19. *Anomalous Dispersion of Elastic Surface Waves.*

By Katsutada SEZAWA,

Earthquake Research Institute.

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1. *Introduction.*

The propagational velocity of elastic surface waves, whether Rayleigh-waves or Love-waves, that are transmitted on a stratified surface, generally increases with increase in wave-length, its nature therefore being similar to that of normal dispersion in the case of problems in optics.[1] The condition implied in the case of elastic surface waves is however such that the velocity of bodily waves in the surface layer is invariably less than that in the subjacent medium. If, on the other hand, the condition in question were reversed, propagation of surface waves would probably be impossible in the majority of cases.

The present investigation shows that even should the velocity of bodily waves in the surface layer be higher than that in the subjacent medium (density of the surface layer being less than that of the subjacent layer), it is possible for Rayleigh-type waves to be transmitted for a certain range of wave-length. Since moreover velocity of propagation of the surface waves now under consideration diminishes with increase in wave-length, the nature of dispersion is quite anomalous. Although the idea of the probable existence of such waves occurred to me about ten years ago when I was studying the dispersion of Rayleigh-waves, having even gone as far as to present a few results of the investigation at one of the meetings of the Institute, yet owing to there being certain uncertainties about my numerical calculations, I refrained from publishing any part of it. Now that with the kind aid of Dr. Kanai, the investigation has progressed favourably, I shall deal with the results in this paper.

2. *Love-waves.*

Let the densities and the rigidities of the surface layer and the subjacent medium be ρ', μ'; ρ, μ respectively, the thickness of the layer being H. In order that the Love-type movements shall be surface waves,

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1) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **3** (1927), 1.

the displacements in the surface layer and the subjacent medium must assume the forms

$$v' = (Ae^{s'y} + Be^{-s'y})\cos(pt + fx), \tag{1}$$

$$v = Ce^{sy}\cos(pt + fx), \tag{2}$$

where

$$s'^2 = f^2 - \kappa'^2, \quad \kappa'^2 = \rho' p^2/\mu',$$
$$s^2 = f^2 - \kappa^2, \quad \kappa^2 = \rho p^2/\mu. \tag{3}$$

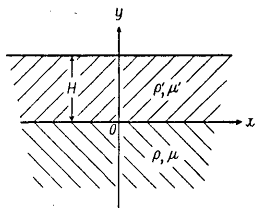

Fig. 1.

Substituting (1), (2) in the boundary conditions

$$y = 0; \qquad v' = v, \qquad \mu'\frac{\partial v'}{\partial y} = \mu\frac{\partial v}{\partial y}, \tag{4}$$

$$y = H; \qquad \mu'\frac{\partial v'}{\partial y} = 0, \tag{5}$$

we get the relation

$$\tanh s'H = -\frac{\mu s}{\mu' s'}, \tag{6}$$

from which it is possible to find the velocity of propagation of the Love-waves.

The problem will now be confined to the case in which, from the condition of the present investigation, $\rho'/\mu' < \rho/\mu$, whence $\kappa'^2 < \kappa^2$, so that from (3), $s'^2 > s^2$. Since s is real, s' is also real: s can never be negative. The relation (6) therefore does not hold. It follows then that were neither s nor s' a complex quantity, Love-waves cannot exist in the layer in the condition assumed.

We shall next suppose the case in which either s or s' is a complex quantity. Writing, generally, $s = a + ib$, $s' = a' + ib'$, we get from (6) the relations

$$\left.\begin{aligned} \frac{\sinh 2a'}{\cosh 2a' + \cos 2b'} &= \frac{\mu(aa' + bb')}{\mu'(a'^2 + b'^2)}, \\ \frac{\sinh 2b'}{\cosh 2a + \cos 2b'} &= \frac{\mu(ab' - a'b)}{\mu'(a'^2 + b'^2)}. \end{aligned}\right\} \tag{7}$$

