52. *Relation between the Thickness of a Surface Layer and the Amplitudes of Dispersive Rayleigh-waves.*

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1. *Introduction.*

Stoneley[1] recently showed that the dispersive Rayleigh-waves of minimum group velocity are likely to predominate in seismograms, the dispersion formula as well as the idea of predominance of waves of certain periods in his case being rather approximate. We shall here attack the same problem from a more appropriate standpoint even though the case in discussion should turn out to be of a rather special kind. As to the dispersion curves for the various conditions, we have already given mathematically accurate results.[2] [3] Although it is possible from these results to determine the amplitudes of waves for any given disturbance, since the numerical calculation is generally extremely difficult, the problem will be restricted here to that case in which the elastic constants of the medium underneath the surface layer are extremely large. The velocity of transmission and the amplitude distribution of the waves in the layer have however already been found.[4] Although the present problem is likely to correspond to that for a thin surface layer of the ground, the result still appears to apply to the case of a thick layer a few hundred kilometer deep, at any rate qualitatively. Our result furthermore shows that the idea of minimum group velocity as applied to the vibrational amplitudes is fairly applicable even in our special case.

Dispersive Rayleigh-waves generated from a source.

We shall now deal with a three-dimensional problem. Let the

1) R. STONELEY, "Surface-waves associated with 20° Discontinuity," *M. N. R. A. S. Geophys. Suppl.*, **4** (1937), 39~43.

2) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **3** (1927), 1.

3) K. SEZAWA and K. KANAI, "Discontinuity in Dispersion Curves of Rayleigh-waves," *Bull. Earthq. Res. Inst.*, **13** (1935), 273~244.

4) K. SEZAWA and K. KANAI, "On the Propagation of Waves along a Surface Stratum of the Earth," *Bull. Earthq. Res. Inst.*, **12** (1934), 263~268.

thickness of the stratum and the depth of the seismic origin from the free surface be η and $\eta-\xi$ respectively, and let the density and the elastic constants of the layer be ρ, λ, μ, the axes of r, z being taken as shown in the sketch.

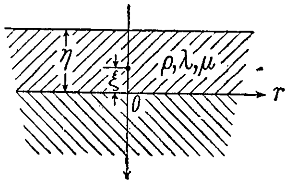

Fig. 1.

If U be the displacement due to the original disturbance, then

$$U=-\frac{1}{h^2}\frac{\partial \Delta_0}{\partial R}=-\frac{1}{h^2}\frac{\partial}{\partial R}\left(e^{-ipt}\frac{he^{ihR}}{R}\right), \quad (1)$$

its horizontal and vertical components being

$$u=-\frac{1}{h^2}\frac{\partial \Delta_0}{\partial r}, \quad w=-\frac{1}{h^2}\frac{\partial \Delta_0}{\partial z}, \quad (2)$$

where $h^2=\rho p^2/(\lambda+2\mu)$, $R^2=r^2+z^2$.

The primary disturbance Δ_0 may be written in such forms as

$$\Delta_0=e^{-ipt}h\int_0^\infty \frac{e^{\alpha(z+\xi)}}{\alpha}J_0(fr)fdf, \qquad [z<-\xi] \quad (3)$$

$$\Delta_0=e^{-ipt}h\int_0^\infty \frac{e^{-\alpha(z+\xi)}}{\alpha}J_0(fr)fdf, \qquad [z>-\xi] \quad (4)$$

where $\alpha^2=f^2-h^2$. Adding the reflected waves at the free surface as well as at the boundary $z=0$, we have

$$\Delta_1=e^{-ipt}h\left[\int_0^\infty \frac{e^{\alpha(z+\xi)}}{\alpha}J_0(fr)fdf+\int_0^\infty \frac{e^{-\alpha(z+2\eta-\xi)}}{\alpha}J_0(fr)fdf + \int_0^\infty \frac{e^{\alpha(z-2\eta+\xi)}}{\alpha}J_0(fr)fdf+\ldots\ldots\right], \quad [z<-\xi] \quad (5)$$

$$\Delta_1=e^{-ipt}h\left[\int_0^\infty \frac{e^{-\alpha(z+2\eta-\xi)}}{\alpha}J_0(fr)fdf+\int_0^\infty \frac{e^{\alpha(z-2\eta+\xi)}}{\alpha}J_0(fr)fdf + \ldots\ldots\right], \quad [z>-\xi] \quad (6)$$

corresponding to the primary waves Δ_0 for $z<-\xi$ shown in (3), and

$$\Delta_2=e^{-ipt}h\left[\int_0^\infty \frac{e^{\alpha(z-\xi)}}{\alpha}J_0(fr)fdf+\int \frac{e^{-\alpha(z+2\eta+\xi)}}{\alpha}J_0(fr)fdf+\ldots\ldots\right], \quad [z<-\xi] \quad (7)$$

$$\Delta_2 = e^{-ipt}h\left[\int_0^\infty \frac{e^{-\alpha(z+\xi)}}{\alpha}J_0(fr)fdf + \int_0^\infty \frac{e^{\alpha(z-\xi)}}{\alpha}J_0(fr)fdf + \int_0^\infty \frac{e^{-\alpha(z+2\eta+\xi)}}{\alpha}J_0(fr)fdf + \ldots\ldots\right], \quad [z>-\xi] \quad (8)$$

corresponding to the primary waves Δ_0 for $z>-\xi$ shown is (4). Equations (5), (7) satisfy the condition $\partial w_0/\partial r + \partial u_0/\partial z = 0$ at $z=-\eta$ and (6), (8) the condition $w_0=0$ at $z=0$, where u_0, w_0 are displacement components corresponding to Δ.

