53. *The Problem of Elastic Stability of the Earth treated in Polar Coordinates.*

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1. *Introduction.*

The problem of elastic stability of the Earth under gravitational forces was first solved by Jeans,[1] whose result then attracted the special attention of Lord Rayleigh.[2] The assumption made in Jeans's paper and also in Lord Rayleigh's is such that the coordinate position of every material point does not shift, even in the deformed condition of the Earth. Love[3] afterwards discussed the problem with some regard to the change of the mean position of the material points. While Love's problem concerns the case of a gravitating uniform sphere, Matuzawa[4] extended the same problem to the case of a stratified Earth. Almost all these problems were solved by using such Cartesian coordinates as were given by Lamb.[5] On the other hand, similar vibrational problems in polar coordinates have frequently appeared in our papers.[6] [7] Thus, our previous solutions will now be extended to the case of the elastic stability of the Earth. The treatment of the problem in the condition that will correspond to Love's idea is extremely complex. But, even should the condition of no shift of material points be allowable, the resulting defect or excess of mass on the moving free surface would be fairly effective on the change in gravitational force. From this con-

1) J. H. Jeans, "On the Vibrations and Stability of a Gravitational Planet," *Phil. Trans. Roy. Soc.*, **201** (1903), 157~184.

2) Lord Rayleigh, "On the Dilatational Stability of the Earth," *Proc. Roy. Soc.*, **77** (1906), 486~499.

3) A. E. H. Love, "On the Gravitational Stability of the Earth," *Phil. Trans. Roy. Soc.*, **207** (1908), 171~241; Some Problems of Geodynamics (Cambridge, 1911).

4) T. Matuzawa, "On the Gravitational Stability of the Earth with a Core," *Proc. Phys.-Math. Soc.*, Japan, [3], **9** (1927), 31~44.

5) H. Lamb, "On the Vibrations of an Elastic Sphere," *Proc. Math. Soc.*, **13** (1882), 192.

6) K. Sezawa, "Further Studies on Rayleigh-waves having Some Azimuthal Distribution" *Bull. Earthq. Res. Inst.*, **6** (1929), 1~18.

7) K. Sezawa and K. Kanai, "Amplitudes of P- and S-waves at Different Focal Distances," *Bull. Earthq. Res. Inst.*, **10** (1932), 299~334.

sideration, while dealing with the differential equation in the same sense as that due to Jeans and Lord Rayleigh, we have modified the boundary conditions as follows. The usual conditions at the free boundary of the sphere in polar coordinates are such that

$$\lambda\Delta+2\mu\frac{\partial u}{\partial r}=0, \qquad \frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}=0, \qquad \frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi}+\frac{\partial w}{\partial r}-\frac{w}{r}=0 \tag{1}$$

on $r=a$. Instead of the first of these equations, we have used the equation

$$\lambda\Delta+2\mu\frac{\partial u}{\partial r}=-\rho g u, \tag{2}$$

that is,

$$\frac{4}{3}\pi\gamma\rho^2 u+\lambda\Delta+2\mu\frac{\partial u}{\partial r}=0 \tag{2'}$$

on $r=a$.

2. *Mathematical solutions.*

The general solution of the problem for the case without gravitational force was solved in previous papers.[8] In the present paper we shall reinvestigate the problem from the beginning of its treatment with special regard to the earth's pressure as well as with the gravitational forces in the Earth.

The stress components in polar coordinates are such that

$$\left.\begin{aligned}
\widehat{rr}&=-p_0+\lambda\Delta+2\mu\frac{\partial u}{\partial r},\\
\widehat{\theta\theta}&=-p_0+\lambda\Delta+2\mu\left(\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r}\right),\\
\widehat{\phi\phi}&=-p_0+\lambda\Delta+2\mu\left(\frac{1}{r\sin\theta}\frac{\partial w}{\partial\phi}+\frac{v}{r}\cot\theta+\frac{u}{r}\right),\\
\widehat{\theta\phi}&=\mu\left\{\frac{1}{r}\left(\frac{\partial w}{\partial\theta}-w\cot\theta\right)+\frac{1}{r\sin\theta}\frac{\partial v}{\partial\phi}\right\},\\
\widehat{\phi r}&=\mu\left\{\frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi}+\frac{\partial w}{\partial r}-\frac{w}{r}\right\},
\end{aligned}\right\} \tag{3}$$

8) *loc. cit.*, 6), 7).

$$\widehat{r\theta}=\mu\left\{\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right\},$$

where p_0 is the pressure at r, θ, ϕ. The equations of motion are written

$$\left.\begin{aligned}
\rho\frac{\partial^2 u}{\partial t^2}&=\rho\frac{\partial V}{\partial r}+\rho\frac{\partial W}{\partial r}+\frac{\partial \widehat{rr}}{\partial r}+\frac{1}{r}\frac{\partial \widehat{r\theta}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \widehat{r\phi}}{\partial \phi}\\
&\qquad+\frac{1}{r}(2\widehat{rr}-\widehat{\theta\theta}-\widehat{\phi\phi}+\widehat{r\theta}\cot\theta),\\
\rho\frac{\partial^2 v}{\partial t^2}&=\rho\frac{1}{r}\frac{\partial V}{\partial \theta}+\rho\frac{1}{r}\frac{\partial W}{\partial \theta}+\frac{\partial \widehat{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \widehat{\theta\theta}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \widehat{\theta\phi}}{\partial \phi}\\
&\qquad+\frac{1}{r}\left\{(\widehat{\theta\theta}-\widehat{\phi\phi})\cot\theta+3\widehat{r\theta}\right\},\\
\rho\frac{\partial^2 w}{\partial t^2}&=\rho\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}+\rho\frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}+\frac{\partial \widehat{r\phi}}{\partial r}+\frac{1}{r}\frac{\partial \widehat{\theta\phi}}{\partial \theta}\\
&\qquad+\frac{1}{r\sin\theta}\frac{\partial \widehat{\phi\phi}}{\partial \phi}+\frac{1}{r}\left\{3\widehat{r\phi}+2\widehat{\theta\phi}\cot\theta\right\},
\end{aligned}\right\} \tag{4}$$

where u, v, w are components of displacement in r, θ, ϕ directions, V is the gravitational potential due to statical distribution of the mass in the Earth and W that due to increased or decreased mass per unit volume in the vibratory condition of the same Earth. On the other hand, we know that

$$\rho\frac{\partial V}{\partial r}=\frac{\partial p_0}{\partial r},\quad \rho\frac{1}{r}\frac{\partial V}{\partial \theta}=\frac{1}{r}\frac{\partial p_0}{\partial \theta},\quad \rho\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}=\frac{1}{r\sin\theta}\frac{\partial p_0}{\partial \phi}. \tag{5}$$

