23. *On the Free Vibrations of a Surface Layer due to an Obliquely Incident Disturbance.*

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1. It has been established that if the primary waves were directed vertically upwards, the damping constant of the vibration would invariably be the same whatever the type of wave form of the initial disturbance.[1] If $V_1=\sqrt{(\lambda'+2\mu')/\rho'}$, $V_2=\sqrt{\mu'/\rho'}$, $\alpha_2=\sqrt{\rho'\mu'/\rho\mu}$, $\alpha_1=\sqrt{\rho'(\lambda'+2\mu)/\rho(\lambda+2\mu)}$, the coefficients of damping are

$$k_1=\frac{V_1}{2H}\log_e\left|\frac{1+\alpha_1}{1-\alpha_1}\right|,\quad k_2=\frac{V_2}{2H}\log_e\left|\frac{1+\alpha_2}{1-\alpha_2}\right| \tag{1}$$

for dilatational and distortional disturbances respectively. Although the problem is greatly complicated, if the disturbance is directed obliquely to the layer, the special case in which the obliquely incident primary disturbance consists of transverse waves with movements orientated horizontally, is particularly simple, the damping coefficient then being uniquely determined as a certain function of the incident angle,[2] namely,

$$k_2=\frac{V_2}{2H\cos e'}\log_e\left|\frac{1+\nu}{1-\nu}\right|, \tag{2}$$

where $\nu=\sqrt{\rho'\mu'/\rho\mu}(\cos e'/\cos e)$.

The nature of damping of free vibration due to obliquely incident longitudinal waves or to obliquely incident transverse waves with movements orientated in a vertical plane, however, is not simple. This is because no point along the layer is ever subjected to the initial disturbance simultaneously, and also because, after every free vibration, whether of longitudinal or transverse type, both longitudinal and transverse vibrations are excited. Discussion of this problem from the elementary

1) K. SEZAWA and K. KANAI, "Decay Constants of Seismic Vibrations of a Surface Layer," *Bull. Earthq. Res. Inst.*, **13** (1935), 255.

2) Ditto, "Damping in Seismic Vibrations of a Surface Layer due to an Obliquely Incident Disturbance," *Bull. Earthq. Res. Inst.*, **14** (1936), 357.

stage of the solutions is however extremely difficult. It appears however that Nishimura's result[3] in connection with the forced vibration of a layer due to an obliquely incident disturbance renders it possible to examine the problem without encountering serious difficulties. Although we studied previously also a case[4] included in Nishimura's solutions, owing to its being a rather limited case, the result is of little avail for the present discussion.

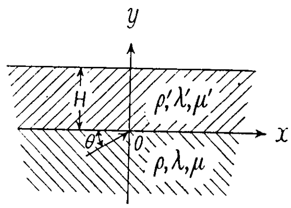

Fig. 1.

2. The elementary solutions are such that

$$\left.\begin{aligned} &\phi_0 = e^{i(fx-ry-pt)}, \\ &\phi = Ae^{i(fx+ry-pt)}, \quad \psi = Be^{i(fx+sy-pt)}, \\ &\phi' = Ce^{i(fx-r'y-pt)} + De^{i(fx+r'y-pt)}, \quad \psi' = Ee^{i(fx-s'y-pt)} + Fe^{i(fx+s'y-pt)}, \\ &r^2 = \frac{\rho p^2}{\lambda+2\mu} - f^2, \; s^2 = \frac{\rho p^2}{\mu} - f^2, \; r'^2 = \frac{\rho' p^2}{\lambda'+2\mu'} - f^2, \; s'^2 = \frac{\rho' p^2}{\mu'} - f^2. \end{aligned}\right\} \quad (3)$$

From the boundary conditions at $y=0$ and $y=H$ it is possible to determine A, B, C, D, E, F, namely,

$$\left.\begin{aligned} A &= \Delta^{-1}\{\alpha_A + \beta_A e^{i(a'+b')fH} + \gamma_A e^{i(a'-b')fH} + \delta_A e^{-i(a'+b')fH} + \varepsilon_A e^{-i(a'-b')fH}\}, \\ B &= \Delta^{-1}\{\alpha_B + \beta_B e^{i(a'+b')fH} + \gamma_B e^{i(a'-b')fH} + \delta_B e^{-i(a'+b')fH} + \varepsilon_B e^{-i(a'-b')fH}\}, \\ C &= \Delta^{-1}\{\alpha_C + \delta_C e^{-i(a'+b')fH} + \varepsilon_C e^{-i(a'-b')fH}\}, \\ D &= \Delta^{-1}\{\alpha_D + \beta_D e^{i(a'+b')fH} + \gamma_D e^{i(a'-b')fH}\}, \\ E &= \Delta^{-1}\{\alpha_E + \gamma_E e^{i(a'-b')fH} + \delta_E e^{-i(a'+b')fH}\}, \\ F &= \Delta^{-1}\{\alpha_F + \beta_F e^{i(a'+b')fH} + \varepsilon_F e^{-i(a'-b')fH}\}, \end{aligned}\right\} \quad (4)$$

where

$$\Delta^{-1} = Me^{ifH[a'(m-2p)+b'(m-2n+2q)]}, \text{ [5]} \quad (5)$$

$$M = \sum_{m=0}^{\infty} (-1)^m \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} \sum_{p=0}^{n} \frac{n!}{p!(n-p)!} \cdot \sum_{q=0}^{p} \frac{p!}{q!(p-q)!} \alpha_\Delta^{-(m+1)} \beta_\Delta^{m-n} \gamma_\Delta^{n-p} \delta_\Delta^{p-q} \varepsilon_\Delta^{q}. \quad (6)$$

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3) G. NISHIMURA, "On the Effect of Discontinuity Surface on the Propagation of Elastic Waves (VII ," *Bull. Earthq. Res. Inst.*, **13** (1935), 540~554.

4) K. SEZAWA and K. KANAI, "Reflection and Refraction of Seismic Waves in a Stratified Body," *Bull. Earthq. Res. Inst.*, **10** (1932), 805~816; **12** (1934), 269~276.

5) G. NISHIMURA also expanded Δ^{-1} in a somewhat similar form for evaluating the integral for free waves but not for obtaining the modulus of damping.

M may be named as *decay modulus* of vibrations. The terms α_Δ, β_Δ, ...; α_A, β_A, ...; α_B, β_B, ...; ......; α_F, β_F, ... in (2), (3), (4) correspond to the respective coefficients of e^o, $e^{i(r'+s')H}$, $e^{i(r'-s')H}$, $e^{-i(r'+s')H}$, $e^{-i(r'-s')H}$, in Δ, Δ_A, Δ_B, Δ_C, Δ_D, Δ_E, Δ_F, in Nishimura's paper[6]; the same notations being arranged in Table I.

Table I.

Coeffient of	in Δ	in Δ_A	in Δ_B	in Δ_C	in Δ_D	in Δ_E	in Δ_F
e^o	α_Δ	α_A	α_B	α_C	α_D	α_E	α_F
$e^{i(r'+s')H}$	β_Δ	β_A	β_B	β_C	β_D	β_E	β_F
$e^{i(r'-s')H}$	γ_Δ	γ_A	γ_B	γ_C	γ_D	γ_E	γ_F
$e^{-(ir'+s')H}$	δ_Δ	δ_A	δ_B	δ_C	δ_D	δ_E	δ_F
$e^{-i(r'-s')H}$	ε_Δ	ε_A	ε_B	ε_C	ε_D	ε_E	ε_F

3. Fourier's integral for a two-dimensional case is

$$\chi(x,\ y)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}df\int_{-\infty}^{\infty}dr\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}\chi(\xi,\ \eta)e^{if(x-\xi)-ir(y-\eta)}d\eta, \tag{7}$$

from which we may write

$$\phi_o=\Phi\{fx-ry-pt\}, \tag{8}$$

Φ being the type of initial disturbance, and

$$\begin{aligned}\phi=M\Big\{\alpha_A\Phi&\Big[f\{x+(m-2p)a'H+(m-2n+2q)b'H\}+ry-pt\Big]\\
+\beta_A\Phi&\Big[f\{x+(m-2p+1)a'H+(m-2n+2q+1)b'H\}+ry-pt\Big]\\
+\gamma_A\Phi&\Big[f\{x+(m-2p+1)a'H+(m-2n+2q-1)b'H\}+ry-pt\Big]\\
+\delta_A\Phi&\Big[f\{x+(m-2p-1)a'H+(m-2n+2q-1)b'H\}+ry-pt\Big]\\
+\varepsilon_A\Phi&\Big[f\{x+(m-2p-1)a'H+(m-2n+2q+1)b'H\}+ry-pt\Big]\Big\},\end{aligned} \tag{9}$$

6) G. Nishimura, *loc. cit.* 3).

$$
\begin{aligned}
\psi = M\Big\{ & \alpha_B \Phi\Big[f\{x+(m-2p)a'H+(m-2n+2q)b'H\}+sy-pt\Big] \\
& + \beta_B \Phi\Big[f\{x+(m-2p+1)a'H+(m-2n+2q+1)b'H\}+sy-pt\Big] \\
& + \gamma_B \Phi\Big[f\{x+(m-2p+1)a'H+(m-2n+2q-1)b'H\}+sy-pt\Big] \\
& + \delta_B \Phi\Big[f\{x+(m-2p-1)a'H+(m-2n+2q-1)b'H\}+sy-pt\Big] \\
& + \varepsilon_B \Phi\Big[f\{x+(m-2p-1)a'H+(m-2n+2q+1)b'H\}+sy-pt\Big]\Big\},
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\phi' = M\Big\{ & \alpha_C \Phi\Big[f\{x+(m-2p)a'H+(m-2n+2q)b'H\}-r'y-pt\Big] \\
& + \delta_C \Phi\Big[f\{x+(m-2p-1)a'H+(m-2n+2q-1)b'H\}-r'y-pt\Big] \\
& + \varepsilon_C \Phi\Big[f\{x+(m-2p-1)a'H+(m-2n+2q+1)b'H\}-r'y-pt\Big] \\
& + \alpha_D \Phi\Big[f\{x+(m-2p)a'H+(m-2n+2q)b'H\}+r'y-pt\Big] \\
& + \beta_D \Phi\Big[f\{x+(m-2p+1)a'H+(m-2n+2q+1)b'H\}+r'y-pt\Big] \\
& + \gamma_D \Phi\Big[f\{x+(m-2p+1)a'H+(m-2n+2q-1)b'H\}+r'y-pt\Big]\Big\},
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\psi' = M\Big\{ & \alpha_E \Phi\Big[f\{x+(m-2p)a'H+(m-2n+2q)b'H\}-s'y-pt\Big] \\
& + \gamma_E \Phi\Big[f\{x+(m-2p+1)a'H+(m-2n+2q-1)b'H\}-s'y-pt\Big] \\
& + \delta_E \Phi\Big[f\{x+(m-2p-1)a'H+(m-2n+2q-1)b'H\}-s'y-pt\Big] \\
& + \alpha_F \Phi\Big[f\{x+(m-2p)a'H+(m-2n+2q)b'H\}+s'y-pt\Big] \\
& + \beta_F \Phi\Big[f\{x+(m-2p+1)a'H+(m-2n+2q+1)b'H\}+s'y-pt\Big] \\
& + \varepsilon_F \Phi\Big[f\{x+(m-2p-1)a'H+(m-2n+2q+1)b'H\}+s'y-pt\Big]\Big\}.
\end{aligned}
\tag{12}
$$

Since the form of decay modulus M in (6) is independent of the type of original disturbance, Φ, the behaviour of the free vibration of the layer is invariably the same, though it is not so simple as that in exponential damping.

In equations (9) ~ (12), α_A, β_A, ...; α_B, β_B, ...; ...... are functions of densities as well as of elastic constants in the layer and the subjacent medium, besides of the incidence angle of the primary waves. Since, on the other hand, ϕ, ψ, ϕ', ψ' have the same and invariable damping modulus M (shown in (6)), the respective vibrational modes, namely dilatational and distortional, including both types of dissipation waves, decay quite in the same way. The only difference between ϕ, ψ, ϕ', ψ' is in the ratios of the general amplitudes, which differ with the differences in α_A, β_A, ...; α_B, β_B, ...; ......, r, s, r', s', so that it is not possible for even one out of ϕ, ψ, ϕ', ψ' to prevail even with lapse of time.

It may seem odd that the vibrations are not separated into two kinds, dilatational and distortional. Such separation however would be hardly possible unless the primary disturbance were composed of dilatational and distortional waves even in the case of simultaneous application of the disturbance at all points on the boundary surface of the layer.

4. It has already been remarked that from the decay modulus shown in (6), namely,

$$M=\sum_{m=0}^{\infty}(-1)^m\sum_{n=0}^{m}\frac{m\,!}{n\,!(m-n)\,!}\sum_{p=0}^{n}\frac{n\,!}{p\,!(n-p)\,!}$$

$$\cdot\sum_{q=0}^{p}\frac{p\,!}{q\,!(p-q)\,!}\alpha_\Delta^{-(m+1)}\beta_\Delta^{m-n}\gamma_\Delta^{n-p}\delta_\Delta^{p-q}\epsilon_\Delta^{q}, \tag{6'}$$

no such simple decay constant as the coefficient of exponential function is deduced. Were the disturbance, however, to be applied simultaneously at all points on the boundary surface of the layer just mentioned, the modulus under consideration would immediately give the damping constant. In that case the direction of propagation of the disturbance should be in the vertical sense, from which results the condition that f tends to zero, while r, s, r', s' remain as finite values. It follows then that Nishimura's values Δ, Δ_A, Δ_B, ..., Δ_F[7] tend to assume the forms

$$\Delta=\frac{ss'^5}{f^{10}}\chi\{(\mu s^2r'-\mu'rs'^2)e^{ir'H}+(\mu s^2r'+\mu'rs'^2)e^{-ir'H}\},$$

7) G. NISHIMURA, *loc. cit.* 3).

$$\left.\begin{aligned}
&\Delta_A=\frac{-ss'^5}{f^{10}}\chi\{(\mu r's^2+\mu' rs'^2)e^{ir'H}+(\mu r's^2-\mu' rs'^2)e^{-ir'H}\},\\
&\Delta_B\to 0, \qquad \Delta_C=\frac{2}{f^{10}}rs^3s'^5\mu e^{-ir'H}\chi,\\
&\Delta_D=\frac{-2}{f^{10}}rs^3s'^5e^{ir'H}\chi, \qquad \Delta_E\to 0, \qquad \Delta_F\to 0,\\
&\chi=(\mu s-\mu' s')e^{is'H}+(\mu s+\mu' s')e^{-is'H}.
\end{aligned}\right\} \tag{13}$$

Whence

$$\left.\begin{aligned}
A&=\frac{-\{(1+\alpha)e^{2ir'H}+(1-\alpha)\}}{(1-\alpha)e^{2ir'H}+(1+\alpha)},\\
C&=\frac{2\beta}{(1-\alpha)e^{2ir'H}+(1+\alpha)},\\
D&=\frac{-2\beta e^{2ir'H}}{(1-\alpha)e^{2ir'H}+(1+\alpha)},
\end{aligned}\right\} \tag{14}$$

where $\alpha=\sqrt{\rho'(\lambda'+2\mu')/\rho(\lambda+2\mu)}$, $\beta=\sqrt{\rho(\lambda'+2\mu')/\rho'(\lambda+2\mu)}$.

If the primary waves

$$\phi_0=\Phi(-ry-pt) \tag{15}$$

are incident on the bottom surface of the layer, free oscillation of the layer of the type

$$\begin{aligned}
\phi'=&2\beta\sum_{m=0}^{\infty}(-1)^m\frac{(1-\alpha)^m}{(1+\alpha)^{m+1}}\Phi(-r'y-pt+2mr'H)\\
&-2\beta\sum_{m=0}^{\infty}(-1)^m\frac{(1-\alpha)^m}{(1+\alpha)^{m+1}}\Phi\{r'y-pt+2(1+m)r'H\},
\end{aligned} \tag{16}$$

and the dissipation waves

$$\begin{aligned}
\phi=&-\sum_{m=0}^{\infty}(-1)^m\frac{(1-\alpha)^{m+1}}{(1+\alpha)^{m+1}}\Phi(ry-pt+2mr'H)\\
&-\sum_{m=0}^{\infty}(-1)^m\frac{(1-\alpha)^m}{(1+\alpha)^m}\Phi\ ry-pt+2(1+m)r'H\}
\end{aligned} \tag{17}$$

are excited, whence the vibration of the layer are of the purely exponential type

$$e^{-kt}$$

with the coefficient of damping

$$k = \frac{V'}{2H} \log_e \left| \frac{1+\alpha}{1-\alpha} \right|. \tag{18}$$

It should be borne in mind that, even should the initial disturbance Φ be of a very sharp type, the functions Φ in ϕ' or ϕ would be deformed to a fairly gradual type after every successive oscillation due to inner resistance,[8] etc., so that it would then be possible for the vibration to assume almost sinusoidal and damped forms.

At all events, the special point in our ideas that differs from the classical theory of free vibration is that in the present case, the vibrations of higher orders are unlikely to occur. From results of vibration experiment, however, it would be rather improbable for the natural vibrations of higher orders in a body to exist under the action of a single shock of very short duration (compared with its first natural period) and of localized type, even though such vibrations might be damped by other causes such as viscous friction, etc. The only possible case of exciting vibrations of higher orders due to apparent short duration is that in which the sources of disturbance are initially so distributed in a body as in the case of statical deformation of the same body that is to be released simultaneously.

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23. 斜のディスターバンスによつて誘起する表面層の自由振動に就て

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縦波が來ても横波が來てもそれが地中の眞下から來る場合には，それによつて誘起する表面層の自由振動は逸散性によつて減衰するとしても，その減衰性が簡單な指數函數的性質卽ち粘性によつて減衰する場合の性質と變らぬことがわかつた．又斜に波動が來る場合でも，振動方向が水平の横波の場合には特に同じ性質のあることがわかつたのである．然るにそれ以外の振動方向の横波や一般的の縱波が斜から來る場合は，減衰性が非常に複雜になるものである．この場合の計

8) K. SEZAWA, "Decay of Waves in a Visco-elastic Solid Bodies," *Bull. Earthq. Res. Inst.*, 3 (1927), 43.

算を根本からやることは可なり面倒であるが，幸にも西村技師により同じやうな場合の強制振動の研究結果が出してあるからその結果を借用して計算を試みた．勿論，西村技師の計算の特別の場合はそれよりも前に筆者等が出したけれども西村技師のもの程に一般的には役に立たぬ．

只今の研究の結果，表面層の振動の減衰性には相當複雑な性質があり，指數函數などでは規定できぬけれども兎も角も一定の方則のある Decay modulus に從て減衰し，それは最初のディスターバンスの型によらぬことがわかつたのである．又その振動は縱波の種類と横波の種類とから成立つてをるけれども，その何れかの型が時間と共に後迄殘るといふやうなことはなく同じ割合で減衰することが知られた．

又，問題を特別の場合，卽ち入射角が鉛直の場合にもつて行くと複雑な計算の後に初め述べたやうな簡單な場合になることがわかるのである．一寸考へると，斜のディスターバンスによつて表面層の縱振動と横振動とが誘起されその一方がより速に減衰するやうに見えるけれども，計算の結果は寧ろ鉛直方向に縱横2種のディスターバンスが同時に働く場合にそのやうな性質の存在し得ることを示すものである．

斜のディスターバンスの働くのは極く近距離に震源のある場合であらうから，實際問題としても現象の分析が稍困難な場合に當つてをる譯である．しかし遠方の震源の場合にはディスターバンスが衝撃型でなく週期的振動の方が勝つてをるから，その方から別の困難が生ずるのである．何れにしても表面層の自由振動といふ問題は餘り簡單でなく寧ろ不明瞭な事柄に近い譯である．尙，この論文等の計算に於て始めの衝擊の型が繰返されるやうに見えるかも知れぬけれども，實際問題としては固體內の粘性等の爲に繰返しの振動が減衰性正弦波に近くなるのは當然といはねばならぬ．

只今のやうな新しい問題と古典的な自由振動の考卽ち節平面が不動であるといふ問題との間の非常な相違は，只今の問題では衝擊によつて高次の自由振動が現れ難いことである．しかしディスターバンスが衝擊型であるにしても適當の間隔で二度與へられれば第2次の減衰自由振動が現れ得るものである．この場合に節平面が振動性になるのはいふまでもない．但し單なる衝擊で高次の自由振動が現れ難いことは古典的考から得られた實驗卽ち固體の第1次振動週期よりも短い時間で働く衝擊の場合の結果から見ると同様ではある．之等に關する概念は週期的强制振動のある場合の共振に於ける節面が逸散のないときの自由振動の場合の節面と同じになることから容易に得られる譯である．別に注意すべき事は，短い時間中に衝擊が働くやうに見えても最初の時間に應力が固體全體に適當に分布せる場合，卽ち始めは靜力學的に力が働く場合には，ディスターバンスの原點が固體中到る所にあるから，このときには高次振動があつても差支がない譯である．