21. *On the Elastic Deformation of a Stratified Body Subjected to Vertical Surface Loads.*

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1. *Introduction.*

The elastic deformation of a semi-infinite body subjected to vertical loads on its surface was discussed some years ago by Boussinesq,[1] Nagaoka,[2] Terazawa,[3] etc. Terazawa[4] furthermore gave a criterion such that the vertical component of surface displacement under vertical loads should invariably be directed downward. On the other hand, some of the results in both Matumura's paper[5] and Anzo's[6] in connexion with the deformation of an elastic foundation seemed to show that in the case of a stratified body the surface deformation may be partly upward. Although Nishimura[7] solved a similar problem more exactly, and obtained a result agreeing with Terazawa's criterion, owing to his case being restricted to a very thin surface layer there is still some doubt whether or not the surface deformation in a body, having a layer of a thickness that is comparable to the width of the load distribution, may still always be in the downward sense. With a view to ascertaining the nature of the problem, we solved it from a different point of view for a wide range of ratio of thickness of layer to width of load distribution, the problem in the present paper, however, being restricted to a two-dimensional case.

2. *Solution of the problem.*

Let ρ, λ, μ, ρ', λ', μ' be the densities and the elastic constants of the surface layer (of thickness H) and the subjacent medium, the axes

1) J. BOUSSINESQ, *Application des Potentials . . . .* , (Paris, 1885).
2) H. NAGAOKA, *Proc. Phys.-Math. Soc., Tokyo,* 6 (1912), 208.
3) K. TERAZAWA, *Journ. Coll. Sci., Tokyo Imp. Univ.,* **37** (1916), 1~64; *Phil. Trans. Roy. Soc.,* **217** (1916), 35.
4) Ditto, *Journ. Phys.-Math. Soc., Japan.* **1** (1927), 141~143.
5) M. MATUMURA, *Journ. Civil Eng., Japan,* **17** (1931), 813~869.
6) Z. ANZÔ, *Bull. Fac. Eng., Kyûsyu Imp. Univ.,* **8** (1933), 232~247.
7) G. NISHIMURA, *Bull. Earthq. Res. Inst.,* **10** (1932), 23~28.

of x, y being taken as shown in Fig. 1. If χ, χ' be Airy's functions, the solutions of the elastic equilibrium of both media can be deduced from the differential equations

$$\nabla^4\chi=0\,,\qquad \nabla^4\chi'=0\,,\tag{1}$$

so that the stresses in both media are such that

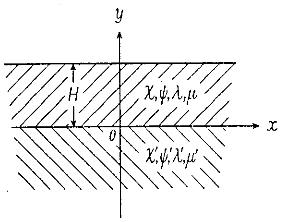

Fig. 1.

$$\left.\begin{aligned}\sigma_x&=\frac{\partial^2\chi}{\partial y^2}\,,\qquad \sigma=\frac{\partial^2\chi}{\partial x^2}\,,\qquad \tau_{xy}=-\frac{\partial^2\chi}{\partial x\partial y}\,,\\ \sigma'_x&=\frac{\partial^2\chi'}{\partial y^2}\,,\qquad \sigma'_y=\frac{\partial^2\chi'}{\partial x^2}\,,\qquad \tau'_{xy}=-\frac{\partial^2\chi'}{\partial x\partial y}\,.\end{aligned}\right\}\tag{2}$$

If complementary functions ψ, ψ' satisfy the conditions

$$\frac{\partial^2\psi}{\partial x\partial y}=\nabla^2\chi\,,\qquad \frac{\partial^2\psi'}{\partial x\partial y}=\nabla^2\chi'\,,\tag{3}$$

$$\nabla^2\psi=0\,,\qquad \nabla^2\psi'=0\,,\tag{4}$$

then the displacements $(u,\ v)$, $(u',\ v')$ in both media are

$$\left.\begin{aligned}2\mu u&=-\frac{\partial\chi}{\partial x}+(1-\sigma)\frac{\partial\psi}{\partial y}\,,\qquad 2\mu v=-\frac{\partial\chi}{\partial y}+(1-\sigma)\frac{\partial\psi}{\partial x}\,,\\ 2\mu' u'&=-\frac{\partial\chi'}{\partial x}+(1-\sigma')\frac{\partial\psi'}{\partial y}\,,\qquad 2\mu' v'=-\frac{\partial\chi'}{\partial y}+(1-\sigma')\frac{\partial\psi'}{\partial x}\,,\end{aligned}\right\}\tag{5}$$

where σ, σ' are the respective Poisson's ratios, namely $\sigma=\lambda/2(\lambda+\mu)$, $\sigma'=\lambda'/2(\lambda'+\mu')$.

The elementary solutions of χ, χ', ψ, ψ' satisfying (1), (4) are such that

$$\left.\begin{aligned}\chi&=\cos mx(Ay\,\mathrm{sh}\,my+B\mathrm{ch}my+Cy\,\mathrm{ch}\,my+D\,\mathrm{sh}\,my)\,,\\ \chi'&=\cos mx(\mathrm{E}ye^{my}+Fe^{my})\,,\end{aligned}\right\}\tag{6}$$

$$\psi=\sin mx(\alpha\,\mathrm{ch}\,my+\beta\,\mathrm{sh}\,my)\,,\qquad \psi'=\sin mx\gamma e^{my}\,.\tag{7}$$

Substituting (6), (7) in (3), we get

$$\alpha=\frac{2}{m}C\,,\qquad \beta=\frac{2}{m}A\,,\qquad \gamma=\frac{2}{m}E\,.\qquad (8),\ (9),\ (10)$$

The boundary conditions are

$$y=0; \qquad u=u', \quad v=v', \quad \sigma_y=\sigma'_y, \quad \tau_{xy}=\tau'_{xy}, \qquad (11),(12),(13),(14)$$

$$y=H; \qquad \tau_{xy}=0, \qquad \sigma_y=\varphi(x). \qquad (15),(16)$$

Substituting (6)∼(10) in (11)∼(16) and using (2), we get

$$\left.\begin{aligned}
\frac{C\Phi}{A}&=\Big[4\mu\mu'(1-\sigma)(1-\sigma')+mH(\mu-\mu')\Big\{(3-4\sigma')\mu+\mu'\Big\}\Big]\mathrm{ch}\,mH\\
&\quad+\Big\{\mu^2(3-4\sigma')-2\mu\mu'\sigma(1-2\sigma')+\mu'^2(1-2\sigma)\Big\}\mathrm{sh}\,mH\,;\\
\frac{D\Phi}{A}&=\frac{-1}{m}\Bigg\{\Big[4\mu\mu'(1-\sigma)(1-\sigma')+mH\Big\{\mu^2(3-4\sigma')-2\mu\mu'\sigma(1-2\sigma')\\
&\quad+\mu'^2(1-2\sigma)\Big\}\Big]\mathrm{ch}\,mH+\Big[\Big\{\mu+\mu'(1-2\sigma)\Big\}\Big\{\mu(3-4\sigma')\\
&\quad-\mu'(1-2\sigma)\Big\}+4mH\mu\mu'(1-\sigma)(1-\sigma')\Big]\mathrm{sh}\,mH\Bigg\},\\
\frac{E\Phi}{A}&=\Big\{4\mu'^2(1-\sigma)^2+2mH\mu'(\mu-\mu')(1-\sigma)\Big\}\mathrm{ch}\,mH\\
&\quad+2\mu'(1-\sigma)\Big[\Big\{\mu+\mu'(1-2\sigma)\Big\}-mH(\mu-\mu')\Big]\mathrm{sh}\,mH\,,\\
\frac{F\Phi}{A}&=\frac{-2\mu'(1-\sigma)}{m}\Bigg\{2\Big\{\mu'(1-\sigma)+mH\mu(1-\sigma')\Big\}\mathrm{ch}\,mH\\
&\quad+\Big[2\mu(1-\sigma')+mH\Big\{\mu(1-2\sigma')+\mu'\Big\}\Big]\mathrm{sh}\,mH\Bigg\},\\
B&=F\,, \quad \alpha=\frac{2}{m}C\,, \quad \beta=\frac{2}{m}A\,, \quad \gamma=\frac{2}{m}E\,,
\end{aligned}\right\} \quad (17)$$

where

$$\Phi=2\mu'(1-\sigma)\Big\{\mu(1-2\sigma')+\mu'\Big\}\mathrm{ch}\,mH$$
$$+\Big[4\mu\mu'(1-\sigma)(1-\sigma')-mH(\mu-\mu')\Big\{(3-4\sigma')\mu+\mu'\Big\}\Big]\mathrm{sh}\,mH. \qquad (18)$$

The elementary value of σ_y at $y=H$ is

$$\sigma_y=-m^2\cos mx\,(AH\,\mathrm{sh}\,mH+B\,\mathrm{ch}\,mH+CH\,\mathrm{ch}\,mH+D\,\mathrm{sh}\,mH). \qquad (19)$$

Hence in the special case

$$y=H; \qquad \sigma_y=P\cos mx\,, \qquad (20)$$

the solution of the corresponding vertical component of displacement at the surface assumes the form

$$v_{y=H}=\frac{PH\cos mx}{\mu}\frac{(1-\sigma)}{mH}\left[\frac{1+2R(mH)e^{-2mH}-Ne^{-4mH}}{1-2\{R(mH)^2+S\}e^{-2mH}+Ne^{-4mH}}\right], \tag{21}$$

where

$$\left.\begin{aligned}
&N=N'/M', \qquad R=R'/M', \qquad S=S'/M',\\
&M'=\mu^2(3-4\sigma')+2\mu\mu'(5-6\sigma'-6\sigma+8\sigma\sigma')+\mu'^2(3-4\sigma),\\
&N'=\mu^2(3-4\sigma')+2\mu\mu'(-3+2\sigma'+2\sigma)+\mu'^2(3-4\sigma),\\
&R'=2(\mu-\mu')\{\mu(3-4\sigma')+\mu'\},\\
&S'=\mu^2(3-4\sigma')+2\mu\mu'(1-2\sigma)(1-2\sigma')+\mu'^2(-5+12\sigma-8\sigma^2).
\end{aligned}\right\} \tag{22}$$

The maximum value of the displacement $v_{y=H}$, which is distributed sinusoidally, for different ratios of the thickness H to the wave length $2\pi/m$ as well as for different values of μ/μ' are shown in Fig. 2. It should be borne in mind that the broken line specially shows the maximum vertical displacement at $y=0$, namely, the bottom boundary of the layer, for the case $\mu/\mu'=1/2$.

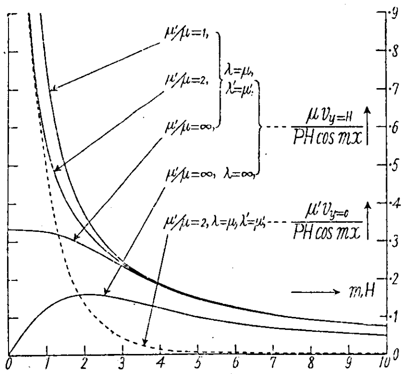

Fig. 2. Maximum values of v for sinusoidally distributed loads.

We shall now generalize the solution in (19) in the form

$$\sigma_y(\text{at } y=H)=\int_0^\infty -m^2\cos mx A\phi(H,m)\,dm\,, \tag{23}$$

where

$$\phi(H,m)=H\,\text{sh}\,mH+\frac{B}{A}\text{ch}\,mH+\frac{C}{A}H\,\text{ch}\,mH+\frac{D}{A}\text{sh}\,mH\,. \tag{24}$$

Comparing (23) with Fourier's integral (for an even function)

$$\sigma_y(\text{at } y=H)=\varphi(x)=\frac{2}{\pi}\int_0^\infty \cos mx\,dm\int_0^\infty \varphi(\lambda)\cos m\lambda\,d\lambda\,,$$

we get

$$A=\frac{-2}{\pi m^2\phi(H,m)}\int_0^\infty \varphi(\lambda)\cos m\lambda\,d\lambda\,. \tag{25}$$

The corresponding vertical component of displacement in the layer is then

$$v=\frac{2}{\pi}\int_0^\infty \frac{\cos mx}{2\mu}\,\frac{\Phi(y,m)}{m^2\varphi(H,m)}\,dm\int_0^\infty \varphi(\lambda)\cos m\lambda d\lambda\,, \tag{26}$$

where

$$\Phi(y,m)=\Big\{my\,\mathrm{ch}\,my-(1-\sigma)\,\mathrm{sh}\,my\Big\}+\frac{B}{A}m\,\mathrm{sh}\,my$$

$$-\frac{C}{A}\Big\{(1-2\sigma)\,\mathrm{ch}\,my-my\,\mathrm{sh}\,my\Big\}+\frac{D}{A}m\,\mathrm{ch}\,my\,. \tag{27}$$

It is possible to write the horizontal component of displacement in the same way as in the foregoing.

3. *Distribution of surface displacements and inclinations in a general case.*

The surface vertical displacement is now clearly of the form

$$v_{y=H}=\int_0^\infty \frac{2\cos mx}{\pi\mu m}(1-\sigma)\left[\frac{1+2R(mH)e^{-2mH}-Ne^{-4mH}}{1-2\{R(mH)^2+S\}e^{-2mH}+Ne^{-4mH}}\right]$$

$$\cdot dm\int_0^\infty \varphi(\lambda)\cos m\lambda d\lambda\,, \tag{28}$$

R, S, N being shown in (22).

In the special case, $\mu=\mu'$, $\sigma=\sigma'$, that is, in the case of a semi-infinite body, there exist the relations $R=0$, $S=0$, $N=0$, so that the integral (28) can be evaluated very readily for any form of $\varphi(\lambda)$.

Let

$$\varphi(x)=\frac{Pa^2}{a^2+x^2}\,, \tag{29}$$

the maximum and the resultant of the pressure being P and πPa respectively. Since the integral (28) is somewhat difficult to evaluate even in the case of pressure distribution (29), we shall for simplicity calculate the surface inclination of the body. The surface inclination in the case of pressure distribution of type (29) assumes the form

$$\frac{\partial v_{y=H}}{\partial x}=-\int_0^\infty \frac{Pa\sin mx\,e^{-ma}}{\mu}(1-\sigma)\left[\frac{1+2R(mH)e^{-2mH}-Ne^{-4mH}}{1-2\{R(mH)^2+S\}e^{-2mH}+Ne^{-4mH}}\right]dm\,. \tag{30}$$

As already mentioned, in the special case where $\mu=\mu'$, $\sigma=\sigma'$, namely, that of a semi-infinite body, the expression within the pair of brackets in (30) tends to unity, the evalution being consequently very simple.

Unless μ'/μ is as large as $\mu'/\mu \gg 2$, the integral (30) can be evaluated by expanding in series the expression within the pair of brackets, the expression of the surface inclination accordingly being

$$\frac{\partial v_{y=H}}{\partial x} = -\int_0^\infty \frac{2Pa(1-\sigma)}{\mu} \sin mx \left\{ \frac{1}{2} e^{-ma} + \gamma_1 e^{-m(a+2H)} + \gamma_2 e^{-m(a+4H)} + \gamma_3 e^{-m(a+6H)} + \dots \right\} dm, \quad (31)$$

where

$$\gamma_1 = S + RHm + RH^2m^2,$$

$$\gamma_2 = (2S^2-N) + 2RSHm + 4RSH^2m^2 + 2R^2H^3m^3 + 2R^2H^4m^4,$$

$$\gamma_3 = S(4S^2-3N) + (4S^2-N)RHm + 3(4S^2-N)RH^2m^2 + 8R^2SH^3m^3 + 12R^2SH^4m^4 + 4R^3H^5m^5 + 4R^3H^6m^6,$$

$$\gamma_4 = (8S^4 - 8NS^2 + N^2) + 4(2S^2-N)RSHm + 16(2S^2-N)RSH^2m^2 + \dots + 8R^4H^8m^8,$$

$$\gamma_5 = S(16S^4-20NS^2+5N^2) + (16S^4-12NS^2+N^2)RHm + 5(16S^4-12NS^2+N^2)RH^2m^2 + \dots + 16R^5H^{10}m^{10},$$

$$\gamma_6 = (32S^6-48NS^4+16N^2S^2-N^3) + 2(16S^4-16NS^2+3N^2)RSHm + 12(16S^4-16NS^2+3N^2)RSH^2m^2 + \dots,$$

$$\gamma_7 = S(64S^6-112NS^4+56N^2S^2-7N^3) + (64S^6-80NS^4+24N^2S^2-N^3)RHm + 7(64S^6-80NS^4+24N^2S^2-N^3)RH^2m^2 + \dots,$$

$$\gamma_8 = (128S^8-256NS^6+160N^2S^4-32N^3S^2+N^4) + 8(16S^6-24NS^4+10N^2S^2-N^3)RSHm + 64(16S^6-24NS^4+10N^2S^2-N^3)RSH^2m^2 + \dots,$$

$$\gamma_9 = S(256S^8-576NS^6+432N^2S^4-120N^3S^2+9N^4) + (256S^8-448NS^6+240N^2S^4-40N^3S^2+N^4)RHm$$

$$+9(256S^8-448NS^6+240N^2S^4-40N^3S^2+N^4)RH^2m^2$$

$$+\cdots\cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (32)$$

In the case where μ'/μ is fairly large, including the case $\mu'/\mu=\infty$, the expanded form shown in (31) diverges. Here we shall alternatively assume that the expression (28) is equivalent to the form

$$v_{y=H}=\frac{H}{\mu\pi}\int_0^\infty \frac{1}{mH}\sum_n A_n e^{-c_n m}dm\int_{-\infty}^{\infty}\varphi(\lambda)\cos m(x-\lambda)d\lambda, \qquad (33)$$

from which it follows that

$$\frac{1}{mH}\sum_n A_n e^{-c_n m}=\frac{(1-\sigma)}{mH}\left[\frac{1+2R(mH)e^{-2mH}-Ne^{-4mH}}{1-2\{R(mH)^2+S\}e^{-2mH}+Ne^{-4mH}}\right], \qquad (34)$$

the pressure distribution being restricted to even type about the plane $x=0$. Since the right-hand side of (34) exactly corresponds to the ordinates of the curves in Fig. 2, the expansion shown on the left-hand side of the same expression is tantamount to analysing the curves under consideration in a series of exponential functions, the values of A_n and c_n being thereby determined. The present method is available for a wide range of cases.

Now, the inclination of the surface corresponding to (33) is written

$$\frac{\partial v_{y=H}}{\partial x}=-\frac{2}{\mu\pi}\int_0^\infty \sum_n A_n e^{-c_n m}\sin mx dm\int_0^\infty \varphi(\lambda)\cos m\lambda d\lambda, \qquad (35)$$

hence assuming the same form of $\varphi(\lambda)$ as that in (29), we get

$$\frac{\partial v_{y=H}}{\partial x}=-\frac{Pa}{\mu}\sum_n\frac{A_n x}{(c_n+a)^2+x^2}. \qquad (36)$$

It may be noted moreover that it is the integral of (36) that gives rise to the displacement distribution, namely,

$$v_{y=H}=-\frac{Pa}{\mu}\sum_n\frac{A_n}{2}\log_e\frac{\left(\frac{c_n}{H}+\frac{a}{H}\right)^2+\left(\frac{x}{H}\right)^2}{\left(\frac{c_n}{H}+\frac{a}{H}\right)^2+\left(\frac{\infty}{H}\right)^2}, \qquad (37)$$

provided the displacement $v_{y=H}$ at $x=\infty$ is zero.

It was found empirically that, for $x>0$, the inclination determined by means of (36) always assumes a positive value for any condition of the elastic constants. This matter will be dealt with more fully in the next section. Using the mathematical results above given, we calculated the distribution of the inclination on the surface for three cases, (i) $\mu'=\mu$,

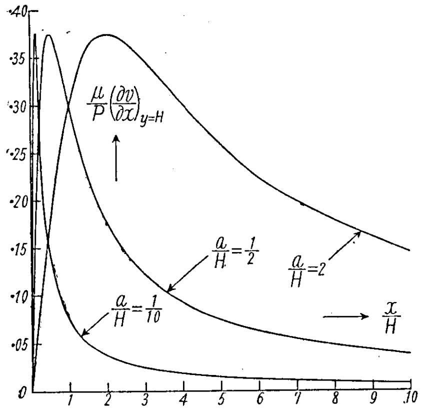

Fig. 3. Distribution of surface displacements. $\mu'=\mu$.

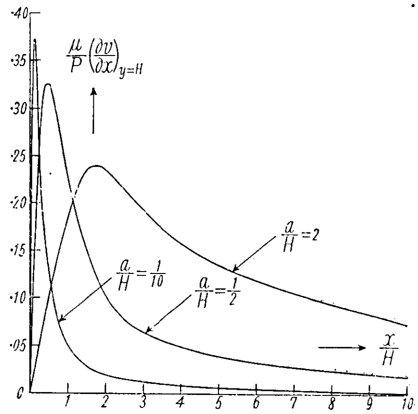

Fig. 4. Distribution of surface displacements. $\mu'/\mu=2$.

(ii) $\mu'/\mu=2$, (iii) $\mu'/\mu=\infty$, and for different values of a/H for these three cases, σ being assumed to be 1/4; the results are shown in Figs. 3, 4, 5. From the form of $\varphi(x)$ in (29), it is shown that the smaller the value of μ/μ' or that of a/H, the greater the concentration of the vertical displacement in the neighbourhood of line $x=0$. It also appears from Figs. 3, 4, 5 that the displacements are invariably directed downward like that due to Terazawa's criterion for the case of a semi-infinite body. From tentative calculations it was found that the present conclusion is also valid for the case of an incompressible body.

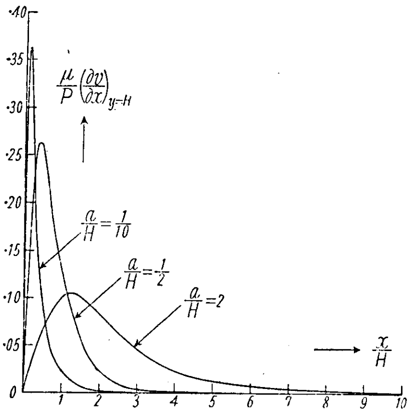

Fig. 5. Distribution of surface displacements. $\mu'/\mu=\infty$.

4. *A more general criterion of the sense of the vertical displacement on the surface.*

Although we have shown some examples of the general deformation of the surface in a stratified body, we have yet given no accurate criterion with regard to the sense of the vertical displacement on the surface.

Now, it is generally possible to write

$$\frac{\partial v_{y=H}}{\partial x}=\frac{-1}{\mu\pi}\int_0^\infty \sum_n A_n e^{-c_n m} \sin mx dm \int_{-\infty}^{\infty} \varphi(\lambda)\cos m\lambda d\lambda \tag{35'}$$

for an even distribution of the surface pressure with respect to x, provided expansion

$$\sum_n A_n e^{-c_n m}=(1-\sigma)\left[\frac{1+2R(mH)e^{-2mH}-Ne^{-4mH}}{1-2\{R(mH)^2+S\}e^{-2mH}+Ne^{-4mH}}\right] \tag{34'}$$

in (34) is possible. Rearranging (35′) in the form

$$\frac{\partial v_{y=H}}{\partial x}=\frac{-1}{\mu\pi}\int_{-\infty}^{\infty}\varphi(\lambda)\, d\lambda \int_0^\infty \sum_n A_n e^{-c_n m} \sin m(x-\lambda) d\lambda\,, \tag{38}$$

and using the integral

$$\int_0^\infty \sum_n A_n e^{-c_n m} \sin m(x-\lambda)\, d\lambda=\sum_n \frac{A_n(x-\lambda)}{c_n^2+(x-\lambda)^2}\,, \tag{39}$$

we have

$$\frac{\partial v_{y=H}}{\partial x}=\frac{-1}{\mu\pi}\int_{-\infty}^{\infty} \sum_n \frac{A_n(x-\lambda)}{c_n^2+(x-\lambda)^2}\varphi(\lambda)d\lambda\,, \tag{40}$$

the integral of which is

$$v_{y=H}=\frac{-1}{\mu\pi}\int_{-\infty}^{\infty} \frac{A_n}{2}\log_e \frac{c_n^2+(x-\lambda)^2}{c_n^2+(\infty-\lambda)^2}\varphi(\lambda)d\lambda\,, \tag{41}$$

under the condition that $v_{y=H}$ at $x=\pm\infty$ is zero. Putting $x-\lambda=-X$, we get

$$v_{y=H}=\frac{-1}{\mu\pi}\int_{-\infty}^{\infty} \sum_n \frac{A_n}{2}\log_e \frac{\left(\frac{c_n}{H}\right)^2+\left(\frac{X}{H}\right)^2}{\left(\frac{c_n}{H}\right)^2+\left(\frac{\infty_1}{H}\right)^2}\varphi(x+X)dX\,, \tag{42}$$

where ∞_1 denotes the condition that X tends to $\mp\infty$.

On the other hand, we find that the displacement distribution for the special type of $\varphi(x)$, namely

$$\varphi(x)=\frac{Pa^2}{a^2+x^2}\,, \tag{29'}$$

is given by

$$v_{y=H} = -\frac{Pa}{\mu}\sum_n \frac{A_n}{2}\log_e \frac{\left(\frac{c_n}{H}+\frac{a}{H}\right)^2+\left(\frac{x}{H}\right)^2}{\left(\frac{c_n}{H}+\frac{a}{H}\right)^2+\left(\frac{\infty}{H}\right)^2}, \tag{37'}$$

where A_n's and c_n's are invariably the same as those in (34′), namely, those in (42). In the special case where the pressure of type $\varphi(x)$ is concentrated quite close to $x=0$, it is possible to put $a/H \to 0$, so that the expression of (37′) may assume the form

$$v_{y=H} = -\frac{Pa}{\mu}\sum_n \frac{A_n}{2}\log_e \frac{\left(\frac{c_n}{H}\right)^2+\left(\frac{x}{H}\right)^2}{\left(\frac{c_n}{H}\right)^2+\left(\frac{\infty}{H}\right)^2}, \tag{43}$$

regardless of whether c_n/H is large or small. The assumed condition $a/H \to 0$ is virtually equivalent to treatment of the case of a semi-infinite body. From Terazawa's criterion,[8] therefore, the expression of $v_{y=H}$ in (43) always assumes a negative value for any value of $x(\lesseqgtr 0)$. Thus x in (43) may be replaced by X, whence it follows that the first factor in the integrand in (42), is greater than zero, that is

$$\sum_n \frac{A_n}{2}\log_e \frac{\left(\frac{c_n}{H}\right)^2+\left(\frac{X}{H}\right)^2}{\left(\frac{c_n}{H}\right)^2+\left(\frac{\infty_1}{H}\right)^2} > 0 \tag{44}$$

for any $X(\lesseqgtr 0)$.

From the condition that the pressure is always directed in the sense of gravity, the second factor in the integrand in (42) is also greater than zero, namely,

$$\varphi(x+X) > 0 \tag{45}$$

for any $x+X(x \lesseqgtr 0,\ X \gtreqless 0)$.

From (44), (45) we conclude that the expression in (42) is always negative, that is,

$$v_{y=H} < 0, \tag{46}$$

the problem thus being proved from general considerations. Whether the material of the solid is compressible or incompressible does not matter.

It is easy, though not rigorous, to prove alternatively from common sense reasoning that $v_{y=H} < 0$. The deformation of a stratified body under

8) K. TERAZAWA, *loc. cit.* 4).

vertical loads, distributed in whatever way, is virtually the same as the resultant in the superposition of different deformations, every one of which is caused by a point load. Since the problem of the deformation of a body of any stratification that is subjected to a point load, is invariably the same as that of a semi-infinite body, the deformation due to every point load is always directed downward. The superposition of the deformations of downward displacements gives a resultant displacement of the same downward sense, the proof of the problem being thus very simply obtained. It thus appears, at all events, that all problems, whether of a semi-infinite body, or even of a multi-layered body, can be reduced to that of the case of a point load. It should however be borne in mind that although this final discussion has resulted also from a special case of Terazawa's criterion, the explanation shown in the beginning of this section, in which the general case of Terazawa's condition was partly availed of, is a more exact one.

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21. 地表層が鉛直荷重を受ける場合の彈性的變形分布に就て

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金井清

半無限體が鉛直荷重を受ける場合の彈性的變形の問題は幾多の學者によつて研究し盡され，その場合に表面變形が必ず下向になるといふ規範を寺澤教授などが態々出した程にもなつてをるのである．然るに表面層のある場合は計算が遙かに困難になる爲に，ある計算の結果では必しも寺澤教授の規範の如くならぬ場合さへも現れてをるのである．しかし之の正確な決定は普通の數學的運算法では寧ろ定らぬのが當然のやうにも見える．

之を正確に決定する爲にこの論文を作つたのであつて，その理論的證明は半無限體のとき程簡單には行かぬけれども，兎も角も特別の應用數學的方法を問題の一般性が失はれぬやうに採用し且つ寺澤教授の規範も部分的に利用することにより，結局半無限體の場合と同じ結論に到達したのである．卽ち層のある場合でも表面變位はやはり下向にしかならぬといふことである．

尙，後で氣のついたことは，層のある場合でもない場合でも問題は表面に點の荷重の働く場合のよせ集めであるといふことである．點の場合には表面層が無限に薄い場合以外はすべて半無限體の場合に持つて行くことができる．從てあらゆる場合の問題は點の荷重の場合の規則さへわかつてをればよいといふことになり，卽ちこの論文の問題に常識的の證明を與へ得ることもわかるのである，但し之は寺澤規範の極く特別の場合を應用したのに過ぎず前の一般的の說明の方が理論的に正確である．