1. *Amplitudes of Rayleigh-waves with Discontinuities in their Dispersion Curves.*

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1. *Introduction.*

It was found in the previous paper[1] that the amplitude of Rayleigh-waves transmitted through a stratified layer is maximum for a certain ratio of wave length to thickness of the surface layer. It was furthermore ascertained that, although there are two peaks in the resonance curve of the waves, one of these peaks represents the prevalent oscillation in Rayleigh-waves, while the other concerns a mere resonance-like condition of bodily waves transmitted along the surface layer.

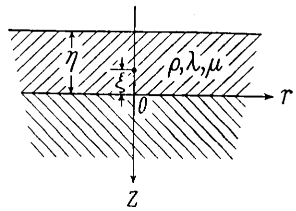

Fig. 1.

The dispersion curves corresponding to the Rayleigh-waves now under consideration are of the type shown in Fig. 2, the two peaks of the resonance curve just mentioned belonging, as a matter of fact, to part AO of the first dispersion curve AOB and part OD of the second dispersion curve COD, respectively, the reason why the two peaks in question differing from each other in their qualities being therefore obvious. In the absence of numerical results for the amplitudes of waves corresponding to part OB of curve AOB or part CO of curve COD, it is not yet inadmissible to conceive of waves (surface waves as well

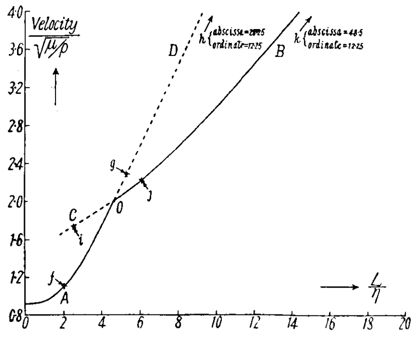

Fig. 2. Dispersion curves with discontinuity. Full and broken lines correspond to waves of the first and second sorts respectively.

1) K. SEZAWA and K. KANAI, "Relation between the Thickness of a Surface Layer and the Amplitudes of Dispersive Rayleigh-waves," *Bull. Earthq. Res. Inst.*, **15** (1937), 845~859.

as bodily waves) for a wide range of vibrational frequencies.[2]

It has already been found that the variation in the ratio of horizontal to vertical displacements of Rayleigh-waves[3] is also discontinuous at the wave length that corresponds to discontinuity O in dispersion curve AOB. The present investigation shows that the values of the horizontal and vertical displacements themselves in the resonance curves are also discontinuous at the same wave-length.

2. *Method of calculation.*

We shall investigate the same problem as in the previous paper,[4] all the conditions of the problem and the mathematical symbols being the same as those in that paper.

The final solutions are such that

$$u_{z=-\eta}=\left[\frac{1}{\sqrt{r\eta}}e^{i(\kappa r-\frac{\pi}{4}-pt)}\right]\frac{1}{\Phi'(\kappa)}\sqrt{\frac{2\pi}{\kappa\eta}}\frac{k^2\eta}{h\gamma}\left[\left\{\frac{2\gamma\delta}{1+\delta^2}\operatorname{sh}\delta\kappa\eta-\operatorname{sh}\gamma\kappa\eta\right\}\operatorname{ch}\gamma\kappa\xi\right.$$

$$\left.+\left\{\operatorname{ch}\gamma\kappa\eta-\frac{2}{1+\delta^2}\operatorname{ch}\delta\kappa\eta\right\}\operatorname{sh}\gamma\kappa\xi\right], \quad (1)$$

$$w_{z=-\eta}=\left[\frac{1}{\sqrt{r\eta}}e^{i(\kappa r+\frac{\pi}{4}-pt)}\right]\frac{1}{\Phi'(\kappa)}\sqrt{\frac{2\pi}{\kappa\eta}}\frac{k^2\eta}{h}\left[\left\{\frac{2}{1+\delta^2}\operatorname{ch}\gamma\kappa\eta-\operatorname{ch}\delta\kappa\eta\right\}\operatorname{ch}\gamma\kappa\xi\right.$$

$$\left.+\left\{\frac{1}{\gamma\delta}\operatorname{sh}\delta\kappa\eta-\frac{2}{1+\delta^2}\operatorname{sh}\gamma\kappa\eta\right\}\operatorname{sh}\gamma\kappa\xi\right], \quad (2)$$

where

$$\Phi'(\kappa)=\left(\frac{k}{\kappa}\right)^2\left\{\frac{4}{(1+\delta^2)^2}-1\right\}\operatorname{ch}\gamma\kappa\eta\operatorname{ch}\delta\kappa\eta$$

$$+2\gamma\delta\left\{4-(1+\delta^2)\left(\frac{1}{\gamma^2}+\frac{1}{\delta^2}\right)\right\}\left\{\frac{1}{4\gamma^2\delta^2}-\frac{1}{(1+\delta^2)^2}\right\}\operatorname{sh}\gamma\kappa\eta\operatorname{sh}\delta\kappa\eta$$

$$+\frac{\kappa\eta}{\delta}\left\{\frac{\left(\frac{h}{\kappa}\right)^2(1+\delta^2)}{2\gamma^2}-\frac{2\left(\frac{k}{\kappa}\right)^2}{1+\delta^2}\right\}\operatorname{ch}\gamma\kappa\eta\operatorname{sh}\delta\kappa\eta$$

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2) The nature of discontinuity in dispersion curves is analogous to that of asymptotic lines as a limiting condition of hyperbolas.