Use the dilatation and rotations

$$\left.\begin{aligned}
\Delta&=\frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2u\sin\theta)+\frac{\partial}{\partial \theta}(rv\sin\theta)+\frac{\partial}{\partial \phi}(rw)\right\},\\
2\varpi_r&=\frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial \theta}(rw\sin\theta)-\frac{\partial}{\partial \phi}(rv)\right\},
\end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned}2\varpi_\theta&=\frac{1}{r\sin\theta}\left\{\frac{\partial u}{\partial\phi}-\frac{\partial}{\partial r}(rw\sin\theta)\right\},\\2\varpi_\phi&=\frac{1}{r}\left\{\frac{\partial}{\partial r}(rv)-\frac{\partial u}{\partial\theta}\right\},\end{aligned}\right\}$$

then from (3), (4), (5) we find

$$\left.\begin{aligned}\rho\frac{\partial^2u}{\partial t^2}&=\rho\frac{\partial W}{\partial r}+(\lambda+2\mu)\frac{\partial\Delta}{\partial r}-2\mu\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(\varpi_\phi\sin\theta)-\frac{\partial\varpi_\theta}{\partial\phi}\right\},\\\rho\frac{\partial^2v}{\partial t^2}&=\rho\frac{1}{r}\frac{\partial W}{\partial\theta}+(\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta}-2\mu\frac{1}{r\sin\theta}\left\{\frac{\partial\varpi_r}{\partial\phi}-\frac{\partial}{\partial r}(r\varpi_\phi\sin\theta)\right\},\\\rho\frac{\partial^2w}{\partial t^2}&=\rho\frac{1}{r\sin\theta}\frac{\partial W}{\partial\phi}+(\lambda+2\mu)\frac{1}{r\sin\theta}\frac{\partial\Delta}{\partial\phi}-2\mu\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial r}(r\varpi_\theta)-\frac{\partial\varpi_r}{\partial\theta}\right\}.\end{aligned}\right\}\quad(7)$$

Eliminating u, v, w in (7) by means of (6) we obtain

$$\left.\begin{aligned}\rho\frac{\partial^2\Delta}{\partial t^2}&=\rho\nabla^2W+(\lambda+2\mu)\nabla^2\Delta,\\\rho\frac{\partial^2\varpi_r}{\partial t^2}&=\mu\left[\frac{\partial^2\varpi_r}{\partial r^2}+\frac{4}{r}\frac{\partial\varpi_r}{\partial r}+\frac{2}{r^2}\varpi_r\right.\\&\qquad\left.+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varpi_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varpi_r}{\partial\phi^2}\right],\\\rho\frac{\partial^2\varpi_\theta}{\partial t^2}&=\mu\left[\frac{1}{r}\frac{\partial^2(\varpi_\theta r)}{\partial r^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varpi_\theta}{\partial\phi^2}\right.\\&\qquad\left.-\frac{1}{r^2\sin^2\theta}\frac{\partial^2(\varpi_\phi\sin\theta)}{\partial\phi\partial\theta}-\frac{1}{r}\frac{\partial^2\varpi_r}{\partial r\partial\theta}\right],\\\rho\frac{\partial^2\varpi_\phi}{\partial t^2}&=\mu\left[\frac{1}{r}\frac{\partial^2(\varpi_\phi r)}{\partial r^2}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial(\varpi_\phi\sin\theta)}{\partial\theta}\right.\\&\qquad\left.-\frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial\varpi_\theta}{\partial\phi}-\frac{1}{r\sin\theta}\frac{\partial^2\varpi_r}{\partial r\partial\phi}\right].\end{aligned}\right\}\quad(8)$$

It is possible for Poisson's equation to hold for increased or decreased mass per unit volume in the Earth, so that

$$\nabla^2W=4\pi\gamma\rho\Delta,\qquad(9)$$

where γ is the gravitational constant. The first term of (8) reduces to

$$\rho\frac{\partial^2 \Delta}{\partial t^2}-4\pi\gamma\rho\Delta=(\lambda+2\mu)\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Delta}{\partial r}\right)\right.$$
$$\left.+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Delta}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Delta}{\partial\phi^2}\right]. \tag{10}$$

Assuming that

$$\Delta\propto\cos pt, \tag{11}$$

the solutions of (8) become

$$\left.\begin{aligned}
\Delta&=A_{mn}\frac{J_{n+\frac{1}{2}}(hr)}{\sqrt{r}}P_n^m(\cos\theta)\cos m\phi\cos pt,\\
2\varpi_r&=B_{mn}\frac{J_{n+\frac{1}{2}}(kr)}{r^{3/2}}P_n^m(\cos\theta)\sin m\phi\cos pt,\\
2\varpi_\theta&=\left[C_{mn}\frac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}}\frac{P_n^m(\cos\theta)}{\sin\theta}\right.\\
&\quad\left.+\frac{B_{mn}}{n(n+1)}\frac{1}{r}\frac{d}{dr}\left\{\sqrt{r}\,J_{n+\frac{1}{2}}(kr)\right\}\frac{dP_n^m(\cos\theta)}{d\theta}\right]\sin m\phi\cos pt,\\
2\varpi_\phi&=\left[C_{mn}\frac{J_{n+\frac{1}{2}}(kr)}{m\sqrt{r}}\frac{dP_n^m(\cos\theta)}{d\theta}\right.\\
&\quad\left.+\frac{mB_{mn}}{n(n+1)}\frac{1}{r}\frac{d}{dr}\left\{\sqrt{r}\,J_{n+\frac{1}{2}}(kr)\right\}\frac{P_n^m(\cos\theta)}{\sin\theta}\right]\cos m\phi\cos pt,
\end{aligned}\right\} \tag{12}$$

where

$$h^2=\frac{\rho p^2+4\pi\gamma\rho^2}{\lambda+2\mu},\qquad k^2=\frac{\rho p^2}{\mu}. \tag{13}$$

