21. *The Effect of Cooling on a Plastic Earth under Gravitational Forces.*

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1. *Introduction.*

It appears to be generally held that in the earlier stage of secular cooling of the solid Earth, the superficial layer of the Earth tended to contract, forming tension cracks within the layer, while in the later stage the substratum contracted more rapidly than the superficial layer, thereby causing wrinkling of the Earth's surface. This problem has been theoretically discussed by Davison,[1] Darwin,[2] and Jeffreys.[3] On the other hand, we have shown that it is possible for the Earth's crust to be subjected to such horizontal compression as to exceed the vertical compression,[4] thus providing an alternative explanation of the wrinkled state of the surface. We shall now ascertain the extent to which the cooling of any special layer affects the plastic condition of the Earth under gravitational forces.

We shall first discuss the case in which a subjacent layer of uniform thickness is cooled, and then deal with the case in which the superficial layer is cooled, the remaining part of the Earth in every case retaining its original temperature, after which the mathematical expressions for the case of more complex distribution of the cooling will be discussed.

Although the plastic condition will differ more or less with the kind of plastic theory applied, since the difference is qualitatively imperceptible, we have used the maximum shear stress theory as we did in our previous papers. The present treatment shows that, whereas in the case of problems concerning a purely plastic Earth, the uniform cooling of any layer has no effect on the condition of the Earth, in the case of such problems concerning a plastic Earth as treated in the manner of elasticity the cooling under consideration slightly affects the condition of

1) C. DAVISON, *Phil. Trans. Roy. Soc.*, **178** (1887), 231~241.
2) G. H. DARWIN, *ibid.*, 242~250.
3) H. JEFFREYS, *The Earth*, Chap. 10; *M. N. R. A. S. Geophys. Suppl.*, **3** (1937), 53~59.
4) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **15** (1937), 878~887.

that Earth. In any case the Earth should be compressible even in a statical state. Since from Jeffreys's paper,[5] it appears that the Earth is in a statically compressed state, our idea cannot be very farfetched.

2. *Cooling of a substratum, the plasticity problem being treated in the manner of elasticity.*

Let a, b, c be the radii of the respective outer boundaries of the superficial layer, substratum, and inner layer, and ρ_1, ρ_2, ρ_3 and λ_1, μ_1, λ_2, μ_2, λ_3, μ_3 the densities and the elastic constants of the three parts under consideration. Let also θ be the temperature rise of the substratum. Then, the equations of equilibrium of the respective layers are written

$$\left.\begin{aligned}
&-\frac{\xi_1}{r^2}-\gamma_1 r+(\lambda_1+2\mu_1)\left(\frac{\partial^2 u_1}{\partial r^2}+\frac{2}{r}\frac{\partial u_1}{\partial r}-\frac{2u_1}{r^2}\right)=0,\\
&-\frac{\xi_2}{r^2}-\gamma_2 r+(\lambda_2+2\mu_2)\left(\frac{\partial^2 u_2}{\partial r^2}+\frac{2}{r}\frac{\partial u_2}{\partial r}-\frac{2u_2}{r^2}\right)-\beta_2\frac{\partial\theta_2}{\partial r}=0,\\
&-\gamma_3 r+(\lambda_3+2\mu_3)\left(\frac{\partial^2 u_3}{\partial r^2}+\frac{2}{r}\frac{\partial u_3}{\partial r}-\frac{2u_3}{r^2}\right)=0,
\end{aligned}\right\} \quad (1)$$

where $\beta_2=(\lambda_2+2\mu_2/3)q$; $\lambda_2+2\mu_2/3$, q being modulus of compression and volume expansion coefficient respectively. When $\partial\theta_2/\partial r=0$, the solutions of (1) are

$$\left.\begin{aligned}
u_1&=-\frac{\xi_1}{2(\lambda_1+2\mu_1)}+\frac{\gamma_1 r^3}{10(\lambda_1+2\mu_1)}+B_1 r+\frac{C_1}{r^2},\\
u_2&=-\frac{\xi_2}{2(\lambda_2+2\mu_2)}+\frac{\gamma_2 r^3}{10(\lambda_2+2\mu_2)}+B_2 r+\frac{C_2}{r^2},\\
u_3&=\frac{\gamma_3 r^3}{10(\lambda_3+2\mu_3)}+B_3 r.
\end{aligned}\right\} \quad (2)$$

The stresses of the respective layers are denoted by

$$\left.\begin{aligned}
\widehat{rr}_1&=-\frac{\lambda_1\xi_1}{r(\lambda_1+2\mu_1)}+\frac{(5\lambda_1+6\mu_1)}{10(\lambda_1+2\mu_1)}\gamma_1 r^2+(3\lambda_1+2\mu_1)B_1-\frac{4\mu_1}{r^3}C_1,\\
\widehat{rr}_2&=-\frac{\lambda_2\xi_2}{r(\lambda_2+2\mu_2)}+\frac{5\lambda_2+6\mu_2}{10(\lambda_2+2\mu_2)}\gamma_2 r^2+(3\lambda_2+2\mu_2)B_2-\frac{4\mu_2}{{}^3r}C_2-\beta_2\theta_2,
\end{aligned}\right\} \quad (3)$$