3) K. SEZAWA and K. KANAI, "Discontinuity in Dispersion Curves of Rayleigh Waves," *Bull. Earthq. Res. Inst.*, **13** (1935), 245~250.

4) K. SEZAWA and K. KANAI, *loc. cit.* 1). It should be borne in mind that the condition of the source is such that the displacement at the same source is constant for any wave length.

$$+\frac{\kappa\eta}{\gamma}\left\{\frac{\left(\frac{k}{\kappa}\right)^2(1+\delta^2)}{2\delta^2}-\frac{2\left(\frac{h}{\kappa}\right)^2}{1+\delta^2}\right\}\operatorname{sh}\gamma\kappa\eta\operatorname{ch}\delta\kappa\eta, \qquad (3)$$

$$\gamma^2=1-\left(\frac{h}{\kappa}\right)^2, \qquad \delta^2=1-\left(\frac{k}{\kappa}\right)^2. \qquad (4)$$

Using these equations we obtained the values of u, w for part CO and part OB, the results of which, together with those in the preceding paper[5], are given in Tables I, II, and plotted in Fig. 3, K in the tables and the figure denoting the value in every first pair of brackets in (1),(2).

Table I.

L/η	$\sqrt{\rho p^2/\mu f^2}$	u/K	$-w/K$	$-u/w$
1·00	0·9307	4·543	6·77	0·671
1·65	1·000	9·19	14·4	0·638
2·667	1·260	4·707	8·54	0·5513
3·12	1·414	1·826	3·505	0·521
3·96	1·732	1·964	4·265	0·4604
4·622	2·000	2·793	−1·613	——
4·949	2·05	2·751	−0·346	−7·95
6·095	2·236	2·817	−0·282	−9·99
7·295	2·45	2·551	−0·2113	−12·07

Table II.

L/η	$\sqrt{\rho p^2/\mu f^2}$	u/K	$-w/K$	$-u/w$
1·8615	1·644	1·5575	−0·889	−1·752
2 525	1·732	2·03	−0·6153	−3·3
3·885	1·898	2·598	−0·434	−5·988
5·333	2·298	8·208	22·57	0·3637
6·15	2·6458	14·27	43·06	0·3315
6·5	2·795	22·73	70·4	0·323
6·98	3·000	62·1	207·3	0·2996
8	3·44	4·582	18·07	0·2536
9	3·8730	3·056	12·67	0·2411

3. *Interpretation of the result for waves corresponding to the first dispersion curve.*

The condition of the present problem is such that the ratio of the elastic constant of the subjacent material to that of the surface layer is infinitely large. Comparing the dispersion curves of the present case with those for which the ratio in question is 5 or 20, it is possible to assume that the branch AOB in Fig. 2 represents the curve for the limiting condition of the first dispersion curve in general Rayleigh-waves.

The values of horizontal and vertical displacements indicated by full lines in Fig. 3 or tabulated in Table I, correspond exactly to the first dispersion curve (Fig. 2). It will be seen that the variations in u as well as in w are both discontinuous at $L/\eta=4{\cdot}622$. While the vertical displacement is larger than the horizontal for $L/\eta<4{\cdot}622$, the reverse is the case for $L/\eta>4{\cdot}622$. At all events, the amplitude of Rayleigh-waves is maximum at $L/\eta=2$ as long as the first dispersion curve is concerned.

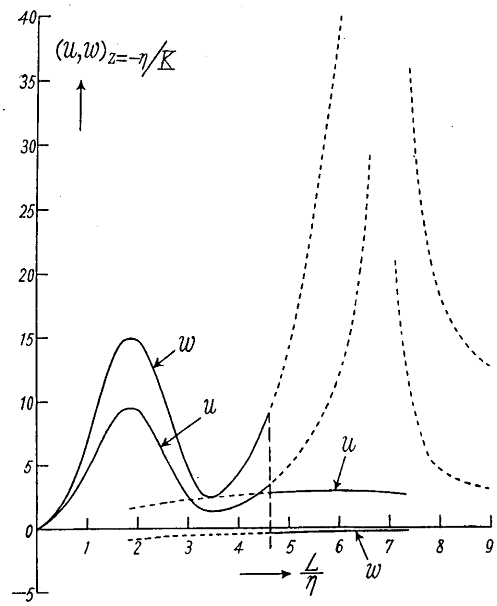

Fig. 3. Resonance curves of Rayleigh-waves with discontinuity in their dispersion curves. Full and broken lines correspond to waves of the first and second sorts respectively.