The solution satisfying $\varpi_r=\varpi_\theta=\varpi_\phi=0$ and corresponding to Δ is

$$\left.\begin{aligned}
u_1&=-\frac{A_{mn}}{h^2}\frac{d}{dr}\frac{J_{n+\frac{1}{2}}(hr)}{\sqrt{r}}P_n^m(\cos\theta)\cos m\phi\cos pt,\\
v_1&=-\frac{A_{mn}}{h^2}\frac{J_{n+\frac{1}{2}}(hr)}{r^{3/2}}\frac{dP_n^m(\cos\theta)}{d\theta}\cos m\phi\cos pt,\\
w_1&=\frac{mA_{mn}}{h^2}\frac{J_{n+\frac{1}{2}}(hr)}{r^{3/2}}\frac{P_n^m(\cos\theta)}{\sin\theta}\sin m\phi\cos pt.
\end{aligned}\right\} \tag{14}$$

The solution satisfying $\Delta=\varpi_r=0$ and corresponding to the respective first terms of ϖ_θ, ϖ_ϕ is

$$\left.\begin{aligned} u_3 &= -\frac{n(n+1)C_{mn}}{mk^2}\frac{J_{n+\frac{1}{2}}(kr)}{r^{3/2}}P_n^m(\cos\theta)\cos m\phi\cos pt, \\ v_3 &= -\frac{C_{mn}}{mk^2}\frac{1}{r}\frac{d}{dr}\left\{\sqrt{r}\,J_{n+\frac{1}{2}}(kr)\right\}\frac{dP_n^m(\cos\theta)}{d\theta}\cos m\phi\cos pt, \\ w_3 &= \frac{C_{mn}}{k^2}\frac{1}{r}\frac{d}{dr}\left\{\sqrt{r}\,J_{n+\frac{1}{2}}(kr)\right\}\frac{P_n^m(\cos\theta)}{\sin\theta}\sin m\phi\cos pt. \end{aligned}\right\} \quad (15)$$

The solution satisfying $\Delta=0$ and corresponding to ϖ_r and the respective second terms of ϖ_θ, ϖ_ϕ is

$$\left.\begin{aligned} u_2 &= 0, \\ v_2 &= \frac{mB_{mn}}{n(n+1)}\frac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}}\frac{P_n^m(\cos\theta)}{\sin\theta}\cos m\phi\cos pt, \\ w_2 &= -\frac{B_{mn}}{n(n+1)}\frac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}}\frac{dP_n^m(\cos\theta)}{d\theta}\sin m\phi\cos pt. \end{aligned}\right\} \quad (16)$$

The solutions u_2, v_2, w_2 are not concerned with the stability problem because of the condition that, besides being independent of γ, it is combined neither with u_1, v_1, w_1 nor u_3, v_3, w_3.

3. *An elastic Earth of uniform density and homogeneous elasticities, the shift of mass position being neglected.*

In this case the boundary conditions are such that

$$\left.\begin{aligned} \widehat{rr} &= \lambda\Delta + 2\mu\frac{\partial u}{\partial r} = 0, \\ \widehat{r\theta} &= \mu\left\{\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right\} = 0, \\ \widehat{r\phi} &= \mu\left\{\frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} + \frac{\partial w}{\partial r} - \frac{w}{r}\right\} = 0 \end{aligned}\right\} \quad (17)$$

at $r=a$. Substituting (14) and (15) in (17) we find

$$\left[\left\{\frac{\lambda}{\mu}(ha)^2-2n(n-1)\right\}J_{n+\frac{1}{2}}(ha)+2(2n+1)haJ_{n+\frac{3}{2}}ha-2(ha)^2J_{n+\frac{5}{2}}(ha)\right]$$

$$\cdot\left[2(n+1)(n-1)J_{n+\frac{1}{2}}(ka)-(2n+1)kaJ_{n+\frac{3}{2}}(ka)+(ka)^2J_{n+\frac{5}{2}}(ka)\right]$$

$$+4n(n+1)\left\{(n-1)J_{n+\frac{1}{2}}(ha)-haJ_{n+\frac{3}{2}}(ha)\right\}\left\{(n-1)J_{n+\frac{1}{2}}(ka)-kaJ_{n+\frac{3}{2}}(ka)\right\}=0, \tag{18}$$

where

$$ha=\sqrt{\frac{\rho p^2+4\pi\gamma\rho^2}{\lambda+2\mu}}a, \quad ka=\sqrt{\frac{\rho p^2}{\mu}}a. \tag{19}$$

The critical condition under which the free vibration of the Earth becomes unstable may be expressed by $p=0$. In that case (18) reduces to

$$\frac{\lambda}{\mu}haJ_{n+\frac{1}{2}}ha+2(n+1)J_{n+\frac{3}{2}}(ha)-2haJ_{n+\frac{5}{2}}(ha)=0, \tag{20}$$

unless $n=1$. Equation (20) may also be transformed into

$$n=0;\quad \tan ha-\frac{4ha}{4-\left(\frac{\lambda}{\mu}+2\right)(ha)^2}=0, \tag{20'}$$

$$n=2;\quad \tan ha-\frac{ha\left\{120-\left(\frac{3\lambda}{\mu}-14\right)(ha)^2\right\}}{120-3\left(\frac{\lambda}{\mu}+18\right)(ha)^2+\left(\frac{\lambda}{\mu}+2\right)(ha)^4}=0, \tag{20''}$$

$$n=3;\quad \tan ha-\frac{ha\left\{1050-5\left(\frac{3\lambda}{\mu}+26\right)(ha)^2+\left(\frac{\lambda}{\mu}+2\right)(ha)^4\right\}}{1050-15\left(\frac{\lambda}{\mu}+32\right)(ha)^2+2\left(\frac{3\lambda}{\mu}+11\right)(ha)^4}=0. \tag{20'''}$$

In the case $n=1$, we have

$$3\frac{\lambda}{\mu}(ha)^2J_{\frac{3}{2}}(ha)+10haJ_{\frac{5}{2}}(ha)-6(ha)^2J_{\frac{7}{2}}(ha)=0 \tag{21}$$

instead of (20). Equation (21) reduces to

$$\tan ha - \frac{3ha\left\{20-\left(\frac{\lambda}{\mu}+2\right)(ha)^2\right\}}{60-\left(\frac{3\lambda}{\mu}+26\right)(ha)^2}=0. \tag{21'}$$

Although it may appear strange, if the number n were assigned, the critical condition would be the same irrespective of the number m, according to which the deformed figure of the Earth will differ.

Using (20′), (20″), (20‴), (21′), we obtain the critical conditions for various cases of n and λ/μ, the result being shown in Table I. In these treatments we shall assume that $\rho=5{\cdot}5$, $a=6370$ km, $\gamma=648.10^{-10}$ (C. G. S.). The velocities of transmission of longitudinal and transverse waves corresponding to the elasticities given in Table I are shown in Tables II, III.

