\cdot}037037\left(\frac{r'}{6{\cdot}15}\right)+0{\cdot}000647\left(\frac{r'}{6{\cdot}35}\right)^{32}\right\}\Big], \quad (33)$$

($1000r'$) being in km. The result of calculation for plastic and radial stresses are shown by curve II both in Fig. 2 and 3 (actual radial stress = ordinate I in Fig. 3 + ordinate II/100 in Fig. 3). With a view to ascertaining the correction due to temperature change, we calculated the gravitational plastic stress for the case of no temperature change, and plotted the results by curve I in Figs. 2 and 3. It will be seen that the effect of cooling on plastic stress and on radial stress under gravitational forces is negligible.

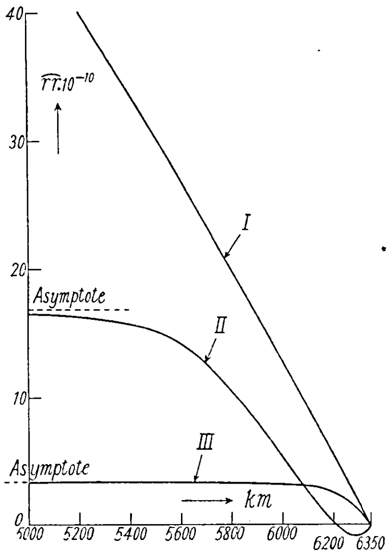

Fig. 3. Distributions of radial stress. Curves II and III represent $100\ \widehat{rr}.10^{-10}$ due to temperature fall only.

(ii) We shall next consider the case in which a superficial part of the earth's crust is cooled in such distribution as that shown by curve III in Fig. 1. In this case the empirical expression for it is extremely simple, namely,

$$\theta \text{ in } C^0=f(r)=-700\left(\frac{r}{6350}\right)^{64}, \quad (34)$$

so that

$$a=-700, \quad A_{64}=\frac{1}{6350^{64}}, \quad (35)$$

the values of β, μ/λ, ρ, γ being assumed the same as those in (30). The expressions for plastic and radial stresses due to both temperature change and gravitational forces are

$$\widehat{rr}-\widehat{\theta\theta}=10^{10}\left[1{\cdot}103r'^2-1{\cdot}07\left(\frac{r'}{6{\cdot}35}\right)^{64}\right], \quad (36)$$

$$\widehat{rr}=10^{10}\left[-122{\cdot}2+3{\cdot}033r'^2+0{\cdot}0334\left\{-1+\left(\frac{r'}{6{\cdot}35}\right)^{64}\right\}\right]. \quad (37)$$

The results of calculation for both stresses are shown by curve III in

Figs. 2 and 3 (actual radial stress = ordinate I in Fig. 3 + ordinate III/100 in Fig. 3). In this case, too, the effect of cooling on both plastic and radial stresses under gravitational forces are negligible.

7. *A numerical example in the case of varying mass density.*

It has been shown in Section 2 that, in the case of varying temperature and varying density, the general solutions of the problem can be obtained by merely superposing the two solutions, namely, the solution corresponding to the condition of thermal stress and that corresponding to the condition of stress under gravitational forces. We shall therefore discuss here the problem of mass density varying only with the radius of the Earth, regardless of thermal conditions. The empirical expressions for continuous distribution of mass density in the Earth differ with their authors, examples of some of them being[2)]

Legendre: $\rho = \rho_0 \sin nr/nr$, ρ_0 = density at the centre $= 11{\cdot}0$, $nR = 2{\cdot}487$.

Roche: $\rho = \rho_0\left\{1-\beta\left(\frac{r}{R}\right)^2\right\}$, $\rho_0 = 10{\cdot}1$, $\beta = 0{\cdot}764$.

Lipschitz: $\rho = \rho_0\left\{1-\beta\left(\frac{r}{R}\right)^\lambda\right\}$, $\rho_0 = 9{\cdot}453$, $\lambda = 2{\cdot}39$, $\beta = 0{\cdot}736$.

Helmert: $\rho = \rho_0\left\{1-\beta_1\left(\frac{r}{R}\right)^2+\beta_2\left(\frac{r}{R}\right)^4\right\}$, $\rho_0 = 11{\cdot}3$, $\beta_1 = 1{\cdot}04$, $\beta_2 = 0{\cdot}275$.

Oekinghaus: $\rho = \rho_0 e^{-\alpha\left(\frac{r}{R}\right)^2}$, $\rho_0 = 13{\cdot}78 \sim 10{\cdot}375$, $\alpha = 1{\cdot}707 \sim 1{\cdot}4$,

Williamson and Adams (from the condition of compression of material):

$$\log\frac{\rho}{\rho_m} = -\int_R^r \frac{k^2 M\rho}{r^2\kappa}dr,$$

where κ is the modulus of compression. More reasonable expressions were obtained by Wiechert, Klussmann, Gutenberg, and Haalck, who took into account of the discontinuities in mass density in the Earth,[3)] every one of which is too well known to require any description here.

