# 40. *Three-dimensional Vibrations of a Framed Structure. I.*

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## 1. *Introduction.*

In the previous paper[1] the aseismic properties of a 'daikokubasira' (extra-thick column) in a Japanese style building was discussed mathematically, in certain parts of which it was necessary to consider the three-dimensional vibration of a framed structure. Although in the same paper the solution of a space-latticed framed structure for a special case was obtained, no general feature of the problem had yet been ascertained. In the present paper, besides discussing the general feature under consideration, we give the general solution for such a different case as concerns the three-dimensional vibration of a structure, which in itself is apparently in a two-dimensional condition. The mathematical results just mentioned were furthermore confirmed by means of model experiments, from which it was possible to show that, notwithstanding the extremely complex forms of the mathematical solutions, there was little error even in the numerical constants of the solutions.

## 2. *The solutions of the problem of a space-latticed framed structure.*

The method of dealing with the problem of a space-latticed framed structure with four columns and four horizontal members (beams) (so as to form a framed structure of the type of a parallelopiped) was shown in the previous paper.[2] When $m_0=m$, $l=l'$, $E'I'=EI=G'J'=GJ$, $E_0I_0=G_0J_0$, where $E_0I_0$, $G_0J_0$ are the bending and torsional stiffnesses of the extra thick column, $EI$, $GJ$ those of ordinary columns, and $E'I'$, $G'J'$ those of horizontal members,

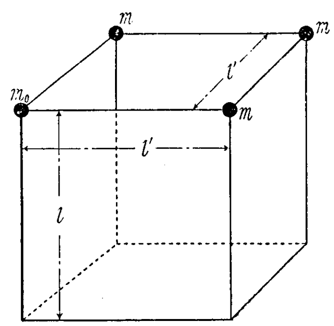


Fig. 1.

1), 2) K. KANAI, "Aseismic Properties of a Daikoku-basira (Principal Column) or Similar Column in a Japanese Style House", *Bull. Earthq. Res. Inst.*, **16** (1938), 256~272.

we get a pair of frequency equations for the free or resonance vibrations of the structure as follows:

$$\left.\begin{aligned}
&\gamma^2(5548\,\gamma_0^2+45044\,\gamma_0+43008)\\
&\quad-\gamma(8322\,\gamma_0^3+310072\,\gamma_0^2+1988902\,\gamma_0+1567744)\\
&\quad+(172353\,\gamma_0^3+3175014\,\gamma_0^2+14889369\,\gamma_0+7709184)=0,\\
&\gamma^2(5548\,\gamma_0^2+45044\,\gamma_0+43008)\\
&\quad-\gamma(8322\gamma_0^3+206104\gamma_0^2+992998\gamma_0+594016)\\
&\quad+(94377\,\gamma_0^3+1455438\,\gamma_0^2+5215377\,\gamma_0+1762488)=0,
\end{aligned}\right\}\qquad(1)$$

where $\gamma=\dfrac{mp^2l^3}{EI}$, $\gamma_0=\dfrac{E_0I_0}{EI}$.

Solving the above equations for six cases of $E_0I_0/EI$ we get[3]

| | | | | | | |
|---|---|---|---|---|---|---|
| (i) | $E_0I_0/EI=0$; | $\gamma=$ | 4·322, | 5·865, | 9·5, | 30·63, |
| (ii) | $E_0I_0/EI=1$; | $\gamma=$ | 8·4, | 8·4, | 10·846, | 33·0, |
| (iii) | $E_0I_0/EI=3$; | $\gamma=$ | 9·5, | 10·5, | 15·3, | 35·77, |
| (iv) | $E_0I_0/EI=10$; | $\gamma=$ | 10·19, | 14·03, | 27·45, | 43·8, |
| (v) | $E_0I_0/EI=20$; | $\gamma=$ | 10·63, | 16·46, | 42·97, | 56·9, |
| (vi) | $E_0I_0/EI=100$; | $\gamma=$ | 12·58, | 22·13, | 161·1, | 170·5, |

these numerical values being plotted in Fig. 2. Since the masses are concentrated at the panel points there are four vibrational frequencies I, II, III, IV.

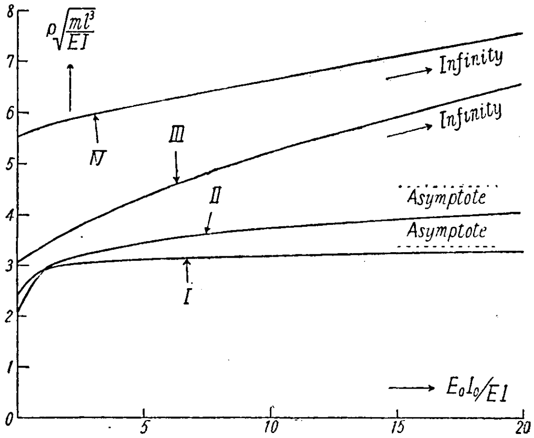


Fig. 2.