Superposing (5) and (7),

$$\left.\begin{aligned} \Delta &= 2e^{-ipt}h\int_0^\infty \frac{\operatorname{ch}\alpha\xi \operatorname{ch}\alpha(\eta+z)}{\alpha \operatorname{sh}\alpha\eta}J_0(fr)fdf, \quad [z<-\xi] \\ \Delta &= 2e^{-ipt}h\int_0^\infty \frac{\operatorname{ch}\alpha z \operatorname{ch}\alpha(\eta-\xi)}{\alpha \operatorname{sh}\alpha\eta}J_0(fr)fdf. \quad [z>-\xi] \end{aligned}\right\} \quad (9)$$

It is possible to write the displacement components corresponding to Δ in (9), namely,

$$\left.\begin{aligned} u_0 &= \frac{2e^{-ipt}}{h}\int_0^\infty \frac{\operatorname{ch}\alpha\xi \operatorname{ch}\alpha(\eta+z)}{\alpha \operatorname{sh}\alpha\eta}J_1(fr)f^2df, \quad [z<-\xi] \\ u_0 &= \frac{2e^{-ipt}}{h}\int_0^\infty \frac{\operatorname{ch}\alpha z \operatorname{ch}\alpha(\eta-\xi)}{\alpha \operatorname{sh}\alpha\eta}J_1(fr)f^2df, \quad [z>-\xi] \end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned} w_0 &= \frac{-2e^{-ipt}}{h}\int_0^\infty \frac{\operatorname{ch}\alpha\xi \operatorname{sh}\alpha(\eta+z)}{\operatorname{sh}\alpha\eta}J_0(fr)fdf, \quad [z<-\xi] \\ w_0 &= \frac{-2e^{-ipt}}{h}\int_0^\infty \frac{\operatorname{sh}\alpha z \operatorname{ch}\alpha(\eta-\xi)}{\operatorname{sh}\alpha\eta}J_0(fr)fdf. \quad [z>-\xi] \end{aligned}\right\} \quad (11)$$

Although the condition of no tangential stress at the free surface and no displacement at the boundary $z=0$ are satisfied by Δ, u_0, w_0 in (9), (10), (11), the condition of zero normal stress on the free surface remains unsatisfied. For annulling the normal stress under consideration we take standing vibrations of types

$$\left.\begin{aligned} u_1 &= e^{-ipt}\frac{f}{h^2}(A \operatorname{ch}\alpha z + B \operatorname{sh}\alpha z)J_1(fr), \\ w_1 &= -e^{-ipt}\frac{\alpha}{h^2}(A \operatorname{sh}\alpha z + B \operatorname{ch}\alpha z)J_0(fr), \end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned} u_2 &= -e^{-ipt}\frac{\beta}{k^2}(C\,\mathrm{sh}\,\beta z + D\,\mathrm{ch}\,\beta z)J_1(fr), \\ w_2 &= e^{-ipt}\frac{f}{k^2}(C\,\mathrm{ch}\,\beta z + D\,\mathrm{sh}\,\beta z)J_0(fr), \end{aligned}\right\} \tag{13}$$

where $h^2=f^2-\alpha^2$, $k^2=f^2-\beta^2$, and then add u_1 and u_2 to u_0 in (10), w_1 and w_2 to w_0 in (11). The expressions in (12), (13) satisfy the vibratory equations of the layer. The respective superpositions of u_1, u_2, u_0 and w_1, w_2, w_0 do not seriously affect the condition of the original disturbance in (1).

The necessary boundary conditions are such that

$$z=0; \qquad w_1+w_2=0, \quad u_0+u_1+u_2=0, \tag{14), (15}$$

$$z=-\eta; \qquad \frac{\partial}{\partial r}(w_1+w_2)+\frac{\partial}{\partial z}(u_1+u_2)=0, \tag{16}$$

$$\lambda(\Delta+\Delta')+2\mu\frac{\partial}{\partial z}(w_0+w_1+w_2)=0, \tag{17}$$

where $\Delta'=\partial u_1/\partial r+u_1/r+\partial w_1/\partial z$. Substituting (9)~(13) in (14)~(17), we get

$$A\Phi=-\frac{2fh}{\alpha\,\mathrm{sh}\,\alpha\eta}\left[\left\{1-\frac{2f^2}{f^2+\beta^2}\mathrm{ch}\,\alpha\eta\,\mathrm{ch}\,\beta\eta+\frac{f^2+\beta^2}{2\alpha\beta}\mathrm{sh}\,\alpha\eta\,\mathrm{sh}\,\beta\eta\right\}\mathrm{ch}\,\alpha(\eta-\xi)\right.$$
$$\left.+\left\{\mathrm{ch}\,\alpha\eta-\frac{f^2+\beta^2}{2f^2}\mathrm{ch}\,\beta\eta\right\}\mathrm{ch}\,\alpha\xi\right],$$

$$B\Phi=-\frac{2fh}{\alpha\,\mathrm{sh}\,\alpha\eta}\left[\left\{\frac{f^2+\beta^2}{2\alpha\beta}\mathrm{ch}\,\alpha\eta\,\mathrm{sh}\,\beta\eta-\frac{2f^2}{f^2+\beta^2}\mathrm{sh}\,\alpha\eta\,\mathrm{ch}\,\beta\eta\right\}\mathrm{ch}\,\alpha(\eta-\xi)\right.$$
$$\left.+\left\{\mathrm{sh}\,\alpha\eta-\frac{f^2+\beta^2}{2\alpha\beta}\mathrm{sh}\,\beta\eta\right\}\mathrm{ch}\,\alpha\xi\right],$$

$$C\Phi=\frac{\alpha k^2}{fh^2}B\Phi,$$

$$D\Phi=\frac{2f^2k^2}{\alpha\beta h\,\mathrm{sh}\,\alpha\eta}\left[\left\{1-\frac{f^2+\beta^2}{2f^2}\mathrm{ch}\,\alpha\eta\,\mathrm{ch}\,\beta\eta+\frac{2\alpha\beta}{f^2+\beta^2}\mathrm{sh}\,\alpha\eta\,\mathrm{sh}\,\beta\eta\right\}\mathrm{ch}\,\alpha(\eta-\xi)\right.$$
$$\left.-\left\{\mathrm{ch}\,\alpha\eta-\frac{f^2+\beta^2}{2f^2}\mathrm{ch}\,\beta\eta\right\}\mathrm{ch}\,\alpha\xi\right],$$