It is an important fact that, although u and $-w$ are in opposite sense for $L/\eta<4{\cdot}622$, both displacement components are of the same sign for $L/\eta>4{\cdot}622$, which indicates that, although the orbital motion of the surface for $L/\eta<4{\cdot}622$, is of the same sense as that in the usual Rayleigh-waves, the same motion for $L/\eta>4{\cdot}622$ is opposite to that of such waves (the usual Rayleigh-waves). But since the vertical displacement for $L/\eta>4{\cdot}622$ is practically zero, the actual displacement of Rayleigh-waves of such wave length would be mainly horizontal.

Although the ratio of horizontal to vertical components of displacements tends to infinity for $L/\eta\to\infty$, the respective components themselves also become zero at the same time.

4. *Interpretation of the result for waves corresponding to the second dispersion curve.*

With the same idea as in the preceding section, the branch COD in Fig. 2 represents the curve for the limiting condition of the second dispersion curve in general Rayleigh-waves. The values of the horizontal and vertical displacements, indicated by broken lines in Fig. 3, correspond exactly to the second dispersion curve (Fig. 2).

In this case, too, the variations in u as well as in w are both discontinuous at $L/\eta=4{\cdot}622$. While the vertical displacement is less than the horizontal for $L/\eta<4{\cdot}622$, the reverse is true for $L/\eta>4{\cdot}622$, in which case the amplitude of the waves is infinitely large at $L/\eta\approx7$. Since, nevertheless, it can hardly be assumed that the branch, COD, represents the resonance curve of the usual Rayleigh-waves, the infinitely large displacement just mentioned is rather of such a nature as to be a resonance-like condition of the bodily waves.

In this case, while the ratio of horizontal to vertical components of displacement tends to zero for L/η, the respective components themselves, on the other hand, become zero.

5. *Some remarks on seismic waves.*

Since the present problem concerns the case of a relatively large ratio of stiffness of the subjacent medium to that of the surface layer, the results apply to waves transmitted along a relatively shallow surface stratum, say, a few hundred meters thick, or even only a few meters thick. Since in such a case the ratio of wave length to thickness of the layer is so large that the relation $L/\eta>4{\cdot}622$ holds, the surface displacement would then be mainly horizontal if the usual Rayleigh-waves were considered. If we were to take an earlier phase of the waves, then a transmission of bodily waves of large vertical displacement should be expected.

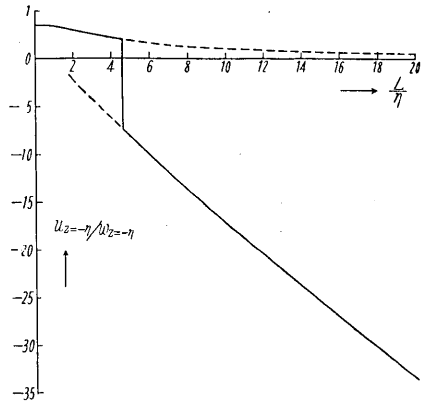

Fig. 4. Ratio of horizontal and vertical displacements. Full and broken lines correspond to the first and second sorts respectively.

At all events, our interest in the present problem is not in the transmission of waves in a layer of very small thickness, but rather in a possible existence of discontinuity in the dispersion curves of Rayleigh-waves.

Finally, with a view to confirming the nature of the present problem from a different point of view, we shall add curves showing the ratio of horizontal to vertical surface displacements for any wave length. The full and broken lines in Fig. 4 represent the respective cases corresponding to the first and second dispersion curves in Fig. 2.

In conclusion I wish to express my sincerest thank to Dr. K. KANAI, through whose assistance the investigation was successfully made.

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1. 不連續性分散曲線を有するレーレー波の振幅

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以前の報告で，地表層の彈性が下層のそれに比して著しく弱くなるとそれに傳はるレーレー波の分散曲線に不連續性が現はれる事を述べて置いた（之は實は双曲線の極限が漸近直線になることと多少 analogous である）. 之等に相當する水平，鉛直の振幅分布を出して見ると種々面白い事實のあることがわかつた.

第１の分散曲線について波長の短い所では，普通のレーレー波の場合と同じ位の水平，鉛直の振幅比があり，且つ $L/\eta=2$ 位の所で振幅が共振的に大きくなり，之が普通の意味に於けるレーレー波の卓越周期を與へる. 波長が相當長くなると殆ど水平の振幅のみとなる. 尙，面白い事は波長の短い所では波の orbit が普通のレーレー波の通りであるが，長い所では逆になり，水の波と同じ傾向を取るものである.

第２の分散曲線については波長の短い所が第１の分散曲線の波長の長い場合と同じ傾向を取り波長の長い所がその反對になる. 且つこの波長の長い所で固體波の共振の如き性質が存在する. 何れの場合にも波長が非常に長くなると水平，鉛直何れの振幅も零に近くなる.