TABLE I. The value of $\mu/10^{11}$ (C. G. S.)

n	$\lambda/\mu=1$	2	10	200
0	5·10	3·34	0·915	0·0507
1	2·62	1·566	0·441	0·0247
2	1·29	0·892	0·252	0·0152
3	0·848	0·593	0·178	0·0102

TABLE II. The value of $\sqrt{(\lambda+2\mu)/\rho}$ in km.

n	$\lambda/\mu=1$	2	10	200
0	5·28	4·95	4·47	1·370
1	3·78	3·38	3·10	0·952
2	2·65	2·55	2·35	0·748
3	2·15	2·08	1·97	0·612

TABLE III. The value of $\sqrt{\mu/\rho}$ in km.

n	$\lambda/\mu=1$	2	10	200
0	3·05	2·46	1·29	0·0959
1	2·18	1·69	0·896	0·0670
2	1·53	1·27	0·677	0·0526
3	1·24	1·04	0·369	0·0431

It will be seen that the higher the order of n, the more diminishes the value of λ or μ corresponding to the critical condition of stability for any ratio of λ/μ.

4. *An elastic Earth of uniform density and homogeneous elasticities, the shift of position of the material point being taken as a stress condition on the free surface.*

We shall next consider the case in which the condition of normal stress on the free surface is

$$\frac{4}{3}\pi\gamma\rho^2 u+\lambda\varDelta+2\mu\frac{\partial u}{\partial r}=0 \tag{2'}$$

on $r=a$, the conditions of shearing stress being the same as those in the previous section. In this case the frequency equation of the vibration assumes the form

$$\begin{aligned}&\Big[\Big\{2n(3n-2)+\frac{\lambda}{\mu}(n-3)(ha)^2\Big\}J_{n+\frac{1}{2}}(ha)-ha\Big\{6(2n+1)\\
&\quad+\Big(\frac{\lambda}{\mu}+2\Big)(ha)^2\Big\}J_{n+\frac{3}{2}}(ha)+6(ha)^2J_{n+\frac{5}{2}}(ha)\Big]\Big[2(n^2-1)J_{n+\frac{1}{2}}(ka)\\
&\quad-(2n+1)kaJ_{n+\frac{3}{2}}(ka)+(ka)^2J_{n+\frac{5}{2}}(ka)\Big]\\
&\quad+2n(n+1)\Big[(n-1)J_{n+\frac{1}{2}}(ha)-haJ_{n+\frac{3}{2}}(ha)\Big]\Big[-\Big\{6(n-1)\\
&\qquad+\Big(\frac{\lambda}{\mu}+2\Big)(ka)^2\Big\}J_{n+\frac{1}{2}}(ka)+6kaJ_{n+\frac{3}{2}}(ka)\Big]=0.\end{aligned} \tag{22}$$

where $4\pi\gamma\rho^2a^2/(\lambda+2\mu)$, arising from the boundary condition, is assumed to be ha under the condition that p is ultimately made zero. In the critical case where $p=0$, (22) finally reduces to

$$n=0;\quad \tan ha-\frac{ha\Big\{12-\Big(\frac{\lambda}{\mu}+2\Big)(ha)^2\Big\}}{4\Big\{3-\Big(\frac{\lambda}{\mu}+2\Big)(ha)^2\Big\}}=0, \tag{22$^{\rm I}$}$$

$$n=1;\quad \tan ha-\frac{3ha}{3-(ha)^2}=0, \tag{22$^{\rm II}$}$$

$$n=2;\quad \tan ha-\frac{ha\Big\{372-\Big(24-\frac{6\lambda}{\mu}\Big)(ha)^2-\Big(\frac{\lambda}{\mu}+2\Big)(ha)^4\Big\}}{372-\Big(148-\frac{6\lambda}{\mu}\Big)(ha)^2-\Big(2+\frac{3\lambda}{\mu}\Big)(ha)^4}=0, \tag{23$^{\rm III}$}$$

$n=3$; $\tan ha -$

$$\frac{ha\left\{6480+3\left(\frac{5\lambda}{\mu}-254\right)(ha)^2-4\left(\frac{\lambda}{\mu}-1\right)(ha)^4\right\}}{6480+3\left(\frac{5\lambda}{\mu}-974\right)(ha)^2-3\left(\frac{3\lambda}{\mu}-38\right)(ha)^4+\left(\frac{\lambda}{\mu}+2\right)(ha)^6}=0. \tag{22$^{\mathrm{IV}}$}$$

In this case, too, notwithstanding the differently deformed figure of the Earth, the critical condition for various cases of m is the same, provided the number n is assigned.

The result of calculation of the critical stability for the same numerical conditions as those in the previous case is shown in Table IV.

TABLE IV. The value of $\mu/10^{11}$ (C. G. S.)

n	$\lambda/\mu=1$	200
0	2·358	0·0325
1	1·010	0·0150
2	2·414	0·0333
3	1·082	0·0165

The volocities of transmisston of logitudinal and transverse waves in the media given in Table IV are shown in Table V.

TABLE V. $\sqrt{(\lambda+2\mu)/\rho}$ and $\sqrt{\mu/\rho}$ in km.

n	$\lambda/\mu=1$		$\lambda/\mu=200$	
	$\sqrt{(\lambda+2\mu)/\rho}$	$\sqrt{\mu/\rho}$	$\sqrt{(\lambda+2\mu)/\rho}$	$\sqrt{\mu/\rho}$
0	1·135	0·655	1·093	0·0769
1	0·742	0·429	0·742	0·0522
2	1·148	0·662	1·106	0·0778
3	0·768	0·444	0·779	0·0548

It will be seen from Table IV that in this case, the value of λ or μ is minimum at $n=1$ and maximum at $n=2$ but tends to decrease with further increase of n.