(18)

where

$$\Phi=2-\left\{\frac{f^2+\beta^2}{2f^2}+\frac{2f^2}{f^2+\beta^2}\right\}\operatorname{ch}\alpha\eta\operatorname{ch}\beta\eta+\left\{\frac{f^2+\beta^2}{2\alpha\beta}+\frac{2\alpha\beta}{f^2+\beta^2}\right\}\operatorname{sh}\alpha\eta\operatorname{sh}\beta\eta, \tag{19}$$

by means of which the resulting displacements on the free surface $z=-\eta$ are expressed by

$$u_{z=-\eta}=e^{-ipt}\int_0^\infty J_1(fr)\frac{k^2}{\Phi\alpha h\operatorname{sh}\alpha\eta}\left[\left\{1-\frac{2f^2}{f^2+\beta^2}\operatorname{ch}\alpha\eta\operatorname{ch}\beta\eta\right.\right.$$

$$\left.\left.+\frac{2\alpha\beta}{f^2+\beta^2}\operatorname{sh}\alpha\eta\operatorname{sh}\beta\eta\right\}\operatorname{ch}\alpha\xi-\left\{\operatorname{ch}\alpha\eta-\frac{2f^2}{f^2+\beta^2}\operatorname{ch}\beta\eta\right\}\operatorname{ch}\alpha(\eta-\xi)\right]df, \tag{20}$$

$$w_{z=-\eta}=e^{-ipt}\int_0^\infty J_0(fr)\frac{fk^2}{\Phi\alpha h\operatorname{sh}\alpha\eta}\left[\left\{\operatorname{ch}\alpha\eta\operatorname{sh}\beta\eta-\frac{\alpha\beta}{f^2}\operatorname{sh}\alpha\eta\operatorname{ch}\beta\eta\right\}\operatorname{ch}\alpha\xi\right.$$

$$\left.+\left\{\frac{2\alpha\beta}{f^2+\beta^2}\operatorname{sh}\alpha\eta-\operatorname{sh}\beta\eta\right\}\operatorname{ch}\alpha(\eta-\xi)\right]df. \tag{21}$$

The expression of Φ is virtually the same as the dispersion equation shown previously.[5]

In order to evaluate the expressions in (20), (21), we use the relations

$$\left.\begin{aligned}J_0(fr)&=\frac{-i}{\pi}\int_0^\infty(e^{ifr\operatorname{ch}j}-e^{-ifr\operatorname{ch}j})\,dj,\\ J_1(fr)&=\frac{-1}{\pi}\int_0^\infty(e^{ifr\operatorname{ch}j}+e^{-ifr\operatorname{ch}j})\operatorname{ch}j\,dj.\end{aligned}\right\} \tag{22}$$

Then

$$u_{z=-\eta}=\frac{-4k^2e^{-ipt}}{\pi h}\int_0^\infty\operatorname{ch}j\,dj\int_0^\infty\frac{1}{\Phi_1(f)}(e^{ifr\operatorname{ch}j}+e^{-ifr\operatorname{ch}j})$$

$$\cdot\left[\left\{e^{\xi\sqrt{f^2-h^2}}+e^{-\xi\sqrt{f^2-h^2}}-e^{(2\eta-\xi)\sqrt{f^2-h^2}}-e^{-(2\eta-\xi)\sqrt{f^2-h^2}}\right\}\right.$$

$$+\frac{1}{2f^2-k^2}\left\{(f^2+\sqrt{f^2-h^2}\sqrt{f^2-k^2})\,(e^{(\eta-\xi)\sqrt{f^2-h^2}+\eta\sqrt{f^2-k^2}}\right.$$

$$+e^{-(\eta-\xi)\sqrt{f^2-h^2}-\eta\sqrt{f^2-k^2}}-e^{-(\eta+\xi)\sqrt{f^2-h^2}+\eta\sqrt{f^2-k^2}}-e^{(\eta+\xi)\sqrt{f^2-h^2}-\eta\sqrt{f^2-k^2}})$$

5) K. SEZAWA and K. KANAI, *loc. cit.* 4).

$$+\left(f^2-\sqrt{f^2-h^2}\sqrt{f^2-k^2}\right)\left(e^{-(\eta-\xi)\sqrt{f^2-h^2}+\eta\sqrt{f^2-k^2}}\right.$$

$$+e^{(\eta-\xi)\sqrt{f^2-h^2}-\eta\sqrt{f^2-k^2}}-e^{(\eta+\xi)\sqrt{f^2-h^2}+\eta\sqrt{f^2-k^2}}$$

$$\left.\left.\left.-e^{-(\eta+\xi)\sqrt{f^2-h^2}-\eta\sqrt{f^2-k^2}}\right)\right\}\right]f^2\sqrt{f^2-k^2}(2f^2-k^2)\,df, \qquad (23)$$

$$w_{z=-\eta}=\frac{-2ik^2e^{-ipt}}{\pi h}\int_0^\infty dj\int_0^\infty\frac{1}{\Phi_1(f)}\left(e^{ifr\,\mathrm{ch}\,j}-e^{-ifr\,\mathrm{ch}\,j}\right)$$

$$\cdot\left[\left(f^2+\sqrt{f^2-h^2}\sqrt{f^2-k^2}\right)\left(-e^{(\eta+\xi)\sqrt{f^2-h^2}-\eta\sqrt{f^2-k^2}}\right.\right.$$

$$\left.+e^{-(\eta-\xi)\sqrt{f^2-h^2}-\eta\sqrt{f^2-k^2}}-e^{(\eta-\xi)\sqrt{f^2-h^2}+\eta\sqrt{f^2-k^2}}+e^{-(\eta+\xi)\sqrt{f^2-h^2}+\eta\sqrt{f^2-k^2}}\right)$$

$$+\left(f^2-\sqrt{f^2-h^2}\sqrt{f^2-k^2}\right)\left(e^{(\eta+\xi)\sqrt{f^2-h^2}+\eta\sqrt{f^2-k^2}}\right.$$

$$\left.-e^{-(\eta-\xi)\sqrt{f^2-h^2}+\eta\sqrt{f^2-k^2}}+e^{(\eta-\xi)\sqrt{f^2-h^2}-\eta\sqrt{f^2-k^2}}-e^{-(\eta+\xi)\sqrt{f^2-h^2}-\eta\sqrt{f^2-k^2}}\right)$$

$$+\frac{4f^2\sqrt{f^2-h^2}\sqrt{f^2-k^2}}{2f^2-k^2}\left(e^{(2\eta-\xi)\sqrt{f^2-h^2}}+e^{\xi\sqrt{f^2-h^2}}-e^{-\xi\sqrt{f^2-h^2}}\right.$$