Writing, on the other hand, $\kappa = \kappa_1 + i\kappa_2$, $\kappa' = \kappa_1' + i\kappa_2'$, and comparing with (3), we find that

$$\left.\begin{aligned} a^2 - b^2 &= f^2 - \kappa_1^2 + \kappa_2^2, \quad & ab &= -\kappa_1\kappa_2, \\ a'^2 - b'^2 &= f^2 - \kappa_1'^2 + \kappa_2'^2, \quad & a'b' &= -\kappa_1'\kappa_2', \end{aligned}\right\} \tag{8}$$

from which the values of a, b, a', b' are determined. This shows that for stationary waves to exist, either a or b (and similarly either a' or b') vanishes. The complex values of s and s' cannot possibly exist.

Generally speaking, were the velocity of propagation of transverse waves in the surface layer higher than that in the subjacent medium, Love-waves do not exist. The nature of the dispersion of the possible Love-waves is thus normal as usually considered. It appears that even in the case of a doubly stratified layer[2), 3)], the possible Love-waves exhibit, as a matter of fact, the nature of normal dispersion.

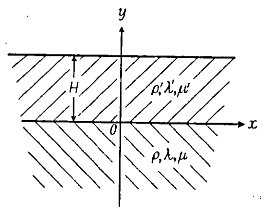

Fig. 2.

3. *Rayleigh-waves.*

Let the densities and the elastic constants of the layer and the subjacent medium be ρ', λ', μ'; ρ, λ, μ (Fig. 2) respectively. The equation necessary for obtaining the velocity of transmission is expressed by a determinant of the form

$$\begin{vmatrix} -\frac{(f^2-s'^2)}{k'^2}Y_2, & 2\frac{\mu' ifs'}{\mu k'^2}Y_1, & 0, & -2\frac{\mu' ifs'}{\mu k'^2}, & 0, & \frac{s'f}{k'^2} \\ -\frac{(f^2-s'^2)}{k'^2}Y_1, & -2\frac{\mu' ifs'}{\mu k'^2}Y_2, & \frac{\mu'(f^2-s'^2)}{\mu k'^2}, & 0, & \frac{if^2}{k'^2}, & 0 \\ 2\frac{ifr'}{h'^2}X_1, & \left(2\frac{\mu' r'^2}{\mu h'^2}-\frac{\lambda'}{\mu}\right)X_2, & -\frac{\mu' 2ifr'}{\mu h'^2}, & 0, & \frac{r'f}{h'^2}, & 0 \\ 2\frac{ifr'}{h^2}X_2, & \left(2\frac{\mu' r'^2}{\mu h'^2}-\frac{\lambda'}{\mu}\right)X_1, & 0, & \left(\frac{\lambda'}{\mu}-2\frac{\mu' r'^2}{\mu h'^2}\right), & 0, & -\frac{if^2}{h'^2} \\ 0, & 0, & -\frac{(f^2+s^2)}{k^2}, & 2\frac{ifs}{k^2}, & -\frac{if^2}{k^2}, & -\frac{sf}{k^2} \\ 0, & 0, & -\frac{2ifr}{h^2}, & \left(2\frac{r^2}{h^2}-\frac{\lambda}{\mu}\right), & -\frac{rf}{h^2}, & \frac{if^2}{h^2} \end{vmatrix} = 0, \quad (9)$$

where $X_1 = \cosh r'H$, $X_2 = \sinh r'H$, $Y_1 = \cos s'H$, $Y_2 = \sin s'H$. This reduces to[4)]

2) T. MATUZAWA, *Proc. Phys.-Math. Soc., Japan*, [iii], **10** (1928), 25~33.
3) R. STONELEY and E. TILLOTSON, *M.N.R.A.S. Geophys. Suppl.*, **1** (1928).
4) K. SEZAWA and K. KANAI, *Bull. Earthq. Res. Inst.*, **13** (1935), 238.