Although, because every vibrational mode is in a coupled condition, it is scarcely possible to indicate accurately the type of vibration that corresponds to each vibrational frequency, the frequency curves in Fig. 2 enable us to know that the type of vibration IV is, roughly, that of a shear of the square formed by four horizontal members (beams), while

3) These numerical values were also shown in the preceding paper, *loc. cit.* 1).

the type of vibration III is that of rotation, about a vertical axis of the same square. Vibrations II and I are somewhat complex. At a smaller value of $E_0I_0/EI$, the type of vibration II is that of translation of the square just mentioned in the direction of the diagonal line between the panel point at the extra thick column and the opposite panel point, whereas the type of vibration I is that of translation in the direction of another diagonal of the same square. At a larger value of $E_0I_0/EI$, from the nature of things, the type of vibration I is that of rotation, whereas the type of vibration II is that of shear.

Notwithstanding that at a larger value of $E_0I_0/EI$, vibrations I and III are both of rotational type, and vibrations II and IV are both of shear type, the vibrational frequencies for III and IV increase with increase in the value of $E_0I_0/EI$, whereas the vibrational frequencies for I and II remain approximately constant, even with increase in $E_0I_0/EI$. Interpreted physically, although in vibrations I and II the movements of the extra thick column are much smaller than those of the remaining columns, in vibrations III and IV the movements of the extra thick column are comparable to those of the remaining columns. It should also be borne in mind that in the condition $E_0I_0/EI=1$, the vibrational frequency of case I is the same as that of case II,—an obvious fact from the nature of the problem.

### 3. *Model experiment confirming the preceding mathematical results.*

With a view to confirming the preceding mathematical results, we conducted simple model experiments. The form of the model was quite similar to that in the sketch in Fig. 1. The column height and beam span are both 40 cm. The members corresponding to the columns and beams were made of steel wire and the concentrated masses were at the panel points, of brass blocks, the mass of every block being 13 gr. Whereas the diameters of the ordinary columns and that of every beam was 1·5 mm, that of the extra thick column differed, namely, (i) 1 mm, (ii) 1·5 mm, (iii) 2 mm, (iv) 2·5 mm, and (v) 3 mm. The masses of the respective wires were (i) 2·63 gr, (ii) 5·58 gr, (iii) 10·01 gr, (iv) 15·23 gr, and (v) 22·08 gr. In the numerical calculation we assumed that the sum of the brass mass and 3/2 the mass of the 1·5 mm wire is effective as inertia mass at every panel point. Whence, in the case of a larger value of $E_0I_0/EI$, the calculated vibrational frequency is liable to be somewhat higher than the observed frequency.

For comparing the calculated and the observed values, we made two preliminary experiments, such that a single wire with an end mass oscillated freely in the sense of torsion on the one hand and in the sense of bending on the other, from which we found the values of the elastic constants of every wire, their values being $E=19{\cdot}48.10^{11}$ dyne/cm$^2$, $G=7{\cdot}84.10^{11}$ dyne/cm$^2$.

The model just mentioned was clamped to the vibration table that was used for experiments in connexion with the vibration damper. The direction of the diagonal was set in various ways. By observing the resonance conditions of the model on the vibration table, we found four vibrational periods in every case, the results being shown with thick lines in Fig. 3.

On the other hand, from the numerical data in the preliminary experiments and the relations in Fig. 2, we calculated the theoretical vibrational periods and plotted the results with broken lines in Fig. 3. It will be seen that the mathematical results fairly agree with the experimental results. Notwithstanding that, as a matter of fact, there is the theoretical relation $E=2(1+\sigma)G$, we calculated a special case in which $EI=GJ$, that is to say, $J=2I$, $E=2G$, from which it appears that the torsional rigidity in every calculation is likely to be overestimated. Furthermore, as already mentioned, at higher values of $E_0I_0/EI$, the $m$ values in the calculation are underestimated. For these two reasons, the calculated vibrational periods are invariably less than the observed vibrational periods.[4]

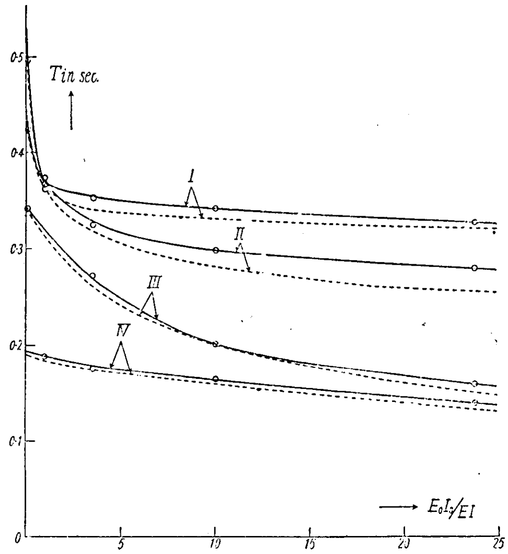

Fig. 3.