$$\left.\left.-e^{-(2\eta-\xi)\sqrt{f^2-h^2}}\right)\right]f(2f^2-k^2)\,df, \qquad (24)$$

where

$$\Phi_1(f)=16f^2\sqrt{f^2-h^2}\sqrt{f^2-k^2}(2f^2-k^2)\left\{e^{\eta\sqrt{f^2-h^2}}-e^{-\eta\sqrt{f^2-h^2}}\right\}$$

$$+\left(f^2-\sqrt{f^2-h^2}\sqrt{f^2-k^2}\right)\left\{(2f^2-k^2)^2-4f^2\sqrt{f^2-h^2}\sqrt{f^2-k^2}\right\}$$

$$\cdot\left\{e^{\eta(2\sqrt{f^2-h^2}+\sqrt{f^2-k^2})}+e^{-\eta\sqrt{f^2-k^2}}-e^{\eta\sqrt{f^2-k^2}}-e^{-\eta(2\sqrt{f^2-h^2}+\sqrt{f^2-k^2})}\right\}. \qquad (25)$$

The expressions in (23), (24) give the displacements at any radial distance on the free surface.

3. *Evaluation of the integrals.*

We shall first evaluate the integrals

$$\int_0^\infty\frac{e^{ifr\,\mathrm{ch}\,j}}{\Phi_1(f)}F_1(f)\,df+\int_0^\infty\frac{e^{-ifr\,\mathrm{ch}\,j}}{\Phi_1(f)}F_1(f)\,df, \qquad (26)$$

$$\int_0^\infty\frac{e^{ifr\,\mathrm{ch}\,j}}{\Phi_1(f)}F_2(f)\,df+\int_0^\infty\frac{e^{-ifr\,\mathrm{ch}\,j}}{\Phi_1(f)}F_2(f)\,df, \qquad (27)$$

contained in (23), (24) respectively, $F_1(f)$ and $F_2(f)$ being the factors within the respective pairs of brackets in (23) and (24). To evaluate the respective first integrals in (26) and (27), we consider the integrals

$$\int_c \frac{e^{iZr\,\mathrm{ch}\,j}}{\Phi_1(Z)}F_1(Z)dZ,\ \int_c \frac{e^{iZr\,\mathrm{ch}\,j}}{\Phi_1(Z)}F_2(Z)dZ \qquad (28),\ (29)$$

taken round the contour in the first quadrant in Fig. 2, and also the integrals

$$\int_c \frac{e^{-iZr\,\mathrm{ch}\,j}}{\Phi_1(Z)}F_1(Z)dZ,\ \int_c \frac{e^{-iZr\,\mathrm{ch}\,j}}{\Phi_1(Z)}F_2(Z)dZ \qquad (30),\ (31)$$

taken round the contours in the fourth quadrant in the same figure.

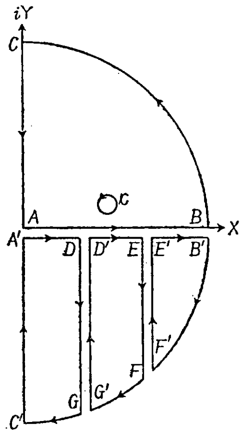

Fig. 2.

Let the roots of $\Phi_1(Z)$ be κ_n, the values of which were found previously.[6] κ_n are thus the singular points in the contours just given. It should however be borne in mind that there are usually a number of roots κ_n. Since, on the other hand, $Z=h$ and $Z=k$ are branch points, we draw the branch lines DG and EF as shown in the figure.

The integrals performed along the circular arcs BC and $B'C'$ vanish when the radii of these arcs are made infinitely large.

The integral of (28) along CA and that of (30) along $C'A'$ cancel each other. Similar conditions exist in (29) as well as in (30).

The integrals worked out along DG, $D'G'$, EF, $E'F'$ are somewhat complex. We write

$$\sqrt{(h+iY)^2-h^2}=\gamma', \qquad \sqrt{(h+iY)^2-k^2}=\delta' \qquad \text{along } DG,$$

$$\sqrt{(h+iY)^2-h^2}=-\gamma', \qquad \sqrt{(h+iY)^2-k^2}=\delta' \qquad \text{along } G'D',$$

$$\sqrt{(k+iY)^2-h^2}=\gamma'', \qquad \sqrt{(k+iY)^2-k^2}=\delta'' \qquad \text{along } EF,$$

$$\sqrt{(k+iY)^2-h^2}=\gamma'', \qquad \sqrt{(k+iY)^2-k^2}=-\delta'' \qquad \text{along } F'E'.$$

The integrals of (30) performed along DG and $G'D'$ then cancel each other, similar relations holding in the integrals of (31) with respect to EF and $F'E'$.

Finally the integrals around the pole κ_n for the cases $u_{z=-\eta}$ and $w_{z=-\eta}$ are

6) K. SEZAWA and K. KANAI, *loc. cit.* 4).

$$\frac{-2ik^2e^{i\kappa r\,\mathrm{ch}\,j}\mathrm{ch}\,j}{h\gamma\Phi'(\kappa)\,\mathrm{sh}\,\gamma\kappa\eta}\left[\left\{1-\frac{2}{1+\delta^2}\mathrm{ch}\,\gamma\kappa\eta\,\mathrm{ch}\,\delta\kappa\eta+\frac{2\gamma\delta}{1+\delta^2}\mathrm{sh}\,\gamma\kappa\eta\,\mathrm{sh}\,\delta\kappa\eta\right\}\mathrm{ch}\,\gamma\kappa\xi\right.$$

$$\left.+\left\{\frac{2}{1+\delta^2}\mathrm{ch}\,\delta\kappa\eta-\mathrm{ch}\,\gamma\kappa\eta\right\}\mathrm{ch}\,\gamma\kappa(\eta-\xi)\right], \quad (32)$$

$$\frac{2k^2e^{i\kappa r\,\mathrm{ch}\,j}}{h\gamma\delta\Phi'(\kappa)\,\mathrm{sh}\,\gamma\kappa\eta}\left[\left\{\mathrm{ch}\,\gamma\kappa\eta\,\mathrm{sh}\,\delta\kappa\eta-\gamma\delta\,\mathrm{sh}\,\gamma\kappa\eta\,\mathrm{ch}\,\delta\kappa\eta\right\}\mathrm{ch}\,\gamma\kappa\xi\right.$$

