36. *On the Plastic Properties of the Earth's Core.*

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1. *Introduction.*

It seems that the Earth's core transmits only longitudinal waves, not transverse waves. It is however uncertain whether this results from the extremely small torsion modulus of the material within the core or from particularly high viscous resistance of the same material against distortional movement. Since, as a matter of fact, the Earth's core is subjected to enormously high pressure, the material in that core is probably in a certain plastic state. The special features of a general plastic body are its increased viscous resistance against distortional force (for any quickness) and its almost incompressible condition in any (statical) deformation, whence it follows from either one of these features of the plastic body that, were the Earth's core a plastic body, transmission of transverse waves (relatively long waves) through the same core would be improbable.

Now it is well known that the velocity of the core waves is about $1/\sqrt{3}$ of that of the longitudinal waves transmitted through the rocky shell immediately next to the core. If the material within the core were in a plastic state, and that in the rocky shell is in an elastic state, both being of the same density, the relation cited above would then be a most likely one, regardless of the denser character of the Earth's core. Whether or not the material forming the core is as dense as iron or nickel is another problem calling for different treatment.

As already mentioned, we shall assume that, although the Earth's central core is in a plastic condition, the outer shell is in a purely elastic state, the density of the material being assumed uniform throughout the Earth. Gravity is taken as a body force, but the effect of the Earth's rotation is neglected. Although there are a number of theories defining the plastic state of materials, yet in accordance with the present vogue among investigators of plastic bodies, we shall take the maximum shear stress theory, which was originated by Saint Venant[1]

1) St. VENANT, *C. R.*, **70** (1870), **73** (1871).

and developed by certain mathematicians eminent in applied mechanics[2]. Although the criterea of the plastic state differ with authors, as the quantitative difference between them is not very marked, we shall use the maximum shear stress theory throughout this paper.

2. *Equilibrium of the elastic shell.*

Assuming that the material displacement is invariably symmetrical with respect to the Earth's centre, that is, that displacement u is perfectly radial, the strain and stress components are then expressed by

$$\left.\begin{aligned} &e_{rr}=\frac{\partial u}{\partial r}, \quad e_{\theta\theta}=e_{\phi\phi}=\frac{u}{r}, \quad \Delta=\frac{\partial u}{\partial r}+\frac{2u}{r}, \\ &\widehat{rr}=\lambda\Delta+2\mu\frac{\partial u}{\partial r}=\lambda\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)+2\mu\frac{\partial u}{\partial r}, \\ &\widehat{\theta\theta}=\widehat{\phi\phi}=\lambda\Delta+2\mu\frac{u}{r}=\lambda\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)+2\mu\frac{u}{r}, \end{aligned}\right\} \tag{1}$$

where λ, μ are Lamé's elastic constants. The gravitational potential V and the resulting radial pressure p are connected by the formulae

$$\left.\begin{aligned} &V=\frac{2}{3}\pi\gamma\rho(3a^2-r^2), \qquad -\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{\partial V}{\partial r}=0, \\ &\frac{\partial p}{\partial r}=-\frac{4}{3}\pi\gamma\rho^2 r, \qquad p=\frac{2}{3}\pi\gamma\rho^2(a^2-r^2), \end{aligned}\right\} \tag{2}$$

where a is the Earth's radius, ρ the Earth's mean density (assmed almost uniform), and γ the gravitational constant. The equation of equilibrium of the elastic shell is

$$\rho\frac{\partial V}{\partial r}+\frac{\partial \widehat{rr}}{\partial r}+\frac{1}{r}(2\widehat{rr}-\widehat{\theta\theta}-\widehat{\phi\phi})=0, \tag{3}$$

the effect of pressure on the change of density being neglected. Substituting from (1), (2) in (3), we obtain

$$-\frac{4}{3}\pi\gamma\rho^2 r+(\lambda+2\mu)\left(\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}-\frac{2u}{r^2}\right)=0. \tag{4}$$

The solution for the present statical problem is then

$$u=\frac{2}{15}\frac{\pi\gamma\rho^2}{\lambda+2\mu}r^3+Br+\frac{C}{r^2}, \tag{5}$$