From these considerations, there does not seem to be any error in the mathematical result in the preceding section.

## 4. *Solution of the problem of a plane framed structure oscillating in the direction normal to the plane of the same structure.*

4) It appears now evident that the smaller vibrational periods in mathematical side did not result from such dynamical condition that approximate calculation shall always give vibrational periods that are somewhat less than the actual periods.

When the framed structure shown in Fig. 4 oscillates in the direction normal to the plane of that structure, the vibration becomes a three-dimensional one for certain conditions of the columns and beams.

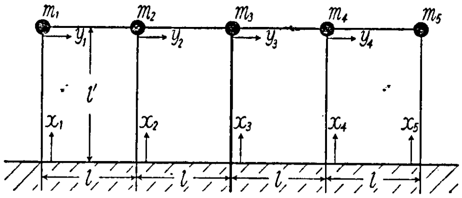

Fig. 4.

Let the bending and torsional stiffnesses of the respective columns be $E_1I_1$, $E_2I_2$, ............ , $E_5I_5$; $G_1J_1$, $G_2J_2$, ......... , $G_5J_5$, and those of the respective beams be $E_1'I_1'$, $E_2'I_2'$, $E_3'I_3'$, $E_4'I_4'$; $G_1'J_1'$, $G_2'J_2'$, $G_3'J_3'$, $G_4'J_4'$. Let also the inertia masses at the panel points corresponding to the tops of the respective columns be $m_1$, $m_2$, ......... , $m_5$. The heights of the columns and the lengths of every span are shown in the sketch.

The expressions for the vibratory motion of the columns and the beams in bending as well as in torsion are of the types

$$\left.\begin{array}{lll}\dfrac{\partial^4 u_s}{\partial x_s^4}=0, & \dfrac{\partial^2\theta_s}{\partial x_s^2}=0, & [s=1,\ 2,\ \ldots\ldots,\ 5] \\[2ex] \dfrac{\partial^4 w_s}{\partial y_s^4}=0, & \dfrac{\partial^2\varphi_s}{\partial y_s^2}=0, & [s=1,\ 2,\ \ldots\ldots,\ 4]\end{array}\right\} \tag{2}$$

where $u_s$, $w_s$ are displacements both in the direction normal to the plane of the structure.

The solutions of the corresponding equations have the forms

$$\left.\begin{array}{r}u_s=(A_s+B_sx_s+C_sx_s^2+D_sx_s^3)\,e^{ipt}, \\ \theta_s=(\alpha_s+\beta_sx_s)\,e^{ipt},\end{array}\right\}\ [s=1,\ 2,\ \ldots\ldots,\ 5] \tag{3}$$

$$\left.\begin{array}{r}w_s=(P_s+Q_sy_s+R_sy_s^2+S_sy_s^3)\,e^{ipt}, \\ \varphi_s=(\lambda_s+\mu_sy_s)\,e^{ipt}.\end{array}\right\}\ [s=1,\ 2,\ \ldots\ldots,\ 4] \tag{4}$$

The boundary conditions are such that

$$x_s=0;\quad u_s=0,\quad \frac{\partial u_s}{\partial x_s}=0,\quad \theta_s=0,\quad [s=1,\ 2,\ \ldots\ldots,\ 5] \qquad (5),\ (6),\ (7)$$

$$x_1=l,\ y_1=0;\qquad u_1=w_1,\quad \frac{\partial u_1}{\partial x_1}=\varphi_1,\quad \frac{\partial w_1}{\partial y_1}=-\theta_1, \qquad (8),\ (9),\ (10)$$

$$-E_1I_1\frac{\partial^2 u_1}{\partial x_1^2}+G_1'J_1'\frac{\partial\varphi_1}{\partial y_1}=0,\quad E_1'I_1'\frac{\partial^2 w_1}{\partial y_1^2}+G_1J_1\frac{\partial\theta_1}{\partial x_1}=0, \qquad (11),\ (12)$$

$$-E_1I_1\frac{\partial^3 u_1}{\partial x_1^3}+E_1'I_1'\frac{\partial^3 w_1}{\partial y_1^3}=m_1p^2u_1, \tag{13}$$