$$\left.+\left\{\frac{2\gamma\delta}{1+\delta^2}\mathrm{sh}\,\gamma\kappa\eta-\mathrm{sh}\,\delta\kappa\eta\right\}\mathrm{ch}\,\gamma\kappa(\eta-\xi)\right], \quad (33)$$

where κ is written in lieu of κ_n, and

$$\Phi'(\kappa)=\left(\frac{k}{\kappa}\right)^2\left\{\frac{4}{(1+\delta^2)^2}-1\right\}\mathrm{ch}\,\gamma\kappa\eta\,\mathrm{ch}\,\delta\kappa\eta$$

$$+2\gamma\delta\left\{4-(1+\delta^2)\left(\frac{1}{\gamma^2}+\frac{1}{\delta^2}\right)\right\}\left\{\frac{1}{4\gamma^2\delta^2}-\frac{1}{(1+\delta^2)^2}\right\}\mathrm{sh}\,\gamma\kappa\eta\,\mathrm{sh}\,\delta\kappa\eta$$

$$+\frac{\kappa\eta}{\delta}\left\{\frac{\left(\frac{h}{\kappa}\right)^2(1+\delta^2)}{2\gamma^2}-\frac{2\left(\frac{k}{\kappa}\right)^2}{1+\delta^2}\right\}\mathrm{ch}\,\gamma\kappa\eta\,\mathrm{sh}\,\delta\kappa\eta$$

$$+\frac{\kappa\eta}{\gamma}\left\{\frac{\left(\frac{k}{\kappa}\right)^2(1+\delta^2)}{2\delta^2}-\frac{2\left(\frac{h}{\kappa}\right)^2}{1+\delta^2}\right\}\mathrm{sh}\,\gamma\kappa\eta\,\mathrm{ch}\,\delta\kappa\eta, \quad (34)$$

$$\gamma=\sqrt{1-(h/\kappa)^2}, \quad \delta=\sqrt{1-(k/\kappa)^2}. \quad (35)$$

In the present case use has been made of Φ instead of Φ_1.

Since it is known that

$$\left.\begin{aligned} H_0^{(1)}(\kappa r)&=\frac{-2i}{\pi}\int_0^\infty e^{i\kappa r\,\mathrm{ch}\,j}dj,\\ H_1^{(1)}(\kappa r)&=\frac{-2}{\pi}\int_0^\infty e^{i\kappa r\,\mathrm{ch}\,j}\mathrm{ch}\,j\,dj,\end{aligned}\right\} \quad (36)$$

we finally get

$$u_{z=-\eta}=\frac{i\pi}{\eta}e^{-ipt}H_1^{(1)}(\kappa r)\frac{k^2\eta}{h\gamma\Phi'(\kappa)\,\mathrm{sh}\,\gamma\kappa\eta}\left[\left\{1-\frac{2}{1+\delta^2}\mathrm{ch}\,\gamma\kappa\eta\,\mathrm{ch}\,\delta\kappa\eta\right.\right.$$
$$\left.+\frac{2\gamma\delta}{1+\delta^2}\mathrm{sh}\,\gamma\kappa\eta\,\mathrm{sh}\,\delta\kappa\eta\right\}\mathrm{ch}\,\gamma\kappa\xi+\left.\left\{\frac{2}{1+\delta^2}\mathrm{ch}\,\delta\kappa\eta-\mathrm{ch}\,\gamma\kappa\eta\right\}\mathrm{ch}\,\gamma\kappa(\eta-\xi)\right], \quad (37)$$

$$w_{z=-\eta}=\frac{i\pi}{\eta}e^{-ipt}H_0^{(1)}(\kappa r)\frac{k^2\eta}{h\gamma\Phi'(\kappa)\,\mathrm{sh}\,\gamma\kappa\eta}\left[\frac{1}{\delta}\left\{\mathrm{ch}\,\gamma\kappa\eta\,\mathrm{sh}\,\delta\kappa\eta\right.\right.$$
$$\left.-\gamma\delta\,\mathrm{sh}\,\gamma\kappa\eta\,\mathrm{ch}\,\delta\kappa\eta\right\}\mathrm{ch}\,\gamma\kappa\xi+\left.\frac{1}{\delta}\left\{\frac{2\gamma\delta}{1+\delta^2}\mathrm{sh}\,\gamma\kappa\eta-\mathrm{sh}\,\delta\kappa\eta\right\}\mathrm{ch}\,\gamma\kappa(\eta-\xi)\right]. \quad (38)$$

When there are a number of κ_n, the superposed expressions of the above types (without any further coefficient) are used.

Taking the asymptotic expansions of $H_0^{(1)}(\kappa r)$, $H_1^{(1)}(\kappa r)$ and also simplifying the resulting expressions, we get

$$u_{z=-\eta}=\left[\frac{1}{\sqrt{r\eta}}e^{i(\chi r-\frac{\pi}{4}-pt)}\right]\frac{1}{\Phi'(\kappa)}\sqrt{\frac{2\pi}{\kappa\eta}}\frac{k^2\eta}{h\gamma}\left[\left\{\frac{2\gamma\delta}{1+\delta^2}\mathrm{sh}\,\beta\eta-\mathrm{sh}\,\alpha\eta\right\}\mathrm{ch}\,\alpha\xi\right.$$
$$\left.+\left\{\mathrm{ch}\,\alpha\eta-\frac{2}{1+\delta^2}\mathrm{ch}\,\beta\eta\right\}\mathrm{sh}\,\alpha\xi\right], \quad (39)$$

$$w_{z=-\eta}=\left[\frac{1}{\sqrt{r\eta}}e^{i(\chi r+\frac{\pi}{4}-pt)}\right]\frac{1}{\Phi'(\kappa)}\sqrt{\frac{2\pi}{\kappa\eta}}\frac{k^2\eta}{h}\left[\left\{\frac{2}{1+\delta^2}\mathrm{ch}\,\alpha\eta-\mathrm{ch}\,\beta\eta\right\}\mathrm{ch}\,\alpha\xi\right.$$
$$\left.+\left\{\frac{1}{\gamma\delta}\mathrm{sh}\,\beta\eta-\frac{2}{1+\delta^2}\mathrm{sh}\,\alpha\eta\right\}\mathrm{sh}\,\alpha\xi\right]. \quad (40)$$

From (39), (40) it is possible to calculate the horizontal and vertical components of the waves for an assigned displacement at the source such as that shown in (1).