$$\frac{4r's'}{f^2}\left(2-\frac{k'^2}{f^2}\right)\eta-\frac{r's'}{f^2}\left\{4\vartheta+\left(2-\frac{k'^2}{f^2}\right)^2\zeta\right\}\cosh r'H\cos s'H$$

$$+\frac{r'}{f}\varphi\left\{\frac{4rs'^2}{f^3}+\frac{s}{f}\left(2-\frac{k'^2}{f^2}\right)^2\right\}\cosh r'H\sin s'H$$

$$+\frac{s'}{f}\varphi\left\{-\frac{4sr'^2}{f^3}+\frac{r}{f}\left(2-\frac{k'^2}{f^2}\right)^2\right\}\sinh r'H\cos s'H$$

$$+\left\{-\frac{4r'^2s'^2}{f^4}\zeta+\left(2-\frac{k'^2}{f^2}\right)^2\vartheta\right\}\sinh r'H\sin s'H=0, \qquad (10)$$

where

$$\left.\begin{aligned}
&\varphi=\frac{\mu'k^2k'^2}{\mu f^4},\quad \zeta=\frac{4rs}{f^2}\left(\frac{\mu'}{\mu}-1\right)^2-\alpha^2,\quad \eta=\frac{2rs}{f^2}\left(\frac{\mu'}{\mu}-1\right)\beta-\alpha\gamma,\\
&\vartheta=\frac{rs}{f^2}\beta^2-\gamma^2,\quad \alpha=\frac{2\mu'}{\mu}-\left(2-\frac{k^2}{f^2}\right),\quad \beta=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-2,\\
&\gamma=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-\left(2-\frac{k^2}{f^2}\right),\\
&r^2=f^2-h^2,\quad s^2=f^2-k^2,\quad r'^2=f^2-h'^2,\quad s'^2=k'^2-f^2,\\
&h^2=\rho p^2/(\lambda+2\mu),\quad h'^2=\rho' p^2/(\lambda'+2\mu'),\quad k^2=\rho p^2/\mu,\quad k'^2=\rho' p^2/\mu'.
\end{aligned}\right\} \qquad (11)$$

In this case, too, the condition of the problem is that the velocity of longitudinal waves (or transverse waves) in the surface layer is higher than that of the longitudinal waves (or transverse waves) in the subjacent medium, so that the relation

$$\left(\frac{k}{f}\right)^2>\left\{\begin{matrix}\left(\frac{h}{f}\right)^2>\left(\frac{k'}{f}\right)^2\\ \left(\frac{k'}{f}\right)^2>\left(\frac{h}{f}\right)^2\end{matrix}\right\}>\left(\frac{h'}{f}\right)^2 \qquad (12)$$

must exist.

(i) If $(k/f)^2<1$; we have $(r/f)^2>0$, $(s/f)^2>0$, $(r'/f)^2>0$, $(s'/f)^2<0$. The expression (10) is then soluble without using complex values for the quantities given in the same expression. The reason why complex values do not fit the problem is the same as that in the case of Love-waves. The relations $(r/f)^2>0$, $(s/f)^2>0$, show the condition that energy of the waves is accumulated near the surface. There is then every likelihood of Rayleigh-type waves being transmitted in the present case.

(ii) If $(k/f)^2>1>\{(k'/f)^2,\ (h/f)^2\}$; then $(r/f)^2>0$, $(s/f)^2<0$, $(r'/f)^2>0$, $(s'/f)^2<0$. The expression (10) then becomes complex owing to (11). These relations show at the same time that the waves are not of surface type. Transmission of Rayleigh-waves is not possible in the present case.

(iii) If $(k'/f)^2>1>(h/f)^2$; then $(r/f)^2>0$, $(s/f)^2<0$, $(r'/f)^2>0$, $(s'/f)^2>0$. In this case, too, the expression (10) becomes complex and the waves are not of surface type owing to the relation $(s/f)^2<0$. There are no Rayleigh-waves in the present case.