$x_{s+1}=l,\ y_s=l',\ y_{s+1}=0;$

$$u_{s+1}=w_s=w_{s+1},\ \frac{\partial u_{s+1}}{\partial x_{s+1}}=\varphi_s=\varphi_{s+1},\ \frac{\partial w_s}{\partial y_s}=\frac{\partial w_{s+1}}{\partial y_{s+1}}=-\theta_{s+1}, \quad (14),(15),(16)$$

$$-E_{s+1}I_{s+1}\frac{\partial^2 u_{s+1}}{\partial x_{s+1}^2}-G'_sJ'_s\frac{\partial \varphi_s}{\partial y_s}+G'_{s+1}J'_{s+1}\frac{\partial \varphi_{s+1}}{\partial y_{s+1}}=0, \quad (17)$$

$$-E'_sI'_s\frac{\partial^2 w_s}{\partial y_s^2}+E'_{s+1}I'_{s+1}\frac{\partial^2 w_{s+1}}{\partial y_{s+1}^2}+G_{s+1}J_{s+1}\frac{\partial \theta_{s+1}}{\partial x_{s+1}}=0, \quad (18)$$

$$-E_{s+1}I_{s+1}\frac{\partial^3 u_{s+1}}{\partial x_{s+1}^3}-E'_sI'_s\frac{\partial^3 w_s}{\partial y_s^3}+E'_{s+1}I'_{s+1}\frac{\partial^3 w_{s+1}}{\partial y_{s+1}^3}=m_{s+1}p^2u_{s+1}, \quad (19)$$

$[s=1, 2, 3]$

$$x_5=l,\ y_4=l';\quad u_5=w_4,\quad \frac{\partial u_5}{\partial x_5}=\varphi_4,\quad \frac{\partial w_4}{\partial y_4}=-\theta_5, \quad (20),(21),(22)$$

$$-E_5I_5\frac{\partial^2 u_5}{\partial x_5^2}-G'_4J'_4\frac{\partial \varphi_4}{\partial y_4}=0,\quad -E'_4I'_4\frac{\partial^2 w_4}{\partial y_4^2}+G_5J_5\frac{\partial \theta_5}{\partial x_5}=0, \quad (23),(24)$$

$$-E_5I_5\frac{\partial^3 u_5}{\partial x_5^3}-E'_4I'_4\frac{\partial^3 w_4}{\partial y_4^3}=m_5p^2u_5. \quad (25)$$

Substituting (3), (4), in (5)$\sim$(25) and eliminating constants, it is possible to get the frequency equations. We now write

$$\left.\begin{array}{l}\dfrac{E_sI_s}{E_{s+1}I_{s+1}}=\varphi_s,\ \dfrac{E'_sI'_s}{E_sI_s}=\eta_s,\ \dfrac{G'_sJ'_s}{E_sI_s}=\vartheta_s,\ [s=1,2,3,4] \\ \dfrac{G_sJ_s}{E_sI_s}=\zeta_s,\ \dfrac{m_sp^2l^3}{E_sI_s}=\gamma_s,\ [s=1,\ 2,\ \ldots\ldots,\ 5]\ \ \dfrac{l'}{l}=\xi.\end{array}\right\} \quad (26)$$

After careful calculations we obtain the set of equations

$$2(\vartheta_1+\xi)C_1+3(\vartheta_1+2\xi)lD_1-2\vartheta_1C_2-3\vartheta_1lD_2=0, \quad (27)$$

$$\begin{aligned}&\xi^3\left\{(4\eta_1+\xi\zeta_1)(2\eta_2+\xi\zeta_2)+8\varphi_1\eta_1(2\eta_1+\xi\zeta_1)-2\xi\varphi_1\zeta_1\eta_1\right\}\left\{\gamma_1C_1+(\gamma_1+6)lD_1\right\}\\&-12\eta_1\left\{(\eta_1+\xi\zeta_1)(2\eta_2+\xi\zeta_2)+\xi\varphi_1\zeta_1\eta_1\right\}(C_1+lD_1)\\&+2\xi^3\eta_1(2\eta_1+\xi\zeta_1)\left\{\gamma_2C_2+(\gamma_2+6)lD_2\right\}\\&+12\eta_1\left\{\eta_2(2\eta_1+\xi\zeta_1)+(\eta_1+\xi\zeta_1)(2\eta_2+\xi\zeta_2)+\xi\varphi_1\zeta_1\eta_1\right\}(C_2+lD_2)\\&\qquad-12\eta_1\eta_2(2\eta_1+\xi\zeta_1)(C_3+lD_3)=0,\end{aligned} \quad (28)$$

$$2\xi\varphi_1C_1+6\xi\varphi_1lD_1+2(\vartheta_2+\xi)C_2+3(\vartheta_2+2\xi)lD_2-2\vartheta_2C_3-3\vartheta_2lD_3=0, \quad (29)$$

$$\xi^3\Big[(2\eta_3+\xi\zeta_3)\Big\{(4\eta_1+\xi\zeta_1)(\eta_2+\xi\zeta_2)+9\varphi_1\eta_1(2\eta_1+\xi\zeta_1)-2\xi\zeta_1\varphi_1\eta_1\Big\}$$