4. *Physical meaning of the integrals.*

As will be seen from (37), (38) or (39), (40), the movement of the ground for any epicentral distance consists only of surface waves. In contrast to this, Rayleigh-waves, in the case of a semi-infinite body, are accompanied by longitudinal and transverse bodily waves of amplitudes both varying as the inverse square of the epicentral distance, and Love-waves in the case of a stratified body by transverse bodily waves of velocity peculiar to the subjacent medium and of amplitudes also varying with the same law as for bodily waves in the case of Rayleigh-waves. The reason why no bodily wave appears in the present case is that, since the subjacent medium is not deformable, the whole vibrational energy in the layer is transmitted in the form of surface waves.

5. *Numerical computation of the mathematical formulae.*

The numerical calculation of the mathematical results in (39), (40) is not simple. For performing the calculation, the expressions within the respective pairs of brackets in (37), (38) will be transformed to

$$[\quad]\text{ for } u_{z=-\eta}=\operatorname{sh}\alpha\eta\operatorname{ch}\alpha\xi\left\{\operatorname{sh}\alpha\eta-\frac{2\alpha\beta}{\kappa^2+\beta^2}\operatorname{sh}\beta\eta\right\}\cdot\left[\frac{\operatorname{ch}\alpha\eta-\dfrac{2\kappa^2}{\kappa^2+\beta^2}\operatorname{ch}\beta\eta}{\operatorname{sh}\alpha\eta-\dfrac{2\alpha\beta}{\kappa^2+\beta^2}\operatorname{sh}\beta\eta}\operatorname{th}\alpha\xi-1\right], \tag{41}$$

$$[\quad]\text{ for } w_{z=-\eta}=-\frac{2\alpha\kappa}{\kappa^2+\beta^2}\operatorname{sh}\alpha\eta\operatorname{ch}\alpha\xi\left\{\operatorname{ch}\alpha\eta-\frac{\kappa^2+\beta^2}{2\kappa^2}\operatorname{ch}\beta\eta\right\}\cdot\left[\frac{\operatorname{sh}\alpha\eta-\dfrac{\kappa^2+\beta^2}{2\alpha\beta}\operatorname{sh}\beta\eta}{\operatorname{ch}\alpha\eta-\dfrac{\kappa^2+\beta^2}{2\kappa^2}\operatorname{ch}\beta\eta}\operatorname{th}\alpha\xi-1\right]. \tag{42}$$

Putting

$$\left.\begin{array}{c}
\dfrac{\operatorname{ch}\alpha\eta-\dfrac{2\kappa^2}{\kappa^2+\beta^2}\operatorname{ch}\beta\eta}{\operatorname{sh}\alpha\eta-\dfrac{2\alpha\beta}{\kappa^2+\beta^2}\operatorname{sh}\beta\eta}\operatorname{th}\alpha\xi-1=\mathrm{I},\quad \dfrac{\operatorname{sh}\alpha\eta-\dfrac{\kappa^2+\beta^2}{2\alpha\beta}\operatorname{sh}\beta\eta}{\operatorname{ch}\alpha\eta-\dfrac{\kappa^2+\beta^2}{2\kappa^2}\operatorname{ch}\beta\eta}\operatorname{th}\alpha\xi-1=\mathrm{I}',\\[2ex]
\operatorname{sh}\alpha\eta-\dfrac{2\alpha\beta}{\kappa^2+\beta^2}\operatorname{sh}\beta\eta=\mathrm{II},\quad \operatorname{ch}\alpha\eta-\dfrac{\kappa^2+\beta^2}{2\kappa^2}\operatorname{ch}\beta\eta=\mathrm{III},\\[2ex]
-\dfrac{2\alpha\kappa}{\kappa^2+\beta^2}=\mathrm{IV},\quad \alpha\eta=\mathrm{V},
\end{array}\right\} \tag{43}$$

then, by means of the volocity equation[7]

$$\frac{\operatorname{ch}\alpha\eta-\dfrac{\kappa^2+\beta^2}{2\kappa^2}\operatorname{ch}\beta\eta}{\operatorname{sh}\alpha\eta-\dfrac{\kappa^2+\beta^2}{2\alpha\beta}\operatorname{sh}\beta\eta}=\frac{\operatorname{sh}\alpha\eta-\dfrac{2\alpha\beta}{\kappa^2+\beta^2}\operatorname{sh}\beta\eta}{\operatorname{ch}\alpha\eta-\dfrac{2\kappa^2}{\kappa^2+\beta^2}\operatorname{ch}\beta\eta}, \tag{44}$$

we have I=I′. Thus

$$\Big[\quad\Big] \text{ for } u_{z=-\eta}=\text{VI.II.I}, \qquad \Big[\quad\Big] \text{ for } w_{z=-\eta}=\text{IV.VI.III.I}. \tag{45), (46}$$

Calculating I, II, III, IV, V, $\Phi'(\kappa)$, and the coefficients as shown in (39), (40), we get $u_{z=-\eta}$, $w_{z=-\eta}$. It should be remembered that the ratio of $u_{z=-\eta}$ to $w_{z=-\eta}$ is

$$\frac{u}{w}=-\frac{(\kappa^2+\beta^2)}{2\alpha\kappa}\frac{\left\{\operatorname{sh}\alpha\eta-\dfrac{2\alpha\beta}{\kappa^2+\beta^2}\operatorname{sh}\beta\eta\right\}}{\left\{\operatorname{ch}\alpha\eta-\dfrac{\kappa^2+\beta^2}{2\kappa^2}\operatorname{ch}\beta\eta\right\}}, \tag{47}$$

which is of the same form as obtained previously.[8]

In the present case we shall take $\xi=\eta$, namely, that the source of the disturbance is on the free surface. If we plot I, II, III, IV, V on the base of L/η where $L=2\pi/\kappa$, we get the result as shown in Fig. 3. It will be seen that the variations of the quantities just mentioned for different L/η's are very marked. Although by combining these quantities it is possible to calculate u, w for any L/η, there are still cases of L/η for which the calculation may be performed in some special way.