(iv) If $(h/f)^2>1>(k'/f)^2$; then $(r/f)^2<0$, $(s/f)^2<0$, $(r'/f)^2>0$, $(s'/f)^2<0$. The relations $(r/f)^2<0$, $(s/f)^2<0$ make the expression (10) complex and at the same time prevent the waves from being of surface type, it being therefore impossible for Rayleigh-waves to exist.

(v) If $\{(h/f)^2,\ (k'/f)^2\}>1>(h'/f)^2$; then $(r/f)^2<0$, $(s/f)^2<0$, $(r'/f)^2>0$, $(s'/f)^2>0$. The relations $(r/f)^2<0$, $(s/f)^2<0$ make the expression (10) complex and they also show that the waves are not of surface type. No Rayleigh wave exists in this case.

(vi) Finally, if $(h'/f)^2>1$; then $(r/f)^2<0$, $(s/f)^2<0$, $(r'/f)^2<0$, $(s'/f)^2>0$. The relations $(r/f)^2<0$, $(s/f)^2<0$, $(r'/f)^2<0$ make the expression (10) complex. We see also from $(r/f)^2<0$, $(s/f)^2<0$, that the waves are not of surface type, it being therefore impossible for Rayleigh waves to exist.

From these conditions it will be seen that, even should there be Rayleigh-waves, their possible velocities are of the range $1>(k/f)^2$. The group velocity, also increasing with decrease in wave length, may however be greater than $\sqrt{\rho/\mu}$ for smaller wave length.

It is a special feature in the dispersion of the present kind that the propagational velocity is never greater than the velocity of transverse waves in the superficial layer nor that in the subjacent medium.

As an example we shall take the case $\rho'/\rho=1/2$, $\lambda=\mu$, $\lambda'=\mu'$, $\lambda=\lambda'$, and solve the velocity equation (10), its treatment being naturally to

Table I.

$\sqrt{\rho p^2/\mu f^2}$	1·000	0·980	0·950	0·930	0·9194
fH	1·351	0·763	0·332	0·1075	0
L/H	4·650	8·240	18·91	58·50	∞

use a trial and error method. The results of calculation are shown in Table I and Fig. 3.

The result of calculation shows that Rayleigh-type waves are possible for the range $L/H = 4{\cdot}650 \sim \infty$, and that the velocity of propagation decreases with increase in wave length; and, furthermore, that for such wave length as $L/H < 4{\cdot}650$, no surface wave exists. Dispersion of the waves is therefore anomalous.

It is possible to get a dispersion curve for every case of stratification that is subjected to the conditions $(k/f)^2 > (k'/f)^2$, $(h/f)^2 > (h'/f)^2$. Although the dispersion curve for any case in question is anomalous, the value of L/H corresponding to the lower critical of the dispersion curve increases with increase in the ratios $(k/f)^2/(k'/f)^2$, $(h/f)^2/(h'/f)^2$.

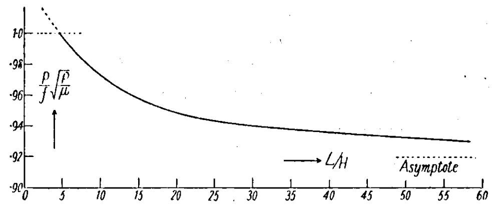

Fig. 3.

4. *Amplitude distribution of Rayleigh-waves in the media of transmission.*

With a view to knowing the distribution of amplitudes of Rayleigh-waves whose dispersion is anomalous, we examined the expressions for displacements, its horizontal and vertical components within the stratum and in the subjacent medium being written[5)]