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2) A. NADAI, *Der bildsame Zustand der Werkstoffe* (Berlin, 1927), *Handbuch d. Physik*, 6, Plastizität u. Erddruck. Nadai's problem and various plastic theories dealt by other people are involved in these books.

where B, C are determined by boundary conditions. The stress components are

$$\left.\begin{aligned}\widehat{rr}&=\frac{\pi\gamma\rho^2(10\lambda+12\mu)}{15(\lambda+2\mu)}r^2+(3\lambda+2\mu)B-\frac{4\mu C}{r^3},\\ \widehat{\theta\theta}=\widehat{\phi\phi}&=\frac{\pi\gamma\rho^2(10\lambda+4\mu)}{15(\lambda+2\mu)}r^2+(3\lambda+2\mu)B+\frac{2\mu C}{r^3}.\end{aligned}\right\}\tag{6}$$

The boundary conditions are such that the normal stress is zero at the free surface, while the maximum shear stress at the boundary between the shell and the core assumes a definite value k. Mathematically these conditions are expressed by

$$\left.\begin{aligned}&r=a; \quad \widehat{rr}=0,\\ &r=b; \quad \widehat{rr}-\widehat{\theta\theta}=\pm 2k, \quad \widehat{rr}-\widehat{\phi\phi}=\pm 2k,\end{aligned}\right\}\tag{7}$$

where b is the radius of the Earth's core. An additional condition is

$$r=c(a>c>b); \quad |\widehat{rr}-\widehat{\theta\theta}|\not> 2k, \quad |\widehat{rr}-\widehat{\phi\phi}|\not> 2k. \tag{8}$$

After reading this paper, these conditions were found to be rather needless.

Substituting (6) in (7) we obtain

$$\left.\begin{aligned}\frac{6\mu C}{b^3}&=\frac{8}{15}\pi\gamma\rho^2\frac{\mu}{\lambda+2\mu}b^2\pm 2k,\\ (3\lambda+2\mu)B&=\frac{-\pi\gamma\rho^2a^2}{15(\lambda+2\mu)}\left\{(10\lambda+12\mu)-\frac{16\mu}{3}\frac{b^5}{a^5}\right\}\pm\frac{4}{3}k\frac{b^3}{a^3}.\end{aligned}\right\}\tag{9}$$

The stress components are therefore

$$\left.\begin{aligned}\widehat{rr}&=\frac{-\pi\gamma\rho^2(10\lambda+12\mu)}{15(\lambda+2\mu)}(a^2-r^2)+\frac{2}{3}b^3\left(\frac{1}{a^3}-\frac{1}{r^3}\right)\left\{\frac{8\pi\gamma\rho^2\mu}{15(\lambda+2\mu)}b^2\pm 2k\right\},\\ \widehat{\theta\theta}=\widehat{\phi\phi}&=\frac{-\pi\gamma\rho^2(10\lambda+12\mu)}{15(\lambda+2\mu)}a^2+\frac{\pi\gamma\rho^2(10\lambda+4\mu)}{15(\lambda+2\mu)}r^2\\ &\qquad+\left(\frac{2}{3}\frac{b^3}{a^3}+\frac{b^3}{3r^3}\right)\left\{\frac{8\pi\gamma\rho^2\mu}{15(\lambda+2\mu)}b^2\pm 2k\right\}.\end{aligned}\right\}\tag{10}$$

It is to be borne in mind that the stresses in (10) should satisfy the condition (8).

In the special case $\mu/\lambda\to 0$, equations in (10) reduce to

$$\left.\begin{aligned}\widehat{rr}&=\frac{-2\pi\gamma\rho^2}{3}(a^2-r^2)\pm\frac{4}{3}kb^3\left(\frac{1}{a^3}-\frac{1}{r^3}\right),\\ \widehat{\theta\theta}=\widehat{\phi\phi}&=\frac{-2\pi\gamma\rho^2}{3}(a^2-r^2)\pm\frac{4}{3}kb^3\left(\frac{1}{a^3}+\frac{1}{2r^3}\right).\end{aligned}\right\}\tag{11}$$