5. *The Earth in a stratified condition in density as well as in elasticities.*

We shall now consider the case wherein the density as well as the elastic constants of the core respectively differ from those of the rocky shell. The expressions of the general solutions for the core are the same as (14), (15), (16), whereas the solutions of the movements of the rocky shell are somewhat complex, as shown in (23)$\sim$(27). Let the density and elastic constants of the rocky shell be ρ', λ', μ'; then the treatments of the differential equations of elastic vibrations and Poisson's equations of gravity potential give rise to

$$\left.\begin{aligned}
\Delta' &= \left\{A'_{mn}\frac{J_{n+\frac{1}{2}}(h'r)}{r^{3/2}}+A''_{mn}\frac{Y_{n+\frac{1}{2}}(h'r)}{r^{3/2}}\right\}P_n^m(\cos\theta)\cos m\phi\cos pt,\\
2\varpi'_r &= \left\{B'_{mn}\frac{J_{n+\frac{1}{2}}(k'r)}{r^{3/2}}+B''_{mn}\frac{Y_{n+\frac{1}{2}}(k'r)}{r^{3/2}}\right\}P_n^m(\cos\theta)\sin m\phi\cos pt,\\
2\varpi'_\theta &= \Bigg[\left\{C'_{mn}\frac{J_{n+\frac{1}{2}}(k'r)}{\sqrt{r}}+C''_{mn}\frac{Y_{n+\frac{1}{2}}(k'r)}{\sqrt{r}}\right\}\frac{P_n^m(\cos\theta)}{\sin\theta}\\
&\quad+\left\{\frac{B'_{mn}}{n(n+1)}\frac{1}{r}\frac{d}{dr}\left(\sqrt{r}\,J_{n+\frac{1}{2}}(k'r)\right)\right.\\
&\quad\left.+\frac{B''_{mn}}{n(n+1)}\frac{1}{r}\frac{d}{dr}\left(\sqrt{r}\,Y_{n+\frac{1}{2}}(k'r)\right)\right\}\frac{dP_n^m(\cos\theta)}{d\theta}\Bigg]\sin m\phi\cos pt,\\
2\varpi'_\phi &= \Bigg[\left\{C'_{mn}\frac{J_{n+\frac{1}{2}}(k'r)}{m\sqrt{r}}+C''_{mn}\frac{Y_{n+\frac{1}{2}}(k'r)}{m\sqrt{r}}\right\}\frac{dP_n^m(\cos\theta)}{d\theta}\\
&\quad+\left\{\frac{B'_{mn}m}{n(n+1)}\frac{1}{r}\frac{d}{dr}\left(\sqrt{r}\,J_{n+\frac{1}{2}}(k'r)\right)\right.\\
&\quad\left.+\frac{B''_{mn}m}{n(n+1)}\frac{1}{r}\frac{d}{dr}\left(\sqrt{r}\,Y_{n+\frac{1}{2}}(k'r)\right)\right\}\frac{P_n^m(\cos\theta)}{\sin\theta}\Bigg]\cos m\phi\cos pt,
\end{aligned}\right\}\tag{23}$$

where

$$h'^2=\frac{\rho' p^2+4\pi\gamma\rho'^2}{\lambda'+2\mu'},\qquad k'^2=\frac{\rho' p^2}{\mu'}.\tag{24}$$

The three kinds of displacement therefore assume the forms

$$\left.\begin{aligned}
u'_1 &= -\left\{\frac{A'_{mn}}{h'^2}\frac{d}{dr}\frac{J_{n+\frac{1}{2}}(h'r)}{\sqrt{r}}+\frac{A''_{mn}}{h'^2}\frac{d}{dr}\frac{Y_{n+\frac{1}{2}}(h'r)}{\sqrt{r}}\right\}P_n^m(\cos\theta)\cos m\phi\cos pt,\\
v'_1 &= -\left\{\frac{A'_{mn}}{h'^2}\frac{J_{n+\frac{1}{2}}(h'r)}{r^{3/2}}+\frac{A''_{mn}}{h'^2}\frac{Y_{n+\frac{1}{2}}(h'r)}{r^{3/2}}\right\}\frac{dP_n^m(\cos\theta)}{d\theta}\cos m\phi\cos pt,\\
w'_1 &= \left\{\frac{mA'_{mn}}{h'^2}\frac{J_{n+\frac{1}{2}}(h'r)}{r^{3/2}}+\frac{mA''_{mn}}{h'^2}\frac{Y_{n+\frac{1}{2}}(h'r)}{r^{3/2}}\right\}\frac{P_n^m(\cos\theta)}{\sin\theta}\sin m\phi\cos pt,
\end{aligned}\right\}\tag{25}$$

$$\left.\begin{aligned}
u_3' &= -\left\{\frac{n(n+1)C'_{mn}}{mk'^2}\frac{J_{n+\frac{1}{2}}(k'r)}{r^{3/2}}\right.\\
&\qquad\left.+\frac{n(n+1)C''_{mn}}{mk'^2}\frac{Y_{n+\frac{1}{2}}(k'r)}{r^{3/2}}\right\}P_n^m(\cos\theta)\cos m\phi\cos pt,\\
v_3' &= -\left\{\frac{C'_{mn}}{mk'^2}\frac{1}{r}\frac{d}{dr}\left(\sqrt{r}\,J_{n+\frac{1}{2}}(k'r)\right)\right.\\
&\qquad\left.+\frac{C''_{mn}}{mk'^2}\frac{1}{r}\frac{d}{dr}\left(\sqrt{r}\,Y_{n+\frac{1}{2}}(k'r)\right)\right\}\frac{dP_n^m(\cos\theta)}{d\theta}\cos m\phi\cos pt,\\
w_3' &= \left\{\frac{C'_{mn}}{k'^2}\frac{1}{r}\frac{d}{dr}\left(\sqrt{r}\,J_{n+\frac{1}{2}}(k'r)\right)\right.\\
&\qquad\left.+\frac{C''_{mn}}{k'^2}\frac{1}{r}\frac{d}{dr}\left(\sqrt{r}\,Y_{n+\frac{1}{2}}(k'r)\right)\right\}\frac{P_n^m(\cos\theta)}{\sin\theta}\sin m\phi\cos pt,
\end{aligned}\right\}\qquad(26)$$

$$\left.\begin{aligned}
u_2' &= 0,\\
v_2' &= \left\{\frac{mB'_{mn}}{n(n+1)}\frac{J_{n+\frac{1}{2}}(k'r)}{\sqrt{r}}\right.\\
&\qquad\left.+\frac{mB''_{mn}}{n(n+1)}\frac{Y_{n+\frac{1}{2}}(k'r)}{\sqrt{r}}\right\}\frac{P_n^m(\cos\theta)}{\sin\theta}\cos m\phi\cos pt,\\
w_2' &= -\left\{\frac{B'_{mn}}{n(n+1)}\frac{J_{n+\frac{1}{2}}(k'r)}{\sqrt{r}}\right.\\
&\qquad\left.+\frac{B'_{mn}}{n(n+1)}\frac{Y_{n+\frac{1}{2}}(k'r)}{\sqrt{r}}\right\}\frac{dP_n^m(\cos\theta)}{d\theta}\sin m\phi\cos pt.
\end{aligned}\right\}\qquad(27)$$

In this case, too, u_2', v_2', w_2' do not participate in the stability problem.