$$+\varphi_2\eta_2\Big\{14\varphi_1\eta_1(2\eta_1+\xi\zeta_1)+\xi\zeta_2(4\eta_1+\xi\zeta_1)-2\xi\zeta_1\varphi_1\eta_1\Big\}\Big]\Big\{\gamma_1C_1+(\gamma_1+6)lD_1\Big\}$$

$$+12\eta_1\Big[(2\eta_3+\xi\zeta_3)\Big\{(\eta_1+\xi\zeta_1)(\eta_2+\xi\zeta_2)+\xi\zeta_1\varphi_1\eta_1\Big\}$$

$$+\xi\varphi_2\eta_2\Big\{\zeta_1\varphi_1\eta_1+\zeta_2(\eta_1+\xi\zeta_1)\Big\}\Big]\Big\{(C_2+lD_2)-(C_1+lD_1)\Big\}$$

$$+\xi^3\eta_1(2\eta_1+\xi\zeta_1)\Big\{3(2\eta_3+\xi\zeta_3)+8\varphi_2\eta_2\Big\}\Big\{\gamma_2C_2+(\gamma_2+6)lD_2\Big\}$$

$$+2\xi^3\eta_1\eta_2(2\eta_1+\xi\zeta_1)\Big\{\gamma_3C_3+(\gamma_3+6)lD_3\Big\}$$

$$+12\eta_1\eta_2\eta_3(2\eta_1+\xi\zeta_1)\Big\{(C_3+lD_3)-(C_4+lD_4)\Big\}=0, \quad (30)$$

$$2\xi\varphi_1\varphi_2(C_1+3lD_1)+2\xi\varphi_2(C_2+3lD_2)+2(\vartheta_3+\xi)C_3$$

$$+3(\vartheta_3+2\xi)lD_3-2\vartheta_3C_4-3\vartheta_3lD_4=0, \quad (31)$$

$$\xi^3\Big\{(\eta_2X_1+\xi\zeta_2X_2)(4\eta_1+\xi\zeta_1)+\varphi_1\eta_1(6X_2+X_3)(2\eta_1+\xi\zeta_1)$$

$$-2\xi\zeta_1\varphi_1\eta_1X_2\Big\}\Big\{\gamma_1C_1+(\gamma_1+6)lD_1\Big\}+12\eta_1\Big\{(\eta_2X_1+\xi\zeta_2X_2)(\eta_1+\xi\zeta_1)$$

$$+\xi\zeta_1\varphi_1\eta_1X_2\Big\}\Big\{(C_2+lD_2)-(C_1+lD_1)\Big\}$$

$$+\eta_1(2\eta_1+\xi\zeta_1)\Big[\xi^3X_3\Big\{\gamma_2C_2+(\gamma_2+6)lD_2\Big\}+2\xi^3\eta_2\eta_3\Big\{\gamma_4C_4+(\gamma_4+6)lD_4\Big\}$$

$$+\xi^3\eta_2\Big\{3(2\eta_4+\xi\zeta_4)+8\varphi_3\eta_3\Big\}\Big\{\gamma_3C_3+(\gamma_3+6)lD_3\Big\}$$

$$+12\eta_2\eta_3\eta_4(C_4+lD_4)-12\eta_2\eta_3\eta_4(C_5+lD_5)\Big]=0, \quad (32)$$

$$3\xi^2(2\eta_1+\xi\zeta_1)\Big\{\xi\eta_2Y_1+(2\varphi_1\eta_1+\xi\zeta_2)Y_2+\varphi_1\eta_1Y_3\Big\}\Big\{\gamma_1C_1+(\gamma_1+6)lD_1\Big\}$$

$$-2\Big[\eta_1(\eta_2Y_1+\zeta_2Y_2)+\zeta_1\Big\{\xi\eta_2Y_1+(\varphi_1\eta_1+\xi\zeta_2)Y_2\Big\}\Big]$$

$$\cdot\Big[\xi^3\Big\{\gamma_1C_1+(\gamma_1+6)lD_1\Big\}+6\eta_1(C_1+lD_1)-6\eta_1(C_2+lD_2)\Big]$$

$$+3\xi\eta_1(2\eta_1+\xi\zeta_1)\Big[\xi\eta_2\eta_3(2\varphi_4\eta_4+\xi\zeta_5)\Big\{\gamma_4C_4+(\gamma_4+6)lD_4\Big\}$$

$$+\xi\eta_2\Big\{\xi\zeta_5\eta_4+(\varphi_4\eta_4+\xi\zeta_5)(2\varphi_3\eta_3+\xi\zeta_4)+\varphi_3\eta_3(2\varphi_4\eta_4+\xi\zeta_5)\Big\}$$

$$\cdot\Big\{\gamma_3C_3+(\gamma_3+6)lD_3\Big\}+\xi Y_3\Big\{\gamma_2C_2+(\gamma_2+6)lD_2\Big\}\Big]=0, \quad (33)$$