5) H. JEFFREYS, *M.N.R.A.S. Geophys. Suppl.*, 4 (1937), 50~60.

$$\widehat{rr}_3=\frac{5\lambda_3+6\mu_3}{10(\lambda_3+2\mu_3)}\eta_3 r^2+(3\lambda_3+2\mu_3)B_3,$$

$$\left.\begin{aligned}
\widehat{\theta\theta}_1=\widehat{\phi\phi}_1&=-\frac{\lambda_1+\mu_1}{\lambda_1+2\mu_1}\frac{\xi_1}{r}+\frac{5\lambda_1+2\mu_1}{10(\lambda_1+2\mu_1)}\eta_1 r^2+(3\lambda_1+2\mu_1)B_1+2\mu_1\frac{C_1}{r^3},\\
\widehat{\theta\theta}_2=\widehat{\phi\phi}_2&=-\frac{\lambda_2+\mu_2}{\lambda_2+2\mu_2}\frac{\xi_2}{r}+\frac{5\lambda_2+2\mu_2}{10(\lambda_2+2\mu_2)}\eta_2 r^2+(3\lambda_2+2\mu_2)B_2+2\mu_2\frac{C_2}{r^3}-\beta_2\theta_2,\\
\widehat{\theta\theta}_3=\widehat{\phi\phi}_3&=\frac{5\lambda_3+2\mu_3}{10(\lambda_3+2\mu_3)}\eta_3 r^2+(3\lambda_3+2\mu_3)B_3.
\end{aligned}\right\} \tag{4}$$

ξ_1, ξ_2, η_1, η_2, η_3 are such that

$$\left.\begin{aligned}
&\xi_1=\frac{4\pi\gamma}{3}\left\{c^3\rho_1(\rho_3-\rho_2)+b^3\rho_1(\rho_2-\rho_1)\right\},\quad \xi_2=\frac{4\pi\gamma}{3}c^3\rho_2(\rho_3-\rho_2),\\
&\eta_1=\frac{4\pi\gamma}{3}\rho_1^2,\quad \eta_2=\frac{4\pi\gamma}{3}\rho_2^2,\quad \eta_3=\frac{4\pi\gamma}{3}\rho_3^2,
\end{aligned}\right\} \tag{5}$$

γ being gravitational constant. From the boundary conditions

$$\left.\begin{aligned}
r&=a; && \widehat{rr}_1=0,\\
r&=b; && \widehat{rr}_1=\widehat{rr}_2,\quad u_1=u_2,\\
r&=c; && \widehat{rr}_2=\widehat{rr}_3,\quad u_2=u_3,
\end{aligned}\right\} \tag{6}$$

we get

$$\begin{aligned}
\frac{B_1\Phi}{\eta_1}=&\frac{-2c^2\mu_2}{5(\lambda_1+2\mu_1)(3\lambda_2+2\mu_2)}\Bigg[\frac{5}{2}\left(\frac{a}{b}\right)^3\left(\frac{a}{c}\right)^2\frac{\lambda_1(3\lambda_2+2\mu_2)}{\mu_1(3\lambda_1+2\mu_1)}\\
&\cdot\left\{\left(1-\frac{\mu_1}{\mu_2}\right)\left(1-\frac{3\lambda_3+2\mu_3}{3\lambda_2+2\mu_2}\right)-\left(\frac{b}{c}\right)^3\left(1+\frac{4\mu_1}{3\lambda_2+2\mu_2}\right)\left(1+\frac{3\lambda_3+2\mu_3}{4\mu_2}\right)\right\}\kappa_1\\
&+\left(\frac{b}{c}\right)^2\frac{3\lambda_2+2\mu_2}{3\lambda_1+2\mu_1}\left\{\left(1-\frac{3\lambda_3+2\mu_3}{3\lambda_2+2\mu_2}\right)-\left(\frac{b}{c}\right)^3\left(1+\frac{3\lambda_3+2\mu_3}{4\mu_2}\right)\right\}\kappa_2\\
&+\left(\frac{b}{c}\right)^2\frac{5\lambda_1+6\mu_1}{3\lambda_1+2\mu_1}\left\{\frac{3\lambda_2+2\mu_2}{4\mu_2}\left(1-\frac{3\lambda_3+2\mu_3}{3\lambda_2+2\mu_2}\right)+\left(\frac{b}{c}\right)^3\left(1+\frac{3\lambda_3+2\mu_3}{4\mu_2}\right)\right\}\kappa_3\\
&-\frac{(\lambda_1+2\mu_1)(3\lambda_3+2\mu_3)}{(3\lambda_1+2\mu_1)(\lambda_2+2\mu_2)}\left(1+\frac{3\lambda_2+2\mu_2}{4\mu_2}\right)\left\{\kappa_4-\frac{5\lambda_2+6\mu_2}{3\lambda_3+2\mu_3}\kappa_5\right\}\Bigg],
\end{aligned} \tag{7}$$

$$\frac{C_1\Phi}{\eta_1}=\frac{a^3c^2}{10(\lambda_1+2\mu_1)}\left[\frac{5}{2}\left(\frac{a}{c}\right)^2\frac{\lambda_1}{\mu_1}\left\{\left(1+\frac{4\mu_2}{3\lambda_1+2\mu_1}\right)\left(1-\frac{3\lambda_3+2\mu_3}{3\lambda_2+2\mu_2}\right)\right.\right.$$