From Jeffreys's recent investigation, the distribution of mass den-

2), 3) The expressions for mass density without discontinuity, or with discontinuity, will be found in the papers of the respective authors; the data here given are from H. SCHMEHL u. K. JUNG, Massenverteilung der Erde, *Handbuch d. Exp. Physik*, 25, 2, 353.

sity has been greatly modified; two discontinuities exist at depths of 400 km and 2900 km from the Earth's surface. Although the following formula gives no discontinuity at any radius, it still appears that the curve corresponding to the formula somewhat resembles that due to Jeffreys's distribution (See Fig. 4).[4]

$$\rho = F(r) = 12{\cdot}09 - 8{\cdot}8\left(\frac{r}{6350}\right)^2 - 50\left(\frac{r}{6350}\right)^4\left\{1-\left(\frac{r}{6350}\right)\right\}\left\{1-\left(\frac{r}{6350}\right)^4\right\}. \quad (36)$$

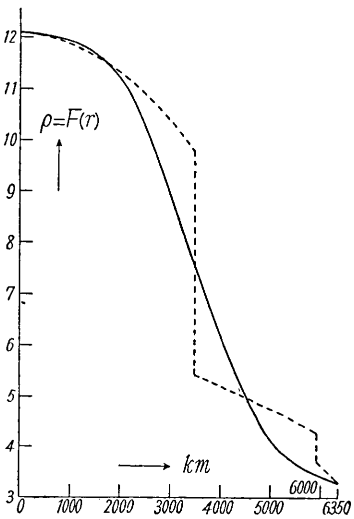

Fig. 4. Density distribution. Broken line: Jeffrey's, full line: ours.

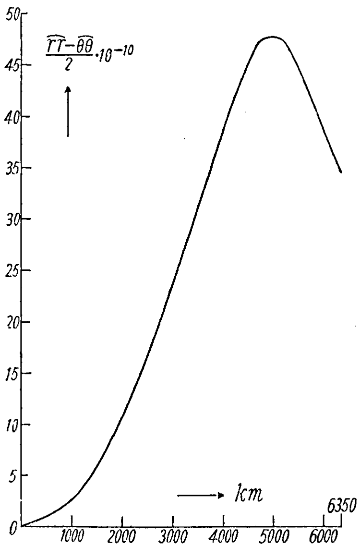

Fig. 5. Distribution of plastic stress due to our density distribution.

Using (16), (25), we have obtained the distribution of plastic stress, with the result as shown in Fig. 5. The distribution of plastic stress for the case in which both temperature and the gravitational forces are effective, is readily obtained by superposing result in this figure and that in Fig. 2.

8. *Concluding remarks.*

It has been ascertained that even in the case of irregular distributions of temperature and mass density, the effect of cooling of any part of the Earth on its plastic deformation is very slight compared with that of the gravitational forces on the same deformation. A very important condition is that the rigidity of the crust shall not be zero,

4) H. JEFFREYS, *M. N. R. A. S. Geophys. Suppl.*, **3** (1937), 53~59.

regardless of whether the problem is treated in the manner of plasticity or in that of elasticity. The present problem, particularly, has another condition that the state of the solid Earth under consideration differs from that of a liquid Earth of nearly equal radius, notwithstanding that the gravitational condition is the same for both cases.

It has been shown that, were the values of $\beta\alpha$ and $\gamma\varepsilon^2$ specified in addition to the distributions of temperature and density, the ratio of plastic stress due to temperature distribution to that due to gravitational forces would invariably be the same for any value of μ, from which it follows that, even should the value of μ be small, the plastic stress due to temperature change never participates much in the total plastic stress.

In calculating the stress condition or displacement of a solid body under the effects of both the gravitational forces and temperature, it is possible to deal with the problem by considering the two effects in question separately.

In the numerical calculation we assumed that $\mu/\lambda=1$. If, on the other hand, we were to assume that $\mu/\lambda \ll 1$, the resulting plastic stresses would be much smaller than those in the examples in the present paper, although it would fit in rather better with actual conditions in the earth's crust.

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39. 地球の冷却が重力の下にあるその プラスチック狀態に及ぼす影響 II

地震研究所 {妹澤克惟 / 金井清}

この前の論文で地球が固體化してからのプラスチック應力の狀態は主として重力から來るものとして差支なく，熱冷却の影響は大したものでないことを述べて置いた．この論文では更に進んで熱冷却の分布や質量分布が不規則な場合をしらべて見たのであるが，プラスチック應力の一般的傾向は依然として同じ事がわかつた．

プラスチック應力があるためには，地殼の剛性がなにがしかなければならぬ．之は地殼を彈性體として取扱つてもプラスチック體として取扱つても同樣である．又，只今の問題では特に，固形の地球の狀態が液狀のそれと如何に異なるかを論じたものである．之は重力の狀態が殆ど同じとして取扱ふからである．

$\beta\gamma$, $\gamma\varepsilon^2$, 溫度の分布，質量密度の分布が一定であれば，溫度分布によるプラスチック應力と重力によるプラスチック應力との比が一定であり，μ 自身の値の大小に關しない事がわかつた．從て溫度によるプラスチック應力は常に重要でない事がわかつた．

應力の計算をなすには，重力による應力の計算と溫度による應力の計算とを別々にやつてその結果を代數的に加へ合せればよいことがわかつた．

只今の計算の例では $\mu/\lambda=1$ と置いた．若し $\mu/\lambda\ll 1$ とすると，その結果として出るプラスチック應力は只今のそれよりも遙かに小くなる．しかし之は寧ろ地殼の實際の狀態に近いものであらう．