3. *The equilibrium of the plastic core.*

The equation of the equilibrium of the plastic core is the same as (3), namely,

$$-\frac{4}{3}\pi\gamma\rho^2 r+\frac{\partial\widehat{rr}}{\partial r}+\frac{1}{r}(2\widehat{rr}-\widehat{\theta\theta}-\widehat{\phi\phi})=0\,, \tag{3'}$$

where $\widehat{\theta\theta}=\widehat{\phi\phi}$, and the equation of the plastic condition is

$$\widehat{rr}-\widehat{\theta\theta}=\mp 2k \tag{12}$$

for any $r(r<b)$. From (3′), (12) we obtain

$$-\frac{4}{3}\pi\gamma\rho^2 r+\frac{\partial\widehat{rr}}{\partial r}\mp\frac{4k}{r}=0\,. \tag{13}$$

Integrating this we get

$$\widehat{rr}\mp 4k\log_e r-\frac{2}{3}\pi\gamma\rho^2 r^2+I=0\,, \tag{14}$$

where I is the integration constant. From (10) the boundary condition is

$$r=b\,;\qquad \widehat{rr}=\frac{-\pi\gamma\rho^2(10\lambda+12\mu)}{15(\lambda+2\mu)}(a^2-b^2)+\frac{2}{3}b^3\left(\frac{1}{a^3}+\frac{1}{b^3}\right)\left(\frac{8}{15}\pi\gamma\rho^2\frac{\mu}{\lambda+2\mu}b^2\pm 2k\right). \tag{15}$$

Determining I in (14) by means of (15), we obtain

$$\left.\begin{aligned}\widehat{rr}&=\pm 4k\log_e\frac{r}{b}-\frac{2}{3}\pi\gamma\rho^2(b^2-r^2)-\pi\gamma\rho^2\frac{10\lambda+12\mu}{15(\lambda+2\mu)}(a^2-b^2)\\&\qquad+\frac{2}{3}\left(\frac{b^3}{a^3}+1\right)\left(\frac{8}{15}\pi\gamma\rho^2\frac{\mu}{\lambda+2\mu}b^2\pm 2k\right).\\ \widehat{\theta\theta}&=\widehat{\phi\phi}=\pm 2k+\widehat{rr}\,.\end{aligned}\right\} \tag{16}$$

When $\mu/\lambda\to 0$, these equations reduce to

$$\left.\begin{aligned}\widehat{rr}&=\pm 4k\log_e\frac{r}{b}-\frac{2}{3}\pi\gamma\rho^2(a^2-r^2)\pm\frac{4}{3}k\left(\frac{b^3}{a^3}+1\right),\\ \widehat{\theta\theta}&=\widehat{\phi\phi}=\pm 2k+\widehat{rr}\,.\end{aligned}\right\} \tag{17}$$

In the case of the Earth's core the upper signs in these solutions should always be used.

By comparing (11) and (17), we find that when $\mu/\lambda\to 0$, the additional stresses in the core are

$$\pm 4k \log_e \frac{r}{b} \pm \frac{4}{3}k\left(1+\frac{b^3}{r}\right), \quad \pm 4k \log_e \frac{r}{b} \pm \frac{4}{3}k\left(1-\frac{b^3}{2r^3}\right)$$

for $\widehat{rr}$ and $\widehat{\theta\theta}(=\widehat{\phi\phi})$ respectively.

4. *Possible slip planes in the plastic core.*

The angle β which the slip surfaces in the plastic core make with any radius of the Earth is obtained by the formula

$$\tan 2\beta = \frac{\widehat{rr}-\widehat{\theta\theta}}{2\widehat{r\theta}} = \frac{\widehat{rr}-\widehat{\phi\phi}}{2\widehat{r\phi}}, \tag{19}$$

where $\widehat{r\theta}$ is the shear stress between r and θ, and $\widehat{r\phi}$ that between r and ϕ.