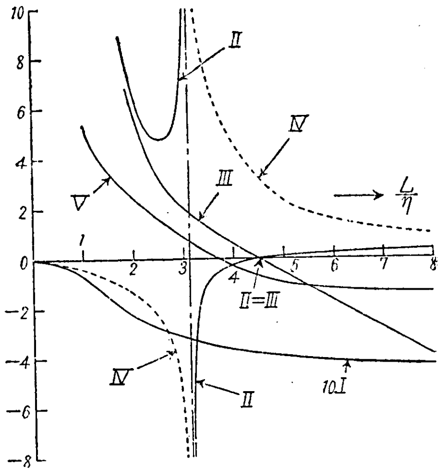

Fig. 3.

(i) In the case $\gamma=0$, namely $h^2=\kappa^2$, at which $L/\eta=3{\cdot}96$, the quantities of the orders $1/\gamma^2$ and $1/\gamma$ in $\Phi'(\kappa)$ are respectively zero, whereas the quantity of the order γ^0 is finite and equal to 1·11. Taking then

7) K. SEZAWA and K. KANAI, *loc. cit.* 4).
8) K. SEZAWA and K. KANAI, *loc. cit.* 4).

the numerators of the finite order in u, w we have

$$\left.\begin{aligned} u&=\frac{6\sqrt{2\pi\kappa\eta}}{\Phi'(\kappa)}(\sqrt{2}\sin\beta\eta+\kappa\eta\cos\beta\eta)=1{\cdot}964, \\ w&=\frac{3\sqrt{2\pi\kappa\eta}}{\Phi'(\kappa)}\left(-2-\cos\beta\eta+\frac{\kappa\eta}{\sqrt{2}}\sin\beta\eta\right)=-4{\cdot}265. \end{aligned}\right\} \tag{48}$$

(ii) In the case $\delta=0$, namely $k^2=\kappa^2$, at which $L/\eta=1{\cdot}65$, the quantity of the order $1/\delta^2$ in $\Phi'(\kappa)$ vanishes, whereas the quantity of the order δ^0 is

$$\frac{\operatorname{ch}\alpha\beta}{12}\left\{151-11(\kappa\eta)^2\right\}-\frac{1}{3}\left\{23+2(\kappa\eta)^2\right\}=-25{\cdot}137,$$

so that

$$\left.\begin{aligned} u&=\frac{-6\sqrt{\pi\kappa\eta}}{\Phi'(\kappa)}\operatorname{sh}\alpha\eta=9{\cdot}19, \\ w&=\frac{2\sqrt{6\pi\kappa\eta}}{\Phi'(\kappa)}(2\operatorname{ch}\alpha\eta-1)=-14{\cdot}4. \end{aligned}\right\} \tag{49}$$

(iii) In the case $1+\delta^2=0$, namely $(k/\kappa)^2=2$, at which $L/\eta=3{\cdot}12$, the numerators of u and w have the factors $1/(1+\delta^2)$, whereas $\Phi'(\kappa)$ is of the form

$$\Phi'(\kappa)=\frac{8}{(1+\delta^2)^2}\left(\operatorname{ch}\frac{\kappa\eta}{\sqrt{3}}\cos\kappa\eta+\frac{1}{\sqrt{3}}\operatorname{sh}\frac{\kappa\eta}{\sqrt{3}}\sin\kappa\eta\right),$$

the numerator of which, in virtue of the velocity equation, is zero. Since, from the condition of the present case, the denominator is also zero, we take the respective derivatives of the numerator as well as the denominator of $\Phi'(\kappa)$, the result being

$$\Phi'(\kappa)=\frac{4}{1+\delta^2}\left[-\frac{1}{\sqrt{3}}\operatorname{sh}\frac{\kappa\eta}{\sqrt{3}}\sin\kappa\eta-2{\cdot}014\operatorname{ch}\frac{\kappa\eta}{\sqrt{3}}\sin\kappa\eta \right.$$
$$\left. -\frac{2{\cdot}014}{\sqrt{3}}\operatorname{sh}\frac{\kappa\eta}{\sqrt{3}}\cos\kappa\eta\right]=\frac{1}{1+\delta^2}(-4{\cdot}910).$$

The expressions for u and w consequently become

$$\left.\begin{aligned} u&=\frac{1}{\Phi'(\kappa)}\frac{1}{(1+\delta^2)}\sqrt{6\pi\kappa\eta}\sqrt{6} \\ &\qquad\cdot\left[-\frac{2}{\sqrt{3}}\sin\beta\eta\,\mathrm{ch}\,\alpha\eta-2\cos\beta\eta\,\mathrm{sh}\,\alpha\eta\right]=1{\cdot}826, \\ w&=\frac{1}{\Phi'(\kappa)}\frac{1}{(1+\delta^2)}\sqrt{6\pi\kappa\eta}\sqrt{2}\,(2+0)=-3{\cdot}505. \end{aligned}\right\}\quad(50)$$

6. *Result of calculation and its interpretation.*

After carefully calculating other cases by means of (39), (40), we obtained the result as shown in Fig. 4, K in the same figure denoting the value in every first pair of brackets in (39), (40). The ratio of u/w for every L/η is always the same as that shown previously.[9]

It will be seen from the figure that within the range of calculation, the amplitude of the surface waves assumes its maximum values for two cases of L/η, namely, $L/\eta\approx2$ and $L/\eta\approx7$. While the maximum at $L/\eta\approx2$ is finite, that at $L/\eta\approx7$ is infinitely large.