$$+\left(\frac{b}{c}\right)^3\frac{4\mu_2}{3\lambda_2+2\mu_2}\left(1-\frac{3\lambda_2+2\mu_2}{3\lambda_1+2\mu_1}\right)\left(1+\frac{3\lambda_3+2\mu_3}{4\mu_2}\right)\Big\}\kappa_1$$

$$-\left(\frac{b}{c}\right)^2\frac{\mu_2}{\mu_1}\left\{\left(1-\frac{3\lambda_3+2\mu_3}{3\lambda_2+2\mu_2}\right)-\left(\frac{b}{c}\right)^3\left(1+\frac{3\lambda_3+2\mu_3}{4\mu_2}\right)\right\}\kappa_2$$

$$-\left(\frac{b}{c}\right)^2\frac{5\lambda_1+6\mu_1}{4\mu_1}\left\{\left(1-\frac{3\lambda_3+2\mu_3}{3\lambda_2+2\mu_2}\right)+\left(\frac{b}{c}\right)^3\frac{4\mu_2}{3\lambda_2+2\mu_2}\left(1+\frac{3\lambda_3+2\mu_3}{4\mu_2}\right)\right\}\kappa_3$$

$$\left.+\frac{(\lambda_1+2\mu_1)(3\lambda_3+2\mu_3)}{4\mu_1(\lambda_2+2\mu_2)}\left(1+\frac{4\mu_2}{3\lambda_2+2\mu_2}\right)\left\{\kappa_4-\frac{5\lambda_2+6\mu_2}{3\lambda_3+2\mu_3}\kappa_5\right\}\right], \quad (8)$$

$$\frac{B_2\Phi}{\eta_1}=\frac{2c^2\mu_2}{5(\lambda_1+2\mu_1)(3\lambda_2+2\mu_2)}\left[\frac{5}{2}\left(\frac{a}{c}\right)^5\frac{\lambda_1}{\mu_1}\left(1+\frac{4\mu_1}{3\lambda_1+2\mu_1}\right)\left(1+\frac{3\lambda_3+2\mu_3}{4\mu_2}\right)\kappa_1\right.$$

$$+\left(\frac{b}{c}\right)^5\left(1+\frac{3\lambda_3+2\mu_3}{4\mu_2}\right)\left\{1-\left(\frac{a}{b}\right)^3\right\}\kappa_2$$

$$-\left(\frac{b}{c}\right)^5\frac{5\lambda_1+6\mu_1}{4\mu_1}\left(1+\frac{3\lambda_3+2\mu_3}{4\mu_2}\right)\left\{\frac{4\mu_1}{3\lambda_1+2\mu_1}+\left(\frac{a}{b}\right)^3\right\}\kappa_3$$

$$+\frac{(\lambda_1+2\mu_1)(3\lambda_3+2\mu_3)}{4\mu_2(\lambda_2+2\mu_2)}\left\{\left(1+\frac{4\mu_2}{3\lambda_1+2\mu_1}\right)\right.$$

$$\left.\left.-\left(\frac{a}{b}\right)^3\left(1-\frac{\mu_2}{\mu_1}\right)\right\}\left\{\kappa_4-\frac{5\lambda_2+6\mu_2}{3\lambda_3+2\mu_3}\kappa_5\right\}\right], \quad (9)$$

$$\frac{C_2\Phi}{\eta_1}=\frac{b^5}{10(\lambda_2+2\mu_2)}\left[\frac{5}{2}\left(\frac{a}{b}\right)^5\frac{\lambda_1(\lambda_2+2\mu_2)}{\mu_1(\lambda_1+2\mu_1)}\left(1+\frac{4\mu_1}{3\lambda_1+2\mu_1}\right)\left(1-\frac{3\lambda_3+2\mu_3}{3\lambda_2+2\mu_2}\right)\kappa_1\right.$$

$$+\frac{\lambda_2+2\mu_2}{\lambda_1+2\mu_1}\left(1-\frac{3\lambda_3+2\mu_3}{3\lambda_2+2\mu_2}\right)\left\{1-\left(\frac{a}{b}\right)^3\right\}\kappa_2$$

$$-\frac{(5\lambda_1+6\mu_1)(\lambda_2+2\mu_2)}{4\mu_1(\lambda_1+2\mu_1)}\left(1-\frac{3\lambda_3+2\mu_3}{3\lambda_2+2\mu_2}\right)\left\{\frac{4\mu_1}{3\lambda_1+2\mu_1}+\left(\frac{a}{b}\right)^3\right\}\kappa_3$$

$$-\left(\frac{c}{b}\right)^2\frac{3\lambda_3+2\mu_3}{3\lambda_2+2\mu_2}\left\{\left(1-\frac{3\lambda_2+2\mu_2}{3\lambda_1+2\mu_1}\right)\right.$$

$$\left.\left.-\left(\frac{a}{b}\right)^3\left(1+\frac{3\lambda_2+2\mu_2}{4\mu_1}\right)\right\}\left\{\kappa_4-\frac{5\lambda_2+6\mu_2}{3\lambda_3+2\mu_3}\kappa_5\right\}\right], \quad (10)$$

$$\frac{B_3\Phi}{\eta_1}=\frac{2c^2\mu_2}{5(\lambda_1+2\mu_1)(3\lambda_2+2\mu_2)}\Bigg[10\left(\frac{a}{c}\right)^5\frac{\lambda_1}{4\mu_1}\left(1+\frac{4\mu_1}{3\lambda_1+2\mu_1}\right)\left(1+\frac{3\lambda_2+2\mu_2}{4\mu_2}\right)\kappa_1$$

$$+\left(\frac{b}{c}\right)^5\left(1+\frac{3\lambda_2+2\mu_2}{4\mu_2}\right)\left\{1-\left(\frac{a}{b}\right)^3\right\}\kappa_2$$

$$-\left(\frac{b}{c}\right)^5\frac{5\lambda_1+6\mu_1}{4\mu_1}\left(1+\frac{3\lambda_2+2\mu_2}{4\mu_2}\right)\left\{\frac{4\mu_1}{3\lambda_1+2\mu_1}+\left(\frac{a}{b}\right)^3\right\}\kappa_3$$

$$+\frac{(\lambda_1+2\mu_1)(3\lambda_2+2\mu_2)}{4\mu_1(\lambda_2+2\mu_2)}\left\{\frac{\mu_1}{\mu_2}\left(1+\frac{4\mu_2}{3\lambda_1+2\mu_1}\right)+\left(\frac{a}{b}\right)^3\left(1-\frac{\mu_1}{\mu_2}\right)\right.$$

$$\left.-\left(\frac{a}{c}\right)^3\left(1+\frac{4\mu_1}{3\lambda_2+2\mu_2}\right)+\left(\frac{b}{c}\right)^3\frac{4\mu_1}{3\lambda_2+2\mu_2}\left(1-\frac{3\lambda_2+2\mu_2}{3\lambda_1+2\mu_1}\right)\right\}\kappa_4$$

$$-\frac{(\lambda_1+2\mu_1)(5\lambda_2+6\mu_2)}{4\mu_1(\lambda_2+2\mu_2)}\left\{\frac{\mu_1}{\mu_2}\left(1+\frac{4\mu_2}{3\lambda_1+2\mu_1}\right)+\left(\frac{a}{b}\right)^3\left(1-\frac{\mu_1}{\mu_2}\right)\right.$$

$$\left.+\left(\frac{a}{c}\right)^3\frac{3\lambda_2+2\mu_2}{4\mu_2}\left(1+\frac{4\mu_1}{3\lambda_2+2\mu_2}\right)-\left(\frac{b}{c}\right)^3\frac{\mu_1}{\mu_2}\left(1-\frac{3\lambda_2+2\mu_2}{3\lambda_1+2\mu_1}\right)\right\}\kappa_5\Bigg], \quad (11)$$

$$\Phi=-\left(1+\frac{4\mu_2}{3\lambda_1+2\mu_1}\right)\left(1-\frac{3\lambda_3+2\mu_3}{3\lambda_2+2\mu_2}\right)+\left(\frac{a}{b}\right)^3\left(1-\frac{\mu_2}{\mu_1}\right)\left(1-\frac{3\lambda_3+2\mu_3}{3\lambda_2+2\mu_2}\right)$$

$$+\left(\frac{a}{c}\right)^3\frac{\mu_2}{\mu_1}\left(1+\frac{3\lambda_3+2\mu_3}{4\mu_2}\right)\left(1+\frac{4\mu_1}{3\lambda_2+2\mu_2}\right)$$

$$-\left(\frac{b}{c}\right)^3\frac{4\mu_2}{3\lambda_2+2\mu_2}\left(1-\frac{3\lambda_2+2\mu_2}{3\lambda_1+2\mu_1}\right)\left(1+\frac{3\lambda_3+2\mu_3}{4\mu_2}\right), \quad (12)$$

where

$$\kappa_1=\left(\frac{c}{a}\right)^3\frac{\rho_3-\rho_2}{\rho_1}+\left(\frac{b}{a}\right)^3\frac{\rho_2-\rho_1}{\rho_1}-\frac{5\lambda_1+6\mu_1}{10\lambda_1}, \quad (13)$$

$$\kappa_2=-1+5\left(\frac{c}{b}\right)^3\frac{\rho_3-\rho_2}{\rho_1}+\frac{5(\rho_2-\rho_1)}{\rho_1}$$

$$-5\left(\frac{c}{b}\right)^3\frac{\rho_2(\rho_3-\rho_2)}{\rho_1^2}\frac{\lambda_1+2\mu_1}{\lambda_2+2\mu_2}+\left(\frac{\rho_2}{\rho_1}\right)^2\frac{\lambda_1+2\mu_1}{\lambda_2+2\mu_2}, \quad (14)$$

$$\kappa_3=-1+\frac{10\lambda_1}{5\lambda_1+6\mu_1}\left\{\left(\frac{c}{b}\right)^3\frac{\rho_3-\rho_2}{\rho_1}+\frac{\rho_2-\rho_1}{\rho_1}\right\}$$

$$-10\left(\frac{c}{b}\right)^3\frac{\rho_2(\rho_3-\rho_2)}{\rho_1^2}\frac{\lambda_2(\lambda_1+2\mu_1)}{(5\lambda_1+6\mu_1)(\lambda_2+2\mu_2)}$$

$$+\left(\frac{\rho_2}{\rho_1}\right)^2\frac{(\lambda_1+2\mu_1)(5\lambda_2+6\mu_2)}{(5\lambda_1+6\mu_1)(\lambda_2+2\mu_2)}-\frac{30\beta_2\theta_2}{4\pi\gamma\rho_1^2 b^2}\frac{\lambda_1+2\mu_1}{5\lambda_1+6\mu_1}, \quad (15)$$

$$\kappa_4=-\left(\frac{\rho_2}{\rho_1}\right)^2+\frac{5\rho_2(\rho_3-\rho_2)}{\rho_1^2}+\left(\frac{\rho_3}{\rho_1}\right)^2\frac{\lambda_2+2\mu_2}{\lambda_3+2\mu_3}, \quad (16)$$

$$\kappa_5=-\left(\frac{\rho_2}{\rho_1}\right)^2+\frac{10\rho_2(\rho_3-\rho_2)}{\rho_1^2}\frac{\lambda_2}{(5\lambda_2+6\mu_2)}+\left(\frac{\rho_3}{\rho_1}\right)^2\frac{(\lambda_2+2\mu_2)(5\lambda_3+6\mu_3)}{(5\lambda_2+6\mu_2)(\lambda_3+2\mu_3)}$$

$$+\frac{30\beta_2\theta_2}{4\pi\gamma\rho_1^2 c^2}\frac{\lambda_2+2\mu_2}{5\lambda_2+6\mu_2}. \quad (17)$$

Substituting the values of these constants in (3), (4), it is possible to get the principal stresses within the Earth. The plastic stresses are equal to the maximum shear stresses, namely,

$$\frac{\widehat{rr}_1-\widehat{\theta\theta}_1}{2}, \quad \frac{\widehat{rr}_2-\widehat{\theta\theta}_2}{2}, \quad \frac{\widehat{rr}_3-\widehat{\theta\theta}_3}{2}. \quad (18)$$

Comparing (3), (4), (7) ~ (18), it will be seen that, were the materials incompressible, that is, $\lambda/\mu=\infty$, the maximum shear stresses $(\widehat{rr}_1-\widehat{\theta\theta}_1)/2$, $(\widehat{rr}_2-\widehat{\theta\theta}_2)/2$, $(\widehat{rr}_3-\widehat{\theta\theta}_3)/2$ would assume zero value, the value of the coefficient β_2 being invariably finite from the thermal nature of the problem. In such a case the horizontal forces $\widehat{\theta\theta}_1$, $\widehat{\phi\phi}_1$ on the free surface $r=a$ are also zero since $\widehat{rr}_1$ on the same surface is zero. We are ascertaining the stress condition of the solid Earth with reference to the state of the liquid Earth, the gravitational conditions for both cases being approximately the same.

Let the thicknesses of the surface layer and the subjacent layer be 100 km and 300 km respectively; then $a=6350$ km, $b=6250$ km, and $c=$ 5950 km. From the ideas of Jeffreys[6] and others it appears that the temperature fall of the second layer is of the order of 2000° since the Earth solidified, that is to say, a volume contraction of 5 per cent. Assume that the temperature fall of the second layer relative to that of the neighbouring layers is 700°C and $q=3.0\cdot 8.10^{-4}$, $k=\lambda+(2/3)\mu=10^{12}$ (C.G.S.), $\rho_1=\rho_2=3{\cdot}49$, $\rho_3=4{\cdot}85$, $\lambda_1=\mu_1=\lambda_2=\mu_2=\lambda_3=\mu_3\equiv\lambda\equiv\mu$. Then $\theta=-700$°C, $\beta=0{\cdot}24.10^5$. Using these values, we calculated the distribution of $(\widehat{rr}-\widehat{\theta\theta})/2$; the results being shown by full lines in Fig. 1.

6) H. Jeffreys, *loc. cit.* 3).

For comparison, the distribution for the case of no temperature fall is shown by a broken line in the same figure. The values of $\widehat{rr}$ for the respective cases are shown in Fig. 2.

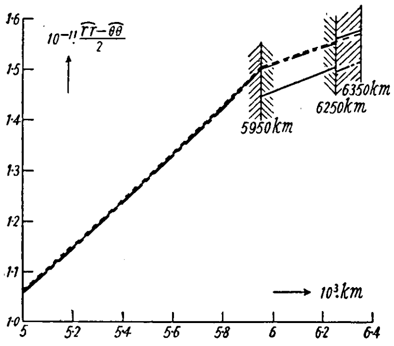

Fig. 1.

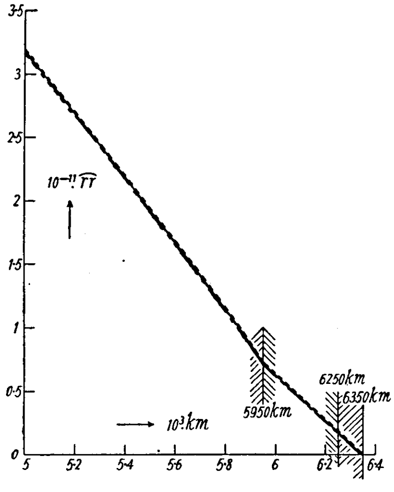

Fig. 2.

It will be seen that the effect of cooling on the plastic state under gravitational forces is negligible in $(\widehat{rr}-\widehat{\theta\theta})/2$ as well as in $\widehat{rr}$.

3. *Cooling of a surface stratum, the plasticity problem being treated in the manner of elasticity.*

Let the temperature rise of a superficial stratum be θ_1, the temperature of the second layer retaining its original value. The equations of equilibrium are then of the forms

$$\left.\begin{aligned}
&-\frac{\xi_1}{r^2}-\eta_1 r+(\lambda_1+2\mu_1)\left(\frac{\partial^2 u_1}{\partial r^2}+\frac{2}{r}\frac{\partial u_1}{\partial r}-\frac{2u_1}{r^2}\right)-\beta_1\frac{\partial\theta_1}{\partial r}=0,\\
&-\frac{\xi_2}{r^2}-\eta_2 r+(\lambda_2+2\mu_2)\left(\frac{\partial^2 u_2}{\partial r^2}+\frac{2}{r}\frac{\partial u_2}{\partial r}-\frac{2u_2}{r^2}\right)=0,\\
&-\eta_3 r+(\lambda_3+2\mu_3)\left(\frac{\partial^2 u_3}{\partial r^2}+\frac{2}{r}\frac{\partial u_3}{\partial r}-\frac{2u_3}{r^2}\right)=0.
\end{aligned}\right\}\qquad(19)^{7)}$$

When $\partial\beta_1/\partial\theta_1=0$, the expressions for the displacements are the same as those in (2), whereas the stresses of the respective layers are expressed by

7) The simultaneous equations of the second and third expressions in (19) are not particularly important except for comparison with the preceding section.

$$\left.\begin{aligned}\widehat{rr}_1&=-\frac{\lambda_1\xi_1}{r(\lambda_1+2\mu_1)}+\frac{5\lambda_1+6\mu_1}{10(\lambda_1+2\mu_1)}\eta_1r^2+(3\lambda_1+2\mu_1)B_1-\frac{4\mu_1}{r^3}C_1-\beta_1\theta_1,\\ \widehat{rr}_2&=-\frac{\lambda_2\xi_2}{r(\lambda_2+2\mu_2)}+\frac{5\lambda_2+6\mu_2}{10(\lambda_2+2\mu_2)}\eta_2r^2+(3\lambda_2+2\mu_2)B_2-\frac{4\mu_2}{r^3}C_2,\\ \widehat{rr}_3&=\frac{5\lambda_3+6\mu_3}{10(\lambda_3+2\mu_3)}\eta_3r^2+(3\lambda_3+2\mu_3)B_3.\end{aligned}\right\}\quad(20)$$

$$\left.\begin{aligned}\widehat{\theta\theta}_1=\widehat{\phi\phi}_1&=-\frac{\lambda_1+\mu_1}{\lambda_1+2\mu_1}\frac{\xi_1}{r}+\frac{5\lambda_1+2\mu_1}{10(\lambda_1+2\mu_1)}\eta_1r^2+(3\lambda_1+2\mu_1)B_1+2\mu_1\frac{C_1}{r^3}-\beta_1\theta_1,\\ \widehat{\theta\theta}_2=\widehat{\phi\phi}_2&=-\frac{\lambda_2+\mu_2}{\lambda_2+2\mu_2}\frac{\xi_2}{r}+\frac{5\lambda_2+2\mu_2}{10(\lambda_2+2\mu_2)}\eta_2r^2+(3\lambda_2+2\mu_2)B_2+2\mu_2\frac{C_2}{r^3},\\ \widehat{\theta\theta}_3=\widehat{\phi\phi}_3&=\frac{5\lambda_3+2\mu_3}{10(\lambda_3+2\mu_3)}\eta_3r^2+(3\lambda_3+2\mu_3)B_3,\end{aligned}\right\}\quad(21)$$

where $\beta_1=(\lambda_1+2\mu_1/3)q$. Using the boundary conditions (6), we find the values of the constants κ_1, κ_2, κ_3, κ_4, κ_5 in the expressions of B_1, C_1, B_2, C_2, B_3 in (7)~(11), the results being shown below.

$$\kappa_1=\left\{\left(\frac{c}{a}\right)^3\frac{\rho_3-\rho_2}{\rho_1}+\left(\frac{b}{a}\right)^3\frac{\rho_2-\rho_1}{\rho_1}\right\}-\frac{5\lambda_1+6\mu_1}{10\lambda_1}+\frac{3\beta_1\theta_1}{4\pi\gamma\rho_1^2a^2}\frac{(\lambda_1+2\mu_1)}{\lambda_1},\quad(22)$$