Since $\widehat{r\theta}=0(\widehat{r\phi}=0)$ in the present case, we get $2\beta=\pi/2$, so that

$$\beta=\pi/4, \tag{20}$$

namely $\tan\beta=1$. The equation of the slip surfaces is such that

$$\frac{rd\theta}{dr}\left(=\frac{rd\phi}{dr}\right)=\tan\beta, \tag{21}$$

from which we have

$$\theta=A \log_e r, \qquad \phi=B \log_e r. \tag{22}$$

The surfaces are therefore of the type of logarithmic spiral. It is immaterial whether the sections of the surfaces are directed in the sense of θ or ϕ or in that of a certain azimuth between θ and ϕ.

5. *The numerical solution and its interpretation.*

The approximate values of the radii of the Earth and its core are given by $a=6370$ km, $a-b=2900$ km, so that $b/a=0{\cdot}545$. It is possible to assume $\gamma\rho^2a^2=8.10^{11}$ in C. G. S. units. The value of the plastic constant k of the Earth's core is not known. According to Griggs's tests[3] of solenhofen limestone and marble under high hydrostatic pressure, the yield points due to a longitudinal force do not differ markedly, though increasing somewhat, in the hydrostatic pressure applied.

(i) If we assume that the materials in the core and in the shell immediately next to it have plastic constants of the same value as their respective plastic constants under atmospheric pressure (the actual temperature and melting point being both raised), we may then approximately put

3) D. T. GRIGGS, *Journ. Geol.*, **44** (1936), 541~577.

$$2k = 5.10^9 \ (\text{C. G. S.})\,. \tag{23}$$

Thus, from (9)

$$\left.\begin{aligned} \frac{6\mu C}{b^3} &= 5.10^9 + 4.10^{11}\frac{\mu}{\lambda}\,, \\ (3\lambda+2\mu)B &= 0{\cdot}436.10^{11}\frac{\mu}{\lambda} - 16{\cdot}8.10^{11}, \end{aligned}\right\} \tag{24}$$

from which μC and $(3\lambda+2\mu)$ B are determined as functions of μ/λ. In both equations of (24), μ/λ is assumed to be relatively small. Comparing these results with the condition (8), we get

$$\mu/\lambda < 1{\cdot}36.10^{-2}\,. \tag{25}$$

The result in (24) is therefore

$$\left.\begin{aligned} &\frac{6\mu C}{b^3} < 10{\cdot}4.10^9\,, && (C > 0) \\ &(3\lambda+2\mu)B > -16{\cdot}8.10^{11}. && (B < 0) \end{aligned}\right\} \tag{26}$$

The condition for the Poisson's ratio is

$$\frac{1}{2} > \sigma > 0{\cdot}493\,. \tag{27}$$

The last condition has resulted from the assumption (23). Although in the present calculation the value of $2k$ has been given, no assumption regarding the absolute values of λ and μ has been made. Although the result well conforms with the properties of a general plastic body and also with the nature of seismic core waves, since the radius of the Earth and that of its core are of comparable order and not so much different as in the case of Mars[4], there still remains some question as to whether the Earth's state shall change from elastic to plastic just on its core's boundary.

(ii) The analysis of seismic waves shows that $\sigma = 1/4$ even at such a deep part of the Earth's shell as immediately next to the core. Thus, if we assume that Poisson's ratio σ is 1/4, namely $\lambda = \mu$, in the rocky shell as well as in the core, we then have

$$\left.\begin{aligned} \frac{6\mu C}{b^3} &= 1{\cdot}33.10^{11} + 2k\,, \\ (3\lambda+2\mu)\beta &= -12{\cdot}3.10^{11} + 0{\cdot}213\,k, \end{aligned}\right\} \tag{28}$$

from which μC and $(3\lambda+2\mu)B$ are determined as functions of k. Re-

4) H. JEFFREYS, "The Density Distributions in the Inner Planets," M. N. R. A. S. *Geophys. Suppl.*, **4** (1937), 62—71.

ferring to the condition (8), we get

$$2k > 1{\cdot}22.10^{11}. \tag{29}$$

The result in (28) is therefore

$$\left.\begin{aligned} &\frac{6\mu C}{b^3} > 2{\cdot}55.10^{11}, \\ &(3\lambda+2\mu)B > -12{\cdot}1.10^{11}. \end{aligned}\right\} \tag{30}$$