$$2\xi\varphi_1\varphi_2\varphi_3(C_1+3lD_1)+2\xi\varphi_2\varphi_3(C_2+3lD_2)+2\xi\varphi_3(C_3+3lD_3)$$
$$+2(\vartheta_4+\xi)C_4+3(\vartheta_4+2\xi)lD_4-\vartheta_4(2C_5+3lD_5)=0\,, \quad (34)$$

$$\varphi_1\varphi_2\varphi_3\varphi_4(C_1+3lD_1)+\varphi_2\varphi_3\varphi_4(C_2+3lD_2)+\varphi_3\varphi_4(C_3+3lD_3)$$
$$+\varphi_4(C_4+3lD_4)+(C_5+3lD_5)=0\,, \quad (35)$$

$$\varphi_1\varphi_2\varphi_3\varphi_4\Big\{\gamma_1C_1+(\gamma_1+6)lD_1\Big\}+\varphi_2\varphi_3\varphi_4\Big\{\gamma_2C_2+(\gamma_2+6)lD_2\Big\}$$
$$+\varphi_3\varphi_4\Big\{\gamma_3C_3+(\gamma_3+6)lD_3\Big\}+\varphi_4\Big\{\gamma_4C_4+(\gamma_4+6)lD_4\Big\}$$
$$+\Big\{\gamma_5C_5+(\gamma_5+6)lD_5\Big\}=0\,, \quad (36)$$

where

$$X_1=(2\eta_4+\xi\zeta_4)(\eta_3+\xi\zeta_3)+\xi\zeta_3\varphi_3\eta_3\,,$$
$$X_2=\eta_3(2\eta_4+\xi\zeta_4)+(\varphi_2\eta_2+\xi\zeta_3)(2\eta_4+\xi\zeta_4+\varphi_3\eta_3),$$
$$X_3=3(2\eta_4+\xi\zeta_4)(\eta_3+3\varphi_2\eta_2+\xi\zeta_3)+\varphi_3\eta_3(3\xi\zeta_3+14\varphi_2\eta_2),$$
$$Y_1=\eta_3\Big\{\zeta_5\eta_4+\zeta_4(\varphi_4\eta_4+\xi\zeta_5)\Big\}+\zeta_3\Big\{\xi\zeta_5\eta_4+(\varphi_4\eta_4+\xi\zeta_5)(\varphi_3\eta_3+\xi\zeta_4)\Big\}\,,$$
$$Y_2=\xi\eta_3\Big\{\zeta_5\eta_4+\zeta_4(\varphi_4\eta_4+\xi\zeta_5)\Big\}$$
$$+(\varphi_2\eta_2+\xi\zeta_3)\Big\{\xi\zeta_5\eta_4+(\varphi_4\eta_4+\xi\zeta_5)(\varphi_3\eta_3+\xi\zeta_4)\Big\}\,,$$
$$Y_3=\xi\eta_3\Big\{\zeta_5\eta_4+\zeta_4(\varphi_4\eta_4+\xi\zeta_5)\Big\}$$
$$+(2\varphi_2\eta_2+\xi\zeta_3)\Big\{\xi\zeta_5\eta_4+(\varphi_4\eta_4+\xi\zeta_5)(\varphi_3\eta_3+\xi\zeta_4)\Big\}$$
$$+\varphi_2\eta_2\Big\{\xi\zeta_5\eta_4+(\varphi_4\eta_4+\xi\zeta_5)(2\varphi_3\eta_3+\xi\zeta_4)+\varphi_3\eta_3(2\varphi_4\eta_4+\xi\zeta_5)\Big\}\,.$$
$$(37)$$

We shall take the special case $l'=l$, $m_s=m$ [$s=1, 2, \ldots\ldots, 5$], $E_3I_3=G_3J_3=E_0I_0$, $E_sI_s=G_sJ_s=EI$ [$s=1, 2, 4, 5$], $E'_sI'_s=G'_sJ'_s=EI$ [$s=1, 2, 3, 4$], that is to say, the stiffnesses of the middle column differ from those of the remaining columns and the beams. In this case we get a pair of frequency equations of such simple forms as

$$(3\gamma-16)(a_2c_1-a_1c_2)+(4\gamma-12)(a_1b_2-b_1a_2)+(2\gamma+96)(b_1c_2-c_1b_2)=0\,,$$
$$(715\gamma-4392)\Big\{(517\kappa+1736)\gamma-(2760\kappa+8508)\Big\}$$
$$-(536\gamma-3036)\Big\{(683\kappa+2269)\gamma-(3852\kappa+11736)\Big\}=0\,,$$
$$(38)$$

where

$$\left.\begin{aligned} &a_1=(4\kappa+35)\gamma-(84\kappa+336), && a_2=(12\kappa+39)\gamma-(90\kappa+360),\\ &b_1=(6\kappa+32)\gamma+(76\kappa+32), && b_2=(18\kappa+38)\gamma+(348\kappa+108),\\ &c_1=-(21\kappa+35)\gamma+(72\kappa+96), && c_2=-(63\kappa+51)\gamma+(198\kappa+144),\\ &\kappa=\frac{E_0I_0}{EI}, && \gamma=\frac{mp^2l^3}{EI}. \end{aligned}\right\} \quad (39)$$