$$\kappa_2=-1+5\left(\frac{c}{b}\right)^3\frac{\rho_3-\rho_2}{\rho_1}+\frac{5(\rho_2-\rho_1)}{\rho_1}-5\left(\frac{c}{b}\right)^3\frac{\rho_2(\rho_3-\rho_2)}{\rho_1^2}\frac{\lambda_1+2\mu_1}{\lambda_2+2\mu_2}+\left(\frac{\rho_2}{\rho_1}\right)^2\frac{\lambda_1+2\mu_1}{\lambda_2+2\mu_2},\quad(23)$$

$$\kappa_3=-1+\frac{10\lambda_1}{5\lambda_1+6\mu_1}\left\{\left(\frac{c}{b}\right)^3\frac{\rho_3-\rho_2}{\rho_1}+\frac{\rho_2-\rho_1}{\rho_1}\right\}-10\left(\frac{c}{b}\right)^3\frac{\rho_2(\rho_3-\rho_2)}{\rho_1^2}\frac{\lambda_2(\lambda_1+2\mu_1)}{(5\lambda_1+6\mu_1)(\lambda_2+2\mu_2)}+\left(\frac{\rho_2}{\rho_1}\right)^2\frac{(\lambda_1+2\mu_1)(5\lambda_2+6\mu_2)}{(5\lambda_1+6\mu_1)(\lambda_2+2\mu_2)}+\frac{15\beta_1\theta_1}{2\pi\gamma\rho_1^2b^2}\frac{\lambda_1+2\mu_1}{5\lambda_1+6\mu_1},\quad(24)$$

$$\kappa_4=-\left(\frac{\rho_2}{\rho_1}\right)^2+\frac{5\rho_2(\rho_3-\rho_2)}{\rho_1^2}+\left(\frac{\rho_3}{\rho_1}\right)^2\frac{\lambda_2+2\mu_2}{\lambda_3+2\mu_3},\quad(25)$$

$$\kappa_5=-\left(\frac{\rho_2}{\rho_1}\right)^2+\frac{10\rho_2(\rho_3-\rho_2)}{\rho_1^2}\frac{\lambda_2}{5\lambda_2+6\mu_2}+\left(\frac{\rho_3}{\rho_1}\right)^2\frac{(\lambda_2+2\mu_2)(5\lambda_3+6\mu_3)}{(5\lambda_2+6\mu_2)(\lambda_3+2\mu_3)}. \tag{26}$$

Assume that the temperature fall of the surface layer relative to that of the neighbouring layer is 700°C, all other conditions being the same as those in the preceding section. The results of calculation are shown by chain-lines in Figs. 1, 2. It will be seen that in this case, too, the effect of cooling on the plasticity problem is negligible.

4. *Cooling of strata, the plasticity problem being treated in the manner of pure plasticity.*

The equations of equilibrium in this case are expressed by

$$\left.\begin{aligned}
&-\frac{\xi_1}{r^2}-\eta_1 r+\frac{\partial\widehat{rr}_1}{\partial r}+\frac{2}{r}(\widehat{rr}_1-\widehat{\theta\theta}_1)-\beta_1\frac{\partial\theta_1}{\partial r}=0,\\
&-\frac{\xi_2}{r^2}-\eta_2 r+\frac{\partial\widehat{rr}_2}{\partial r}+\frac{2}{r}(\widehat{rr}_2-\widehat{\theta\theta}_2)-\beta_2\frac{\partial\theta_2}{\partial r}=0,\\
&-\eta_3 r+\frac{\partial\widehat{rr}_3}{\partial r}+\frac{2}{r}(\widehat{rr}_3-\widehat{\theta\theta}_3)=0,
\end{aligned}\right\} \tag{27}$$

the conditions of plasticity being of the forms

$$\widehat{rr}_1-\widehat{\theta\theta}_1=\pm 2k_1,\quad \widehat{rr}_2-\widehat{\theta\theta}_2=\pm 2k_2,\quad \widehat{rr}_3-\widehat{\theta\theta}_3=\pm 2k_3. \tag{28}$$

The solutions of (26) are therefore of the types

$$\left.\begin{aligned}
&\widehat{rr}_1+4k_1\log r+\frac{\xi_1}{r}+\frac{\eta_1 r^2}{2}+I_1+\beta_1\theta_1=0,\\
&\widehat{rr}_2+4k_2\log r+\frac{\xi_2}{r}+\frac{\eta_2 r^2}{2}+I_2+\beta_2\theta_2=0,\\
&\widehat{rr}_3+4k_3\log r+\frac{\eta_3 r^2}{2}+I_3=0.
\end{aligned}\right\} \tag{29}$$

If we write

$$I_1+\beta_1\theta_1=I_1',\quad I_2+\beta_2\theta_2=I_2', \tag{30}$$

and put the solutions in the boundary conditions

$$r=a;\ \widehat{rr}_1=0,\quad r=b;\ \widehat{rr}_1=\widehat{rr}_2,\quad r=c;\ \widehat{rr}_2=\widehat{rr}_3, \tag{31}$$

it is possible to make the solutions fit the present problem. From the nature of the problem the solutions are the same whatever may be the forms of

$$I_1+\beta_1\theta_1, \qquad I_2+\beta_2\theta_2.$$

The temperature fall of any layer has no effect on the plasticity condition, not at any rate from the present criterion of plasticity.