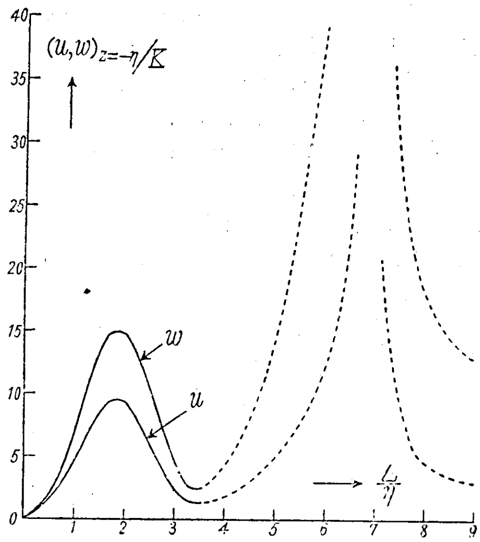

Fig. 4.

$L/\eta\approx2$, namely $\kappa\eta\approx\pi$, appears to correspond to the case in which the amplitudes of Rayleigh-waves become maximum in the usual sense. Stoneley,[10] using the idea of the minimum group velocity, showed that $\kappa\eta=3\sim2{\cdot}5$ corresponds to the case of maximum amplitude of Rayleigh-waves transmitted through a layer. Although the condition of the subjacent medium in his case differs considerably from ours, the nature of the problem, qualitatively, for the two cases, that is, Stoneley's and ours, seems the same. In the case of Love-waves transmitted over a surface layer as defined by Jeffreys, Stoneley's value[11] for $\kappa\eta$ that corresponds to the maximum amplitude, is 3·6, the value

9) K. SEZAWA and K. KANAI, *loc. cit.* 4).
10), 11) R. STONELEY, *loc. cit.* 1).

obtained by us being of the range 2∼3 (for $\xi>0$). It appears therefore that our value $\kappa\eta\gtrsim\pi(L/\eta\gtrsim2)$ for Rayleigh-waves is a plausible one for almost any condition of the subjacent medium. Furthermore, the case treated in the present paper concerns Rayleigh-waves of extremely shallow origin, that is to say, the origin is at the free surface. In the case of Love-waves, the greater the depth of origin, the greater the value of $\kappa\eta$ at which the amplitude of the waves is maximum, tends to increase. Therefore, the value of $\kappa\eta\gtrsim\pi$ in the case of Rayleigh-waves may still be somewhat too small.

To interpret the maximum of u, w at $L/\eta\gtrsim7$ is not a simple matter. Previously,[12] we found the dispersion curves of the case in which the elastic constants of the subjacent medium are extremely large, the curves under consideration being of the type in Fig. 5.

The dispersion curve AOB bends discontinuously at O. In the present case there are actually two dispersion curves, AOB and COD. When $\mu/\mu'=1/\infty$, as in the present case, part AO of curve AB joins part OD of curve CD, while, on the other hand, another part OB of curve AB joins another part CO of curve CD, the value of L/η at point O being 4·619. The infinitely large maximum value at $L/\eta\gtrsim7$ in Fig. 3 belongs, as a matter of fact, to the second dispersion curve COD. No numerical result for the case corresponding to curve OB is given in the present paper. It is probably inadmissible to conceive of waves for such parts as OB and OD in the dispersion curves. The movements of the ground for the case corresponding to OD represent the layer waves with mere transverse (vertical) displacements, and in those for the case corresponding to OB the layer waves of nearly longitudinal (horizontal) displacements, in both of which energy is hardly diffused in the subjacent medium. The infinitely large displacement just mentioned is thus of such a nature as to be but slightly related to the case of ordinary Rayleigh-waves, but are the results of a resonance-like condition of bodily waves.

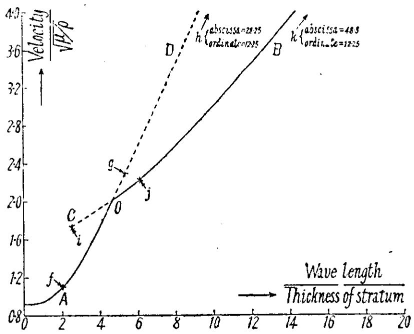

Fig. 5.

Finally, we may conclude that, although in the present paper we have dealt with a very special case of dispersive Rayleigh-waves, the

12) K. SEZAWA and K. KANAI, loc. cit. 3).

result still appears to represent some of the fundamental characters of the general dispersive Rayleigh-waves.

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52. 地表層の厚さと分散性レーレー波の振幅との關係

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Stoneley は分散性レーレー波が地表層を傳はるときに卓越せる振動周期のあることを指摘し之は波動の極小群速度に相當するものであるとなした．極小群速度といふことには相當の假定が含まれてをるので，玆に數學的に嚴密な方法で卓越振動周期をしらべて見たのである．

一般の場合はあまりに複雜であるから，ここには下部の層が無限に剛い場合を取つて計算を試みた．この場合には表面波に伴つて傳るや諸固體波が全然存在しないことになる．

今，L を波長，η を地表層の厚さとすると，$L/\eta=2$ ($\kappa\eta=\pi$) 位のときにレーレー波の振幅が極大値を取り，それよりも長い波長でも短い波長でも振幅が小さくなる傾向をもつものである．但し震原は地表面にあり且つその部分の振幅は一定として考へたものである．Stoneley の考によると下部の層の彈性が表面層の彈性と少しく違ふ場合に $\kappa\eta=3\sim2{\cdot}5$ 位のときにレーレー波の振幅が極大になる．我々の結果は，條件が相當違ふのに拘らず，Stoneley の結果と可なりよく似た傾向をもつことがわかる．同じやうな條件の場合を取扱ふとき，ラブ波の極大振幅が Stoneley によると $\kappa\eta=3{\cdot}6$ となり，我々によると $\kappa\eta=2\sim3$ 位となるのである．尙，只今の場合には震原が地表面にある場合を取扱つてある．然るにラブ波の場合を考へると震原が深くなるにつれて振幅の極大になる $\kappa\eta$ の値が增加する傾向があつたのである．從てレーレー波の場合の $\kappa\eta=\pi$ は未だ寧ろ小さ過ぎるかも知れぬ．

我々の計算によると $L/\eta=7$ 位のところで更に極大になり，且つその場合は無限大の振幅になる．しかし我々が以前に試みた研究と比較するとこの位の波長の波は表面波でなく，且つ無限大の振幅は一種の固體波の共振的現象の結果であることがわかるのである．

只今の研究では分散性レーレー波の特別の場合のみを取扱つたとはいへ，その結果はレーレー波の一般性の傾向だけは表してをるものと思はれる．