Solving these equations for the five cases of $E_0I_0/EI$, we get

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| (i) | $E_0I_0/EI=0$; | $\gamma=2{\cdot}186$, | 4·071, | 6·335, | 16·72, | 38·703, |
| (ii) | $E_0I_0/EI=1$; | $\gamma=3{\cdot}00$, | 4·21, | 8·09, | 17·95, | 39·03, |
| (iii) | $E_0I_0/EI=10$; | $\gamma=4{\cdot}444$, | 4·596, | 16·523, | 21·53, | 57·45, |
| (iv) | $E_0I_0/EI=30$; | $\gamma=4{\cdot}692$, | 4·727, | 21·675, | 23·31, | 112·2, |
| (v) | $E_0I_0/EI=\infty$; | $\gamma=4{\cdot}823$, | 4·823, | 24·84, | 24·84, | $\infty$, |

these numerical values being plotted in Fig. 5. Since the masses are concentrated at five panel points, there are five vibrational frequencies, namely, I, II, III, IV, V.

In this case, too, since every vibrational frequency is in a coupled condition, it is scarcely possible to determine the accurate type of vibration that corresponds to each vibrational frequency. In this case, however, to estimate the respective types of vibrations is a simple matter. Since the masses are arranged in a straight line and the movements are only in the direction normal to the said straight line, the vibrational modes would be somewhat similar to those of flexural vibrations of a bar with free ends. Consequently the modes of the vibrations I, II, III, IV, V in the present case are similar to those of no-nodal, one-nodal, two-nodal, three-nodal, and quasi-four-nodal[5] vibrations in the case of the flexural

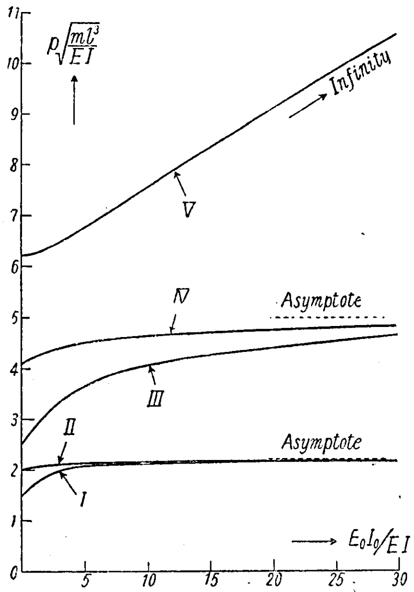

Fig. 5.

5) Case V is of somewhat different nature. In the case of five masses four nodal vibration is scarcely possible. In this case the middle thick column alone participates in the vibrational energy.

vibrations of a bar. When the vibration is observed from the top, namely, in the lengthwise direction of the columns, modes I, III, V represent the vibrations that are symmetrical with respect to the middle column, while modes II, IV correspond to the vibrations that are asymmetrical with respect to the middle column.

With increase in $E_0I_0/EI$, the vibrational frequency for I and that for II tend to approach a common asymptotic line of no inclination and the vibrational frequency for III and that for IV also tend to approach another common asymptotic line of no inclination. This fact shows that at such a high value of $E_0I_0$, every set of ordinary columns on any one side of the middle thick column vibrates with little effect on the columns on the other side of the middle thick column under consideration. The reason that the asymptotes of these cases have no inclination is that, at such a high value of $E_0I_0$, the middle thick column merely serves as the fixed edge of a flexible framed structure. The condition in the case of V is somewhat special: the vibration is not a simple four-nodal vibration, but in this condition the vibrational amplitudes of the middle thick column is comparable to the remaining one, that is to say, the principal vibration is that of the middle thick column.

It must be remembered that the vibrational frequency of mode I at $E_0I_0/EI=1$ is nearly $p\sqrt{ml^3/EI}=\sqrt{3}$, which corresponds to the vibrational frequency of a canti-lever beam with a concentrated mass at its free end.