5. *Cooling, the distribution of which with radius is irregular.*

Let us assume for simplicity that the density and elasticity are uniform throughout the Earth, the temperature rise being distributed in the manner

$$\theta=f\ (r). \tag{32}$$

The equation of equilibrium in the manner of elasticity, then, is of the form

$$-\gamma r+(\lambda+2\mu)\left(\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}-\frac{2u}{r^2}\right)-\beta f'(r)=0. \tag{33}$$

The radial displacement is thus

$$u=\frac{\gamma r^3}{10(\lambda+2\mu)}+Br+\frac{C}{r^2}+\frac{\beta/(\lambda+2\mu)}{D^2+\frac{2}{r}D-\frac{2}{r^2}}\ f'(r), \tag{34}$$

D being operator in r. The stresses are of the forms

$$\left.\begin{aligned}\widehat{rr}&=\lambda\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)+2\mu\frac{\partial u}{\partial r}-\beta f(r)\,,\\ \widehat{\theta\theta}=\widehat{\phi\phi}&=\lambda\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)+2\mu\frac{u}{r}-\beta f(r)\,.\end{aligned}\right\} \tag{35}$$

Using the boundary condition

$$r=a\,; \qquad \widehat{rr}=0, \tag{36}$$

it is possible to determine B, C. In this case the plastic stress and normal stresses are affected by $f(r)$, the numerical determination of which will be reserved for future study. From the results in the preceding section, the influence of the temperature fall would be trifling.

In the case of a purely plasticity problem, the equation of equilibrium is

$$-\gamma r+\frac{\partial\widehat{rr}}{\partial r}+\frac{2}{r}(\widehat{rr}-\widehat{\theta\theta})-\beta f'(r)=0, \tag{37}$$

with a plastic condition

$$\widehat{rr}-\widehat{\theta\theta}=\pm 2k, \tag{38}$$

so that

$$\left.\begin{aligned}\widehat{rr}&=4k\log\frac{a}{r}-\frac{\eta}{2}(a^2-r^2)+\beta\Big\{f(a)-f(r)\Big\},\\ \widehat{\theta\theta}&=4k\log\frac{a}{r}-\frac{\eta}{2}(a^2-r^2)+\beta\Big\{f(a)-f(r)\Big\}-2k.\end{aligned}\right\}\tag{39}$$

Although in this case, the plastic stress

$$\frac{\widehat{rr}-\widehat{\theta\theta}}{2}\tag{40}$$

is independent of temperature fall, the normal stresses $\widehat{rr}$, $\widehat{\theta\theta}$ at a certain depth are affected, though probably only very slightly, by the temperature fall under consideration.

6. *Concluding remarks.*

We have ascertained that the effect of cooling of any layer on the plastic deformation of the Earth under gravitational forces is not very marked, even should the accounted temperature fall be that of the Earth after solidification. Whereas in the case of treatments like elasticity, the correction due to temperature fall on the plastic behaviour is extremely small, in the case of pure plasticity treatments, there is no need for temperature correction, not, at any rate, from such criterion of plasticity as concerns the maximum shear stress.

It has been shown that the horizontal compression in the surface crust of the Earth is the result of the usual gravitational forces, the effect of temperature fall of the substratum being negligible compared with the gravitational action. The crack-forming condition in the early stage of the solid Earth can never be large compared with the compression due to gravity. It should, however, be borne in mind that our theory assumes that the Earth is compressible even in the statical state. If, on the other hand, the Earth is incompressible in the statical state, shear stresses at any part of the Earth are zero so that there is no plastic stress; and also no horizontal compression exists, particularly, on the free surface of the Earth. It is possible that, in the earliest stage of the Earth's solidification, the Earth was incompressible; the mechanical condition on the surface would have been under temperature stress, in which case the explanation of the tension cracks in the superficial layer of the Earth might then be quite simple.

21. 地球の冷却が重力の下にあるそのプラスチック狀態に及ぼす影響

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地球が固體化してからの初期の狀態では表面の薄い層が冷却によつて收縮して表面層に機械的作用を及ぼし，次の時期には第2の層がよく冷却する爲にその結果として表面層を水平に壓縮し從て地表の皺が生じたといふ理論は相當に信用されてをるやうである。我々が最近出したところの，地球の殼が壓縮性である以上は重力の爲に地殼内に剪斷應力(プラスチック應力)を生じ，且つ地表層が水平に壓され得るといふ理論と兩立することになるから，重力と冷却とが同時に存在する場合に何れがよくきくかを確めて見たのである。

地球の第2層又は第1層が殘りの層に比較して700°Cだけ餘計に冷却したとして計算して見ると，冷却の爲のプラスチックの應力が重力のそれに比して著しく小さいことがわかつたのである。故に地表に働く大體の横壓力は重力の作用の結果としてよいと考へられる。

只今の理論は地球が壓縮性であるとするとき成り立ち得るのであつて，非壓縮性であるとすると地殼内のプラスチック應力や表面の横壓力は全然存在しない事になる。しかし Jeffreys が最近研究したところによると地球はその核部に到る迄も普通の Poisson 比が示す程度近くに壓縮されてをるやうであるから，只今の議論は相當正しいものと見てよい。しかし地球が固體化した極く初期にだけ非壓縮性であつたとすると地表の引張りの割目を說明し得るから更に好都合かも知れぬ。