For simplicity, we shall assume that the rigidity of the core is zero, —a rather probable state of the actual core. We neglect here the effect of mass defect or excess at the free surface of the Earth resulting from the assumption of no shift of material position. The boundary conditions in this case then assume the forms

$$\left.\begin{aligned}
r=a;\quad \lambda'\Delta'+2\mu'\frac{\partial u'}{\partial r}=0,\quad &\frac{\partial v'}{\partial r}-\frac{v'}{r}+\frac{1}{r}\frac{\partial u'}{\partial\theta}=0,\\
\frac{1}{r\sin\theta}\frac{\partial u'}{\partial\phi}+\frac{\partial w'}{\partial r}-\frac{w'}{r}=0,&
\end{aligned}\right\}\qquad(28)$$

$$\left.\begin{aligned} r=b;\quad \lambda'\Delta'+2\mu'\frac{\partial u'}{\partial r}=\lambda\Delta,\quad \mu'\Big(\frac{\partial v'}{\partial r}-\frac{v'}{r}+\frac{1}{r}\frac{\partial u'}{\partial\theta}\Big)=0,\\ \mu'\Big(\frac{1}{r\sin\theta}\frac{\partial u'}{\partial\phi}+\frac{\partial w'}{\partial r}-\frac{w'}{r}\Big)=0, \end{aligned}\right\} \tag{29}$$

where a, b are radius of the Earth and that of its inner core respectively.

Substituting (23), (25), (26) in (28), (29), we have the relations

$$\left\{\frac{A'_{mn}}{h'^2}a'_1+\frac{A''_{mn}}{h'^2}a''_1\right\}-\left\{\frac{C'_{mn}}{k'^2}\frac{2n(n+1)}{m}c'_1+\frac{C''_{mn}}{k'^2}\frac{2n(n+1)}{m}c''_1\right\}=0, \tag{30}$$

$$\left\{\frac{A'_{mn}}{h'^2}2a'_2+\frac{A''_{mn}}{h'^2}2a''_2\right\}+\left\{\frac{C'_{mn}}{k'^2}\frac{1}{m}c'_2+\frac{C''_{mn}}{k'^2}\frac{1}{m}c''_2\right\}=0, \tag{31}$$

$$\mu'\left[\left\{\frac{A'_{mn}}{h'^2}\alpha'_1+\frac{A''_{mn}}{h'^2}\alpha''_1\right\}-\left\{\frac{C'_{mn}}{k'^2}\frac{2n(n+1)}{m}\gamma'_1+\frac{C''_{mn}}{k'^2}\frac{2n(n+1)}{m}\gamma''_1\right\}\right]$$
$$=\mu\left[\frac{A_{mn}}{h^2}\alpha_1-\frac{C_{mn}}{k^2}\frac{2n(n+1)}{m}\gamma_1\right], \tag{32}$$

$$\mu'\left[\left\{\frac{A'_{mn}}{h'^2}2\alpha'_2+\frac{A''_{mn}}{h'^2}2\alpha''_2\right\}+\left\{\frac{C'_{mn}}{k'^2}\frac{1}{m}\gamma'_2+\frac{C''_{mn}}{k'^2}\frac{1}{m}\gamma''_2\right\}\right]$$
$$=\mu\left[\frac{A_{mn}}{h^2}2\alpha_2+\frac{C_{mn}}{k^2}\frac{1}{m}\gamma_2\right], \tag{33}$$

$$\left\{\frac{A'_{mn}}{h'^2}\alpha'_3+\frac{A''_{mn}}{h'^2}\alpha''_3\right\}+\left\{\frac{C'_{mn}}{k'^2}\frac{n(n+1)}{m}\gamma'_3+\frac{C''_{mn}}{k'^2}\frac{n(n+1)}{m}\gamma''_3\right\}$$
$$=\frac{A_{mn}}{h^2}\alpha_3+\frac{C_{mn}}{k^2}\frac{n(n+1)}{m}\gamma_3, \tag{34}$$

$$\left\{\frac{A'_{mn}}{h'^2}\alpha'_4+\frac{A''_{mn}}{h'^2}\alpha''_4\right\}+\left\{\frac{C'_{mn}}{k'^2}\frac{1}{m}\gamma'_4+\frac{C''_{mn}}{k'^2}\frac{1}{m}\gamma''_4\right\}=\frac{A_{mn}}{h^2}\alpha_4+\frac{C_{mn}}{k^2}\frac{1}{m}\gamma_4, \tag{35}$$

where the constants in (30) $\sim$ (35) are as follows:

$$\begin{aligned} a_1=&\left\{\frac{\lambda}{\mu}(ha)^2-2n(n-1)\right\}J_{n+\frac{1}{2}}(ha)\\ &+2(2n+1)haJ_{n+\frac{3}{2}}(ha)-2(ha)^2J_{n+\frac{5}{2}}(ha), \end{aligned}$$

$$\left.\begin{aligned} a_1' &= \left\{\frac{\lambda'}{\mu'}(h'a)^2 - 2n(n-1)\right\} J_{n+\frac{1}{2}}(h'a) \\ &\qquad + 2(2n+1)h'aJ_{n+\frac{3}{2}}(h'a) - 2(h'a)^2 J_{n+\frac{5}{3}}(h'a), \\ a_1'' &= \left\{\frac{\lambda'}{\mu'}(h'a)^2 - 2n(n-1)\right\} Y_{n+\frac{1}{2}}(h'a) \\ &\qquad + 2(2n+1)h'aY_{n+\frac{3}{2}}(h'a) - 2(h'a)^2 Y_{n+\frac{5}{2}}(h'a), \end{aligned}\right\} \tag{36}$$

$$\left.\begin{aligned} \alpha_1 &= \left\{\frac{\lambda}{\mu}(hb)^2 - 2n(n-1)\right\} J_{n+\frac{1}{2}}(hb) \\ &\qquad + 2(2n+1)hbJ_{n+\frac{3}{2}}(hb) - 2(hb)^2 J_{n+\frac{5}{2}}(hb), \\ \alpha_1' &= \left\{\frac{\lambda'}{\mu'}(h'b)^2 - 2n(n-1)\right\} J_{n+\frac{1}{2}}(h'b) \\ &\qquad + 2(2n+1)h'bJ_{n+\frac{3}{2}}(h'b) - 2(h'b)^2 J_{n+\frac{5}{2}}(h'b), \\ \alpha_1'' &= \left\{\frac{\lambda'}{\mu'}(h'b)^2 - 2n(n-1)\right\} Y_{n+\frac{1}{2}}(h'b) \\ &\qquad + 2(2n+1)h'bY_{n+\frac{3}{2}}(h'b) - 2(h'b)^2 Y_{n+\frac{5}{2}}(h'b), \end{aligned}\right\} \tag{37}$$

$$\begin{aligned} c_1,\ c_1',\ c_1'', &= (n-1)\left\{J_{n+\frac{1}{2}}(ka),\ J_{n+\frac{1}{2}}(k'a),\ Y_{n+\frac{1}{2}}(k'a)\right\} \\ &\qquad - a\left\{kJ_{n+\frac{3}{2}}(ka),\ k'J_{n+\frac{3}{2}}(k'a),\ k'Y_{n+\frac{3}{2}}(k'a)\right\}, \end{aligned} \tag{38}$$

$$\begin{aligned} \gamma_1,\ \gamma_1',\ \gamma_1'', &= (n-1)\left\{J_{n+\frac{1}{2}}(kb),\ J_{n+\frac{1}{2}}(k'b),\ Y_{n+\frac{1}{2}}(k'b)\right\} \\ &\qquad - b\left\{kJ_{n+\frac{3}{2}}(kb),\ k'J_{n+\frac{3}{2}}(k'b),\ k'Y_{n+\frac{3}{2}}(k'b)\right\}, \end{aligned} \tag{39}$$

$$\begin{aligned} a_2,\ a_2',\ a_2'', &= -(n-1)\left\{J_{n+\frac{1}{2}}(ha),\ J_{n+\frac{1}{2}}(h'a),\ Y_{n+\frac{1}{2}}(h'a)\right\} \\ &\qquad + a\left\{hJ_{n+\frac{3}{2}}(ha),\ h'J_{n+\frac{3}{2}}(h'a),\ h'Y_{n+\frac{3}{2}}(h'a)\right\}, \end{aligned} \tag{40}$$

$$\alpha_2,\ \alpha_2',\ \alpha_2'', = -(n-1)\left\{J_{n+\frac{1}{2}}(hb),\ J_{n+\frac{1}{2}}(h'b),\ Y_{n+\frac{1}{2}}(h'b)\right\}$$

$$+b\left\{hJ_{n+\frac{3}{2}}(hb),\ h'J_{n+\frac{3}{2}}(h'b),\ h'Y_{n+\frac{3}{2}}(h'b)\right\}, \tag{41}$$

$$\begin{aligned} c_2,\ c_2',\ c_2'',=&-2(n^2-1)\left\{J_{n+\frac{1}{2}}(ka),\ J_{n+\frac{1}{2}}(k'a),\ Y_{n+\frac{1}{2}}(k'a)\right\} \\ &+(2n+1)a\left\{kJ_{n+\frac{3}{2}}(ka),\ k'J_{n+\frac{3}{2}}(k'a),\ k'Y_{n+\frac{3}{2}}(k'a)\right\} \\ &-a^2\left\{k^2J_{n+\frac{5}{2}}(ka),\ k'^2J_{n+\frac{5}{2}}(k'a),\ k'^2Y_{n+\frac{5}{2}}(k'a)\right\}, \end{aligned} \tag{42}$$

$$\begin{aligned} \gamma_2,\ \gamma_2',\ \gamma_2'',=&-2(n^2-1)\left\{J_{n+\frac{1}{2}}(kb),\ J_{n+\frac{1}{2}}(k'b),\ Y_{n+\frac{1}{2}}(k'b)\right\} \\ &+(2n+1)b\left\{kJ_{n+\frac{3}{2}}(kb),\ k'J_{n+\frac{3}{2}}(k'b),\ k'Y_{n+\frac{3}{2}}(k'b)\right\} \\ &-b^2\left\{k^2J_{n+\frac{5}{2}}(kb),\ k'^2J_{n+\frac{5}{2}}(k'b),\ k'^2Y_{n+\frac{5}{2}}(k'b)\right\}, \end{aligned} \tag{43}$$

$$\begin{aligned} \alpha_3,\ \alpha_3',\ \alpha_3'',=&n\left\{J_{n+\frac{1}{2}}(hb),\ J_{n+\frac{1}{2}}(h'b),\ Y_{n+\frac{1}{2}}(h'b)\right\} \\ &-b\left\{hJ_{n+\frac{3}{2}}(hb),\ h'J_{n+\frac{3}{2}}(h'b),\ h'Y_{n+\frac{3}{2}}(h'b)\right\}, \end{aligned} \tag{44}$$

$$\gamma_3,\ \gamma_3',\ \gamma_3'',=J_{n+\frac{1}{2}}(kb),\ J_{n+\frac{1}{2}}(k'b),\ Y_{n+\frac{1}{2}}(k'b), \tag{45}$$

$$\alpha_4,\ \alpha_4',\ \alpha_4'',=J_{n+\frac{1}{2}}(hb),\ J_{n+\frac{1}{2}}(h'b),\ Y_{n+\frac{1}{2}}(h'b), \tag{46}$$

$$\begin{aligned} \gamma_4,\ \gamma_4',\ \gamma_4'',=&(n+1)\left\{J_{n+\frac{1}{2}}(kb),\ J_{n+\frac{1}{2}}(k'b),\ Y_{n+\frac{1}{2}}(k'b)\right\} \\ &-b\left\{kJ_{n+\frac{3}{2}}(kb),\ k'J_{n+\frac{3}{2}}(k'b),\ k'Y_{n+\frac{3}{2}}(k'b)\right\}. \end{aligned} \tag{47}$$

Eliminating A'_{mn}, A''_{mn}, ..., C'_{mn} between (30)$\sim$(35), it is possible to get the frequency equation of the free vibration.

It is an outstanding fact that, were $\mu=0$, the volume elasticity λ as well as density ρ in the core have no place in the problem of elastic stability of the Earth.

Using the equations (30)$\sim$(47) and assuming that the density of the rocky shell is $\rho'=4{\cdot}5$, we get the values of λ', μ' at which the stability is critical, the result being shown in Table VI. The velocities of transmission of longitudinal and transverse waves in the media shown in Table VI are tabulated in Table VII.

TABLE VI. The values of $\mu'/10^{11}$ in (C. G. S.)

n	$\lambda'/\mu'=1$	2	200
0	5·75	4·58	2·715
1	0·691	——	——
2	0·770	——	——

TABLE VII. $\sqrt{(\lambda'+2\mu')/\rho'}$ and $\sqrt{\mu'/\rho'}$ in km.

n	$\lambda'/\mu'=1$		2		200	
	$\sqrt{(\lambda'+2\mu')/\rho'}$	$\sqrt{\mu'/\rho'}$	$\sqrt{(\lambda'+2\mu')/\rho'}$	$\sqrt{\mu'/\rho'}$	$\sqrt{(\lambda'+2\mu')/\rho'}$	$\sqrt{\mu'/\rho'}$
0	1·959	1·131	2·01	1·0059	11·04	0·779
1	0·680	0·392	——	——	——	——
2	0·716	0·414	——	——	——	——

In this case, provided the ratis of λ'/μ' is not too large, the general properties with respect to the critical condition of stability is fairly similar to those of the condition shown in Section 3 and somewhat similar to those of the condition shown is Section 4. For a large ratio of λ'/μ', say $\lambda'/\mu'=200$, the lowest values of λ', μ' are $\lambda'=5{\cdot}43{\cdot}10^{13}$, $\mu'=2{\cdot}715{\cdot}10^{11}$ respectively. Thus, were the rocky shell incompressible, namely $\mu'/\lambda'\to 0$, the Earth would be gravitationally unstable, even at such a large value of $\lambda'=5{\cdot}43{\cdot}10^{13}$ as hardly obtains in the actual rocks of the Earth's crust.

6. *General discussion of the result.*

Our mathematical solutions show that for the case in which the Earth is entirely uniform in its density as well as elasticities and we neglect the mass defect or excess on the free surface resulting from the assumption that the position of the mass never shifts, the greater the value of n, greater the decrease in the values of λ and μ at which the elastic stability of the Earth is critical, whereas, on the other hand, in the case where the mass effect on the free surface under considera-tion is taken into account, the value of λ or μ which is minimum at

$n=1$, and maximum at $n=2$, tends to diminish with further increase of n.

It is now possible to imagine that the past condition of the Earth has gradually changed from an extremely deformable state to a more and more rigid one. Thus, if in the past stage of the Earth, its elasticities were less than those of any of the n's in Table IV, the vibrational motion of the Earth for the mode corresponding to every n would have been unstable. We shall assume that the Poisson's ratio is 1/4. Then, the respective vibrational modes $n=0$, 1, 2, 3, ... are unstable, provided $\lambda(=\mu)$ is less than 2·358, 1·010, 2·414, and 1·082 respectively. Similar conditions exist for different Poisson's ratios. Had the elastic constants of the Earth gradually increased, the next stage to be expected would be such that, while the vibrational modes corresponding to $n=1$ is stable, every other mode will be still unstable. With further increase in elastic constants, the modes $n=3$ then becomes stable, and so on. Since, on the other hand, the deformed state is likely to remain unchanged in the condition of higher elastic constants, owing to the probably lower temperature of the solid in that condition, the traces of the unstable state of such a mode as became stable in a relatively later stage will be marked. It follows then that the traces of the past unstable conditions for $n=0$ and $n=2$ would be pronounced. Among other modes traces of instability for the cases $n=3$ as well as $n=1$ would be quite marked. It is impossible for us at present to discuss the cases in which n is greater than 4. But so far as our present investigation is concerned, the conclusion at which we arrive is that traces of the past unstable character of case $n=0$ and $n=2$, that is the cases corresponding to the spherical type and the spheroidal (or particular ellipsoidal) type, will probably be very marked. The particular ellipsoidal type in question indicates such a condition as ($n=2$, $m=1$) or ($n=2$, $m=2$). It should be borne in mind that the above explanation is probable for every case of Poisson's ratio.

From the results in Sections 3, 4, 5 it appears that the problem for the case in which there is a core of no rigidity is likely to assume the character just mentioned, even without regard to the effect of mass defect or excess on the free surface.

Since, furthermore, both cases $n=1$, $n=3$ are also unstable conditions at a stage somewhat earlier than that for $n=0$ or $n=2$, it is impossible to conclude that no trace of the deformation of a pear-shaped type or tetrahedral type can ever remain in the present figure of the Earth.

The question may be raised as to whether or not the condition of elastic stability might be affected by such a high viscosity of the Earth as is extremely possible at a high-temperature stage of the same Earth. Since, however, the present problem concerns a divergent stability, and not oscillatory stability, the damping effect has virtually very little place in the criteria already given.

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53. 極座標上から出した地球の彈性的安定の問題

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地球が重力を受けて彈性的に不安定になる問題は Jeans が始めて考へ Lord Rayleigh も之に興味を持つた. しかし物質の點の平均位置が動く事まで考慮に入れて問題を解いたのは Love である. 之を核の場合迄延長した松澤博士の計算がある. 之等は主として Lamb の考へた直角座標を應用する方法であるが，我々の場合には極座標を用ひて解いた所の似た問題があつたから，之を只今の安定の問題に延長して見たのである.

物理的の考方としてはJeansのやうな場合と，物質點の動く影響を入れたものとの二種を試み，始めの種類では核のある場合をも計算して見たのである. しかし何れの場合にも彈性的變形の結果として現れる重力に關する Poisson の式が大い物を言ふのである.

計算の結果，變形に可なりの違ひを與へる m の如何に拘らず，n が一定でありさへすれば，安定極限が一致することがわかつた. 又，層のある場合に核の剛性が零であるとすると核の變容彈性率や，質量（質量の方は近似的）に關係なく安定極限のきまることもわかつたのである.

計算の結果を綜合して見るのに，若し地球が高温から低温に向ひ，同時に軟い狀態から剛い狀態に移つたものであるとすると，最も最近に變形の安定の狀態に達した振動狀態は $n=0$ と $n=2$ の場合である. 卽ち球形の儘の變形と楕圓體型の變形とであり，且つ之等は殆ど同時であるといつてよい. それより少しく以前の階梯では $n=1$ と $n=3$ とが殆ど同時に安定の狀態に達したのである. 卽ち，西洋梨型の變形と四面體型の變形とである. 後の階梯程，地球の溫度が低いと考へられ，溫度が低い程一度できた變形が後迄殘ると考へられるから，$n=0$ と $n=2$ との場合が現在の地球の形に最もよく痕跡を殘すものといひ得る. $n=1$ と $n=3$ の場合は上述のもの程ではなくても，他の形から見ると遙かによくその痕跡が殘るといひ得るのである.

地球の歷史的階梯を論ずる位であるから，その高温時に於ける粘性が彈性的安定を大いに支配するのでないかといふ疑が起るかも知れぬ. しかしここに論ずるのは divergent stability を論ずるのであつて電氣振動にあるやうな oscillatory stability を問題にするのでないから，粘性の影響は自然に消去されるものである.