From the first hypothesis it follows that the material in the rocky shell, owing to some transformation of that material, changes its state from compressible to incompressible in the immediate vicinity of the boundary of the core, that is, σ changes from 1/4 to 1/2, the other properties of the same material changing from elastic to plastic also in the same vicinity. An alternative explanation resulting from the second hypothesis, though impossible to confirm it even with the data of high pressure experiments that have been made, would be such that while the plastic constant k in a fairly deep part in the Earth is as large as $2k=10^{11}$ (C. G. S.), the Poisson's ratio remains nearly $\sigma=1/4$ even within the same core. It is now possible to assume that the actual condition of the Earth's core (and the shell next to it) is probably intermediating between that in (i) and that in (ii).

Added July. 7, 1937.—From our continued study (the paper read July 6, 1937), it was found that the condition (8) is rather needless. The plastic state of the core is naturally accompanied by the plastic state of the Earth's surface staratum of thickness of a few hundred kilometers, the layer intermediating between the surface stratum in question and the core being possibly elastic. The full explanation will be shown in next Bulletin.

36. 地球の核のプラスチック性に就て

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地球の核には縦波のみが通つて横波が通らぬやうである．しかし之は核の物質の剛性が極めて小さい爲であるか，或は捩り振動に對して粘性が特にきく爲であるかは明瞭でない．地球の核が高壓を受けてをる爲にプラスチックの狀態にある事は明かである．一般のプラスチック體の特性の一つは捩りの力に對して粘性が強く働く事であり，他の一つは非壓縮性である．從つて地球の核がプラスチックであるとすれば横波が通り難い事は當然なやうにも思はれる．殊に核のみがプラスチック性であり從つて非壓縮性であるとすると，核の內部とその外部との縦波の速度の差も說明できるやうである．

今，地球の核のみがプラスチック性であり，外殼は彈性であるとし，重力を正確に考へて理論的計算を試みて見る．プラスチックの理論は最も多く用ひられてをる所の極大剪應力の說に從つて試みる．他の理論でも數値の程度はそれ程には違はない．核の中ではすべての狀況が對數函數的の性質を具備する事は容易に證明される．

理論に計算をあてはめて見る場合に，米國のある實驗結果に從ひ，プラスチックの常數が地球の核の附近でも地表面附近のそれと同じであると假定して見ると（溫度の方は，融點が壓力とともに增加するから態と見逃して置いた），核の性質は非壓縮性でなければならぬ事がわかるのである．之は一般のプラスチック體と共通の性質であり，且つ地球の核を通る地震波の性質とも一致してをり，一寸面白いともいひ得る譯である．しかしプラスチック論からいふと，彈性からプラスチック性に變る事が，必ずしも實際の核の表面でなくてもよいといふ事の自由さが殘されをるものである．之は地球の半徑とその核の半徑が同じ程度の値を持つて居り，火星のそれ等のやうな相違がないからである．それで別の考として Poisson 比が核の中までも 1/4 であるとして問題を取扱ふ事もやつて見たのである．

そのやうにして出發して見ると，地球の核の面が恰度彈性からプラスチックに變る面であるとするときに，プラスチックの常數が前に假定した地表面での値よりも 100倍も大きくなければならぬ事が知られるのである．米國のあまり高くない壓力での實驗でも常數が增加する傾向にはあるが，そのやうな高壓のもとでの實驗結果は未だないから，斯る大きなプラスチック性がおかしいとは斷言できない．しかし如何にも大き過ぎるやうに見える事は事實である．

實際問題としては，地球の核及びその附近では Poisson 比が 1/4 と 1/2 との間にあり，プラスチックの常數は地表面で測定したそれの何倍か又は何十倍かのものであると考へる方がよいのではなからうか．

1937 年7月7日追記――この研究を進めた結果（7月6日發表），核のプラスチック性を前記のやうに大きく取らず，普通の値にすると，地球の表面に數百粁の厚さのプラスチック性の層があり，この層と地球の核との間は彈性であればよい事がわかつた．この方が反て種種の事柄をよく說明するやうである．委しい事は次の報告で述べる．