$$
\begin{aligned}
u'_y = -\frac{s'}{f}\Bigg[& -\left(\frac{k'^2}{f^2}-2\right)\eta \sin ir'(H-y) - \frac{2ir's'}{f^2}\eta \sin s'(H-y) \\
& + \frac{ir's}{f^2}\varphi\left\{\left(\frac{k'^2}{f^2}-2\right)\cos ir'H \cos s'y + 2\cos s'H \cos ir'y\right\} \\
& + \frac{ir's}{f^2}\zeta\left\{\left(\frac{k'^2}{f^2}-2\right)\cos ir'H \sin s'y + 2\sin s'H \cos ir'y\right\} \\
& - \frac{rs'}{f^2}\varphi\left\{2\sin s'H \sin ir'y + \left(\frac{k'^2}{f^2}-2\right)\sin ir'H \sin s'y\right\} \\
& + \vartheta\left\{\left(\frac{k'^2}{f^2}-2\right)\sin ir'H \cos s'y + 2\cos s'H \sin ir'y\right\}\Bigg], \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
v'_y = i\Bigg[& \frac{ir's'}{f^2}\eta\left\{\left(\frac{k'^2}{f^2}-2\right)\cos ir'(H-y) - 2\cos s'(H-y)\right\} \\
& + \frac{s'}{f}\varphi\left\{\frac{r}{f}\left(\frac{k'^2}{f^2}-2\right)\sin ir'H \cos s'y + \frac{2r'^2 s}{f^3}\cos s'H \sin ir'y\right\}
\end{aligned}
$$

5) K. SEZAWA and K. KANAI, *Bull. Earthq. Res. Inst.*, **13** (1935), 240.

$$+\left\{\frac{2r'^2s'^2}{f^4}\zeta\sin s'H\sin ir'y+\left(\frac{k'^2}{f^2}-2\right)\vartheta\sin ir'H\sin s'y\right\}$$

$$+\frac{ir's'}{f^2}\left\{2\vartheta\cos s'H\cos ir'y-\left(\frac{k'^2}{f^2}-2\right)\zeta\cos ir'H\cos s'y\right\}$$

$$-\frac{ir'}{f}\varphi\left\{\frac{2rs'^2}{f^3}\sin s'H\cos ir'y-\frac{s}{f}\left(\frac{k'^2}{f^2}-2\right)\cos ir'H\sin s'y\right\}\Bigg], \quad (14)$$

$$u=\frac{i\mu's'k'^2}{\mu f^3}\left[Pe^{ry}+\frac{s}{f}Qe^{sy}\right], \quad (15)$$

$$v=-\frac{\mu's'k'^2}{\mu f^3}\left[\frac{r}{f}Pe^{ry}+Qe^{sy}\right], \quad (16)$$

where

$$\left.\begin{aligned}P=&\frac{2sr'}{f^2}\beta\cos s'H+\frac{2r's'}{f^2}\alpha\sin s'H\\&-\frac{2sr'}{f^2}\left(2-\frac{k'^2}{f^2}\right)\left(\frac{\mu'}{\mu}-1\right)\cosh r'H+\left(2-\frac{k'^2}{f^2}\right)\gamma\sinh r'H,\\Q=&-\frac{2r'}{f}\gamma\cos s'H-\frac{4rr's'}{f^3}\left(\frac{\mu'}{\mu}-1\right)\sin s'H\\&+\frac{r'}{f}\left(2-\frac{k'^2}{f^2}\right)\alpha\cosh r'H-\frac{r}{f}\left(2-\frac{k'^2}{f^2}\right)\beta\sinh r'H.\end{aligned}\right\} \quad (17)$$

The distribution of horizontal and vertical displacements at different depths for two cases, namely (i) $\sqrt{\rho p^2/\mu f^2}=0{\cdot}980$, $L/H=8{\cdot}24$ and (ii) $\sqrt{\rho p^2/\mu f^2}=0{\cdot}930$, $L/H=58{\cdot}50$, are shown in Figs. 4, 5.

These results show that while, for shorter waves, the amplitudes converge fairly sharply with depth, for longer waves, on the other hand, the convergency is very gradual. In any case, there is a nodal point in the horizontal component of displacement, whereas none exists in the vertical component. It appears that the ratio of horizontal to vertical displacements on the free surface is nearly 2/3 in the present condition of the problem. At all events, these features are quite similar to those in the case of Rayleigh-waves transmitted on a semi-infinite body or even in certain special cases of waves of normal dispersion.

