# 45. *The Effect of Differences in the Media on the Distribution of Displacements in a Seismic Wave Front.*

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## 1. *Introduction.*

Some years ago[1] I gave formulae relating to the energy separation of elastic waves into dilatational and distortional waves when they are radiated from a spherical origin, the forces on the spherical surface being distributed in the form of zonal harmonics. As the object of that paper was merely to show the mechanism of energy separation under consideration, we gave no numerical example or any varification of the correctness of the symbolic expressions. Although another paper[2] was later written with a view to showing accurately the corrected expressions of the solution, the nature of the original problem was not studied. Inouye[3], who recently made use of our formulae in his study of initial motion of seismic waves, showed that the distribution of initial motion in a wave front is possible only in the case of short waves —a condition familiar to us in the case of acoustic waves—although no study to speak of has been made of it in the field of seismology. The results of his studies therefore seem to have thrown some light on the problem of initial motion that is being discussed frequently now in Japan.

The distribution of the initial motion, that is, the distribution of the displacements in a wave front at different radial distances is also affected by the nature of the medium through which the waves are transmitted. Unless both the density and the elasticity of the seismic region are about the same for different earthquakes, the study of the mechanism of a seismic origin from the data of the initial motion would be impossible even if the frequency of the initial motion were specified.

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1) K. SEZAWA, "Dilatational and Distortional Waves generated from a Cylindrical or Spherical Origin", *Bull. Earthq. Res. Inst.*, **2** (1927), 13～20.

2) K. SEZAWA and K. KANAI, "Amplitudes of P- and S-waves at Different Focal Distances", *Bull. Earthq. Res. Inst.*, **10** (1932), 299～334; *Sindôgaku* (1932), 656.

3) W. INOUYE, "On the Mechanism of a Seismic Origin", *Disin*, **8** (1936), 387～403, (in Japanese).

With the aid of Kanai[4] the formulae for calculating the foregoing has recently been improved to such a degree of accuracy as to entirely exclude errors, even in the algebraic signs of the resultant displacement. The present paper is the outcome of Inouye's suggestion to publish in a form available for ready use the somewhat inconvenient formulae that I had originally obtained.

Although the generation of dilatational and distortional waves from a spherical origin may appear to be preferable from the practical point of view, the treatment of such a problem is somewhat difficult. Since in contrast to it, radiation of pure distortional waