6. *The Possibility of Determining the Geoid by means of Levelling.*

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1. We have two methods of determining the height of one triangulation point relative to another. While the relative height is measured by trigonometrical levelling, those of successive triangulation points are determined by measuring the zenith distances from one another. In the latter case, by correcting the deviation of the vertical for each point, the relative height thus determined will show the height referred to the ellipsoid. At the same time, the relative height can also be determined with spirit levels, the height thus obtained being that referred to the geoid.

The difference between the relative heights determined by these two methods is therefore the height of the geoid above the reference ellipsoid.

The reason the foregoing methods for determining the height of the geoid are considered impractical, is that the height of the geoid may be masked by the effects of atmospheric refraction.

The writer however found that these methods could be used for a certain place where the form of the geoid deviated markedly from the reference ellipsoid.

The writer's studies in these respects will now be described.

2. THE DIFFERENCE IN HEIGHT BETWEEN TWO TRIANGULATION POINTS DETERMINED BY TRIGONOMETRICAL LEVELLING, THE DEVIATION OF THE VERTICAL BEING TAKEN INTO CONSIDERATION[1].

To determine the relative height between triangulation points P_1 and P_2, the zenith distance of P_2, Z_{12}, from the zenith of P_1, and that of P_1, Z_{21}, from the zenith of P_2, are measured. When the two points P_1, P_2 are situated on different level surfaces above the reference ellipsoid, Z_{12} and Z_{21} are given by

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1) F. R. HELMERT, *Höhere Geodäsie*, II, 574.

$$\left.\begin{aligned} Z_{12} &= Z'_{12} + \xi_1 \cos a_{12} + \eta_1 \sin a_{12} \\ Z_{21} &= Z'_{21} + \xi_2 \cos a_{21} + \eta_2 \sin a_{21}, \end{aligned}\right\} \tag{1}$$

where

Z'_{12}, Z'_{21} = the observed values of the zenith distances, i.e., the zenith distances referred to the true vertical at each of the points, P_1 and P_2.

ξ_1, ξ_2 = the meridian components of the deviation of the vertical at P_1 and P_2 (Ast.—Geo.).

η_1, η_2 = the prime vertical components of the deviation of the vertical at P_1 and P_2 (Ast.—Geo.).

a_{12}, a_{21} = the azimuths of the sight lines $\overrightarrow{P_1 \ P_2}$, $\overrightarrow{P_2 \ P_1}$ referred to N.

When the triangulation points P_1, P_2 are sighted, the one from the other, the relative height h between P_1 and P_2 on the reference ellipsoid is given in terms of the observed zenith distances in the form

$$h = \mathfrak{H}_2 - \mathfrak{H}_1 = S_m \tan\frac{Z_{21} - Z_{12}}{2} + \frac{k_2 - k_1}{4}\frac{S_m^2}{\rho_{12}}\sec^2\frac{Z_{21} - Z_{12}}{2}. \tag{2}$$

In the above formula (2),

$\mathfrak{H}_1$, $\mathfrak{H}_2$ = the heights of the triangulation points P_1, P_2 above the reference ellipsoid,

S_m $= S_0\left(1 + \dfrac{\mathfrak{H}_1 + \mathfrak{H}_2}{2\rho_{12}}\right)$, where S_0 is the horizontal distance between P_1 and P_2 measured on the surface of the reference ellipsoid.

ρ_{12} = the radius of curvature of the arc $\widehat{P_1 \ P_2}$, the arc of the geodetic line on the reference ellipsoid joining the points at which the straight lines passing through P_1, P_2 normally intersect the surface of the ellipsoid.

k_1, k_2 = the coefficients of atmospheric refraction for the sight lines $\overrightarrow{P_1 \ P_2}$ and $\overrightarrow{P_2 \ P_1}$, which, for simplicity, are assumed to be equal, that is,

$$k_1 = k_2,$$

whence we have from equation (2),

$$h = S_m \tan\frac{Z_{21} - Z_{12}}{2}, \tag{3}$$

$$\mathfrak{H}_2 = \mathfrak{H}_1 + S_m \tan\frac{Z_{21} - Z_{12}}{2}. \tag{4}$$

The height determined with the aid of equation (4) is that referred to the reference ellipsoid. Since the height measured by spirit levels is that referred to the geoid, by subtracting the height of the same triangulation point measured with spirit levels from the height thus determined, we get the height of the geoid above the reference ellipsoid.

Let N be the height of the geoid above the reference ellipsoid, and H the height of the triangulation point under consideration above the geoid, we then have

$$N = \mathfrak{H} - H. \tag{5}$$

3. THE RELATIVE HEIGHT MEASURED BY TRIGONOMETRICAL LEVELLING WITHOUT THE DEVIATION OF THE VERTICAL BEING TAKEN INTO CONSIDERATION[2].

If, in equation (1), the deviation of the vertical is excluded, the equation (3) becomes

$$h = S_m \tan \frac{Z'_{21} - Z'_{12}}{2}. \tag{6}$$

The relative height determined with equation (6) is that referred to the reference ellipsoid. The relative height determined by using the data of another pair of triangulation points is the relative height referred to another ellipsoid, although the orientation of the last-named differs

If, in such a case $\overrightarrow{P_1 \quad P_2}$ be denoted by S, the difference between the relative height referred to the ellipsoid and that referred to the geoid is given by

$$-\int \frac{\Lambda''}{\rho''} dS, \tag{7}$$

where Λ is the deviation of the vertical, the angle between the tangent of the level surface and that of the ellipsoid being measured counter-clockwise with reference to the direction of the tangent of the level surface.

As to the pair of triangulation points for which the relative height is measured, we have

$$\Lambda_1 + \Lambda_2 = 0. \tag{8}$$

If Λ is a linear function of S, as given by $\Lambda = \lambda + \lambda' S$, within the region between P_1 an P_2, the value of (7) becomes 0. Should, however, Λ be a function of S of a higher order, it usually does not be-

2) F. R. HELMERT, *Höhere Geodäsie*, II, 607.

come 0. For example, if Λ is $\lambda+\lambda' S+\lambda'' S^2$, the result of integral (7) is $\frac{1}{6}\frac{\lambda''}{\rho''}S^3$.

In trigonometrical levelling of the Military Land Survey, the length of the sight line is about 4 km. For such a short distance the variation in Λ may be assumed to be a linear function of S, approximately, whence the integral

$$-\int\frac{\Lambda''}{\rho''}dS=h-(H_2-H_1)$$

may consequently be 0, i. e., the height measured by trigonometrical levelling excluding the deviation of the vertical, may be regarded as being equal to that measured by spirit levels, provided the distance between the triangulation points is sufficiently short.

4. Numerical Example.

In the trigomometrical levellings of the Military Land Survey, the height of the triangulation point is determined with reference to the height of a certain bench-mark on the precise level route, that part of the survey connecting the triangulation point with the bench-mark on the precise level route being carried out by means of second order levels. The heights of the surrounding triangulation points are then determined by means of equation (6).

Heights of triangulation points over wide areas are determined not only by trigonometrical levelling, but also by spirit levels.

The writer at first tried to determine the relative height between two triangulation points separated by a long distance, by using data already obtained by trigonometrical levelling.

Fig. 1 shows the distribution of triangulation points between Mitaka, west of Tokyo, and Hatukari-mura, about 30 km east of Kôhu. Table I gives the zenith distances Z_{12}, Z_{21}, the meridian and prime vertical components of the deviations of the verticals ξ, η, the direct azimuth of the sight line a, and the distances between successive points, in which the values of Z are those observed, while the values of ξ, η, and a have been estimated from Figs, 1, 2, 3. The relative height between these two triangulation points was determined by using the data of relative heights for successive triangulation points distributed between these two points, that were obtained by trigonometrical levelling.

In Table II, the numerals in column I are the heights determined with equation (3), for which the values of the deviation of the vertical were estimated from their geographical distribution as shown in

Figs. 2, 3 as mentioned above. The numerical values in column II are the heights determined with equation (6). As standard datum for the heights given in columns I, II, in Table II, the height of the point at Mitaka, which was obtained by second order levelling, was taken. The heights given in columns I, II are those determined by means of trigonometrical levelling with reference to the height of the triangulation point at Mitaka. The values of the height given in column III comprise several (marked *) that were determined by spirit levels, the others having been determined by trigonometrical levelling with reference to certain points whose heights are known.

The differences (I–III) are $N=\mathfrak{H}-H$, namely, the heights of the geoid above the reference ellipsoid; while the differences (II–III) are errors that may arise from observations, deviations from the assumption $k_1=k_2$, etc.

The amount of error permitted in trigonometrical levelling by the Military Land Survey is so small that the discordance between the heights of a certain triangulation point determind by trigonometrical levelling from two other points should not exceed 20 cm. Seeing that the error here falls short of the said limit, the results of trigonometrical levelling for the triangulation points under consideration may be taken as being fairly precise.

Heretofore, the effect of atmospheric refraction was believed to be so

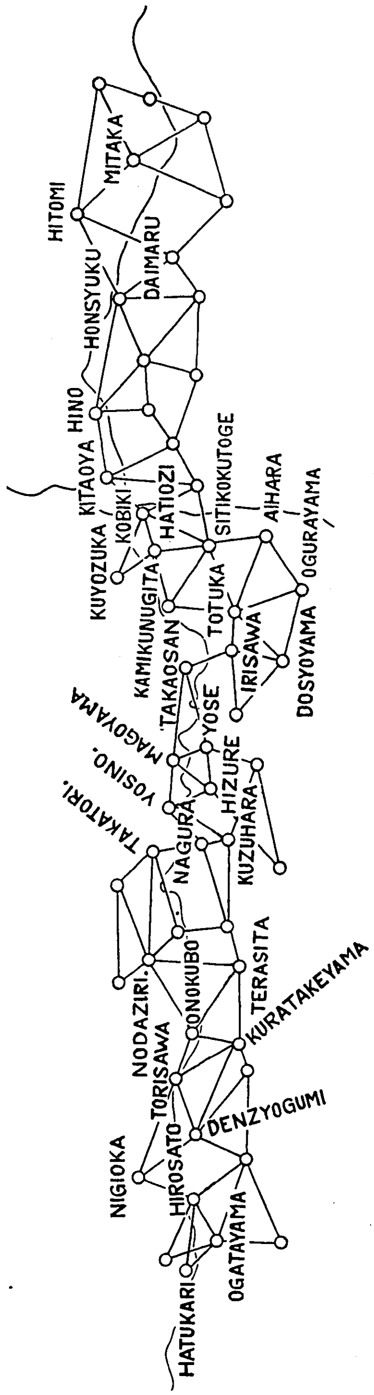

Fig. 1.

TABLE I.

	Z_{12}			Z_{21}			ξ_{a-g}	η_{a-g}	α_{12}	log s
	°	′	″	°	′	″	″	″	°	
Mitaka	89	38	31	90	23	49	+0·4	+ 5·9	312	3,56254
Hitomi	89	28	52	90	33	43	+1·0	+ 4·0	110	3,64486
Daimaru	90	53	25	89	9	27	+1·5	+ 6·4	319	3,61428
Honsyuku	89	43	31	90	19	16	+2·2	+ 4·8	278	3,78317
Hino	89	34	55	90	26	44	+5·3	+ 4·0	255	3,51775
Kitaoya	89	28	40	90	32	57	+6·9	+ 4·0	218	3,49427
Kobiki	89	7	31	90	54	25	+7·4	+ 5·6	201	3,61440
Sitikokutoge	90	46	41	89	14	56	+7·7	+ 8·8	351	3,54323
Hatiozi	89	42	11	90	18	59	+8·0	+ 5·6	318	3,37492
Kuyozuka	88	8	39	91	52	55	+8·3	+ 4·0	205	3,50712
Kamikunugita	89	5	31	90	56	17	+8·1	+ 5·6	180	3,55294
Totuka	92	35	10	87	27	10	+7·8	+ 8·8	110	3,65909
Aiharamura	87	3	45	92	57	49	+7·2	+11·2	233	3,54364
Ogurayama	89	23	57	90	37	47	+7·3	+12·0	280	3,60504
Dosyoyama	87	55	1	92	6	21	+7·1	+10·4	9	3,45574
Irisawa	87	15	9	92	46	5	+7·5	+ 8·8	335	3,42958
Takaosan	92	38	16	87	23	50	+7·8	+ 6·0	250	3,63741
Yose	86	2	31	93	58	18	+7·0	+ 6·4	330	3,29564
Magoyama	93	35	29	86	25	33	+7·0	+ 4·8	212	3,42976
Hizuremura	87	20	38	92	40	42	+6·3	+ 6·4	335	3,45041
Yosinoeki	91	51	14	88	10	25	+6·8	+ 4·0	207	3,59758
Kuzuharamura	91	40	10	88	20	49	+6·3	+ 5·6	354	3,15776
Naguramura	87	11	54	92	49	34	+6·4	+ 4·8	350	3,44002
Takatoriyama	90	42	46	89	20	11	+6·6	+ 3·2	270	3,76294
Nodazirieki	84	50	50	95	11	37	+6·0	+ 2·4	189	3,70337
Terasitamura	94	17	43	85	44	41	+5·4	+ 4·8	302	3,65454
Onokubomura	80	11	15	99	50	24	+5·4	+ 3·2	190	3,43099
Kuratakeyama	98	21	3	81	40	53	+5·0	+ 4·0	328	3,58834
Torisawa	85	18	6	94	43	22	+5·1	+ 1·6	248	3,48503
Denzyogumi	89	25	43	90	35	56	+4·9	+ 1·6	319	3,58882
Nigiokamura	86	23	20	93	38	0	+5·0	+ 0·8	196	3,50128
Hirosatomura	80	21	10	99	40	16	+4·5	+ 0·8	240	3,43454
Ogatayama	99	53	19	80	7	52	+4·2	+ 0·8	315	3,39324
Hatukarimura							+4·2	+ 0·0		

great that it would practically be impossible[3] to determine the form of the geoid from levellings, but from data obtained by means of trigonometrical and spirit levels, the effect from this cause does not seem to be so serious as to prevent determination of the height of the geoi-

3) F. R. HELMERT, *Höhere Geodäsie*, I, 521; II, 552.

TABLE II.

	I m	II m	III m	I-III m	II-III m
Mitaka	70·39 + 24·12	70·39	70·39*	0·00	0·00
Hitomi	94·51 + 41·56	94·42	94·42	+0·09	0·00
Daimaru	136·07 − 61·56	136·19	136·34	−0·27	−0·15
Honsyuku	74·51 + 31·66	74·56	74·73	−0·22	−0·17
Hino	106·17 + 24·92	106·10	106·41*	−0·24	−0·31
Kitaoya	131·09 + 29·31	131·00	131·09*	0·00	−0·09
Kobiki	160·40 + 64·19	160·32	160·27*	+0·13	+0·05
Sitikokutoge	224·59 − 46·72	224·21	224·24	+0·35	−0·03
Hatiozi	177·87 + 12·66	177·61	177·64	+0·23	−0·03
Kuyozuka	190·53 +105·51	190·32	190·24	+0·29	+0·08
Kamikunugita	296·04 + 57·42	295·15	295·18	+0·86	−0·03
Totuka	353·46 −204·64	352·74	352·72	+0·74	+0·02
Aiharamura	148·82 +180·48	148·28	148·28	+0·54	0·00
Ogurayama	329·30 + 43·47	328·50	328·44	+0·86	+0·06
Dosyoyama	372·77 +104·34	371·71	371·66	+1·11	+0·05
Irisawa	477·11 +129·49	476·15	476·10*	+1·01	+0·05
Takaosan	606·60 −198·41	605·68	605·66	+0·94	+0·02
Yose	408·19 +136·89	407·08	407·09*	+1·10	−0·01
Magoyama	545·08 −168.36	544·01	544·00	+1·08	+0·01
Hizuremura	376·78 +131·37	375·56	375·57	+1·21	−0·01
Yosinoeki	508·15 −126·94	507·02	507·01	+1·16	+0·01
Kuzuharamura	381·21 − 41·75	379·80	379·80	+1·41	0·00
Naguramura	339·46 +135·31	338·09	338·24	+1·22	−0·13

(to be continued.)

TABLE II. (*continued.*)

	I m	II m	III m	I-III m	II-III m
Takatoriyama	474·77 − 69·52	473·48	473·54	+1·23	−0·06
Nodazirieki	405·25 +457·49	403·85	403·94*	+1·31	−0·09
Terasitamura	862·74 −337·52	861·20	861·34	+1·40	−0·14
Onokubomura	525·31 +467·37	523·83	523·96*	+1·35	−0·13
Kuratakeyama	992·68 −568·76	991·10	991·18	+1·50	−0·08
Torisawa	423·92 +251·83	423·23	423·29	+0·64	−0·06
Denzyogumi	675·75 + 39·57	674·98	675·08	+0·67	−0·10
Nigiokamura	715·32 −201·21	714·56	714·71	+0·61	−0·15
Hirosatomura	514·11 +463·00	513·82	514·00*	+0·11	−0·18
Ogatayama	977·11 −430·81	976·66	976·93	+0·18	−0·27
Hatukarimura	546·30	546·01	546·22*	+0·08	−0·21

Heights given in this Table are those of the centre of the vertical circle of the theodolite at the stations.

The numerals with signs in column I denote the relative heights between the successive points.

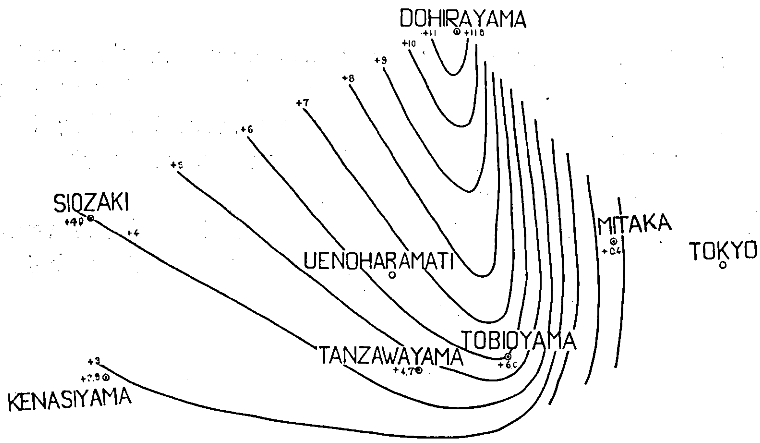

Fig. 2. Distribution of ξ.

dal surface above the surface of the reference ellipsoid. In fact, the

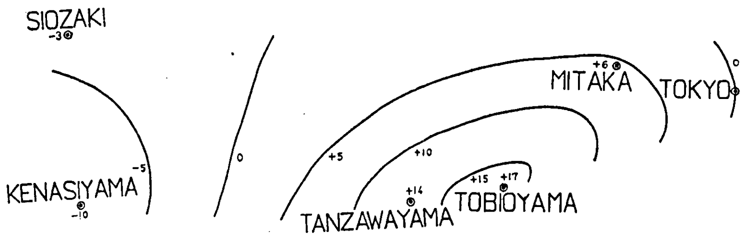

Fig. 3. Distribution of η.

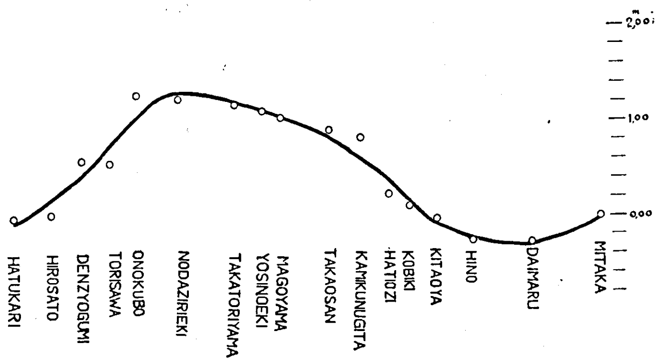

Fig. 4. Profile of Geoid.

result of III–I in Table I (see Fig. 4), agree well with the relative height of the geoidal surface in the region under consideration as obtained by Y. Kawahata[4], which were computed from astro-geodetic data.

5. Resume.

The results of the foregoing study may be summarized as follows:

(i) The height of the triangulation point as determined by the Military Land Survey by means of trigonometrical levelling is of the same kind of height as that measured by means of precise levelling, that is, it is the height above the geoid.

(ii) By taking into consideration the deviation of the vertical, it is possible to determine the form of the geoid from data of heights obtained by trigonometrical and spirit levellings.

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4) Y. Kawahata, *Bull. Earthq. Res. Inst.*, **13** (1935), 78.

6. 水準測量によるヂオイドの形の決定に就て

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直接水準測量及び三角術的間接水準測量結果を比較して，基準楕圓體上ヂオイドの高さを決定し得る事は既に知られてゐるが，實際には視準時に於ける空氣中の光の屈折の影響が不明であるが爲に，かゝる方法によつてヂオイドの形を決定せんとする試みは未だ無いやうである.

筆者は，測量部の觀測値に就き，兩水準測量の結果を比較して見た處，視準線の長さが 4km 程度で天頂距離が兩點より相互に觀視されてゐる場合には，屈折系數 $k_1=k_2$ として，充分ヂオイドの高さを推算し得る見込のあることを知つた. この場合各點に於ける鉛直線偏倚は挿入法により求めた.

また測量部の間接水準測量による高さは，直接水準測量結果卽ちヂオイド面からの高さと見做して差支へない事が分つた.