## 5. *Model experimental confirmation of the preceding mathematical result.*

In this case, too, for confirming the mathematical result, model experiments were conducted. The model tested was similar in form to that shown in the sketch in Fig. 4. The column height and beam span were both 23 cm. The brass mass at every panel point weighed 13 gr. The usual columns and the beams were 1·5 mm in diameter, whereas the extra thick middle column was (i) 1 mm, (ii) 1·5 mm, (iii) 2 mm, (iv) 2·5 mm, and (v) 3 mm, the masses of the respective columns being (i) 2·00 gr, (ii) 3·84 gr, (iii) 6·25 gr, (iv) 9·39 gr, and (v) 13·40 gr. In the numerical calculation we assumed that the sum of the brass mass and $(1/2+4/5)$ of the mass of the 1·5 mm wire, is effective as an inertia mass at every panel point. The inertia mass in this case, too, is likely to be underestimated. The values of $E$ and $G$ are the same as those shown in Section 3. For the same reason shown in Section

3, the value of $G$ in the numerical calculation is an overestimate.

The model just mentioned was clamped to the vibration table with the lengthwise direction of the model at right angles to the direction of the vibratory motion of the table. By observing the resonance conditions of the forced vibrations, we found five vibrational periods in every case, the results being plotted with thick lines in Fig. 6.

At the same time, we calculated the theoretical vibrational periods, and plotted the results with broken lines in the same figure. It will be seen that both results, mathematical and experimental, fairly agree, not exactly. The reason for the calculated periods being somewhat smaller than the experimented ones is explained in Section 3.[6]

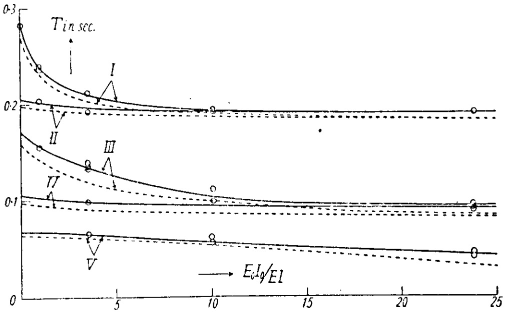

Fig. 6.

## 6. *Concluding remarks.*

From mathematical calculation and model experiments we ascertained certain cases of three-dimensional vibrations of a framed structure. Even should the framed structure be apparently of two-dimentional form, the vibration is liable to be three-dimensional according to certain conditions of the structure as well as the direction of the movements of the seismic vibrations. When a horizontally long house is subjected to seismic disturbance with amplitudes orientated in the direction normal to the length of the house, there would probably be bending vibrations of various modes of the house, even should there be no asymmetrical relations between the structural conditions and the seismic movements. The occurrence of asymmetrical vibrations in the case of a three-dimensional framed structure is too evident to call for remarks here.

The detailed nature of the vibrations for different cases has been dealt with in their respective parts in the paper.

The present investigation was made at Professor Sezawa's suggestion in connection with his research work as member of the Investiga-

6) As explained in Section 3, the condition never resulted from such error that might be involved in numerical calculation.

tion Commitee for Earthquake-proof Construction, of the Japan Society of the Promotion of Scientific Research, to whom I wish to express my sincerest thank for his many kind advices.

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**Note.** Notwithstanding that the problem of a three-dimensional vibration is very important in earthquake-proof construction, the problem has received rather scant attention probably owing to mathematical difficulties. Dr. Kanai, in my opinion, being the person most fitted for attacking the problem, I requested him to make the investigation. This problem, in future, will be extended, I hope, to the problem of decay of the vibrational energy in the seismic vibration of a structure and also to the dynamic damper of the seismic vibration of the same structure.

Finally, I wish to express my warmest thanks to the Council of the Japan Society for the Promotion of Scientific Research for aid given to a series of investigations of which this is a part. (K. Sezawa)

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## 40. 架構の立體的振動

地震研究所 金 井 清

この前論じた大黒柱の問題に關聯して立體的架構又は平面的架構の立體的振動を研究して見たのである.

立體的架構の最も簡單なる例として四本柱の一本の柱の剛度が種々變化する場合を取ると質量が4點に分布する場合には4種の振動數が出るが，その型は角點を結ぶ2つの斜線の方向の移動的振動，梁線で作られる正方形が菱形になる振動，鉛直軸の周圍の囘轉的振動とになる．しかし1本の柱の剛度が著しく高くなると，始めの2つの振動も後の2つの各の型の振動に變化する．但し，型は同じでも周期は著しく高いものが現れる．以上の數理的結果を振動臺の上で共振れする模型の實驗で確めると可なりよく一致する性質を示す．即ち困難なる計算結果が先づ正確な事が證明されたのである．

平面的架構が架構面に直角な方向に振動するときには，對稱的構造であつてもその振動が立體的となる．之も數理的と實驗的とに研究して見たがよく一致する事がわかつた．架構を上から見ると兩端自由なる棒の振動と似てをる．しかし（中央にある）1本の柱の剛度が高くなると互に別の型に屬する一對宛の振動型の振動周期が近づくやうになる．尙，特に剛い柱特有の自己振動數に相當する振動のあることもわかつた．

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