The special feature in amplitude distribution in the present condition of the problem would be the relatively large horizontal displacement near the boundary between the stratum and the subjacent medium, which results from the accumulation of the vibrational energy of the waves in the said boundary.

5. *Concluding remarks.*

From mathematical calculation we found that, while the dispersion of Love-waves is normal, the dispersion of Rayleigh-waves in certain special conditions of stratification is anomalous. It is possible for this condition to exist for a certain range of lengths of waves that are

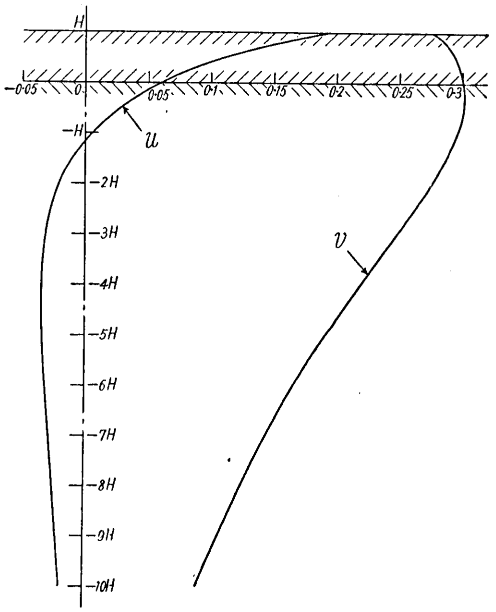

Fig. 4. $\sqrt{\rho p^2/\mu f}=0{\cdot}980$, L/H=8·24.

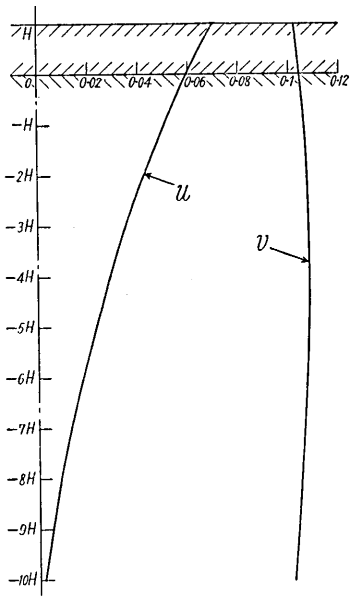

Fig. 5. $\sqrt{\rho p^2/\mu f}=0{\cdot}930$, $L/H=58{\cdot}5$.

transmitted through such a surface layer that the velocity of bodily waves in the same layer is higher than that in the subjacent medium.

It is a special feature in the dispersion of the present kind that the propagational velocity of the surface waves is greater than neither the velocity of transverse waves in the superficial layer nor that in the subjacent medium. Excepting a few conditions, the distribution of displacements with depth does not greatly differ from that of the usual Rayleigh-waves.

In conclusion I wish to express my sincerest thank to Dr. K. Kanai, with whose kind aid the investigation was brought to a successful close.

19. 彈性表面波の異常分數

地震研究所 妹澤克惟

表面層の物質の密度が下層のそれよりも少い場合は實際問題として寧ろ多いかも知れぬ．このやうな場合に彈性波の異常分散といふ事が問題になり得るのである．

數學的研究により，ラブ波の分散は常にノーマルであるけれども，或特別の場合のレーレー波の分散は異常であることがわかつた．このやうな異常分散の狀態は地表層中の固體波の速度が下層のそれよりも速いときに起り得るのである．しかしあまり短い波はこの場合に存在し得ないものである．尙，この場合に於けるレーレー波の速度が，地表層の横波の速度及び地中層の横波の速度の何れよりも低いといふ事は極めて特別な性質である．異常分散をなす場合の波動振幅の地中に於ける分布は，一二の特別の條件，卽ち兩層の境界附近の振幅が比較的に大きいといふこと以外は，ノーマル分散の場合と大して違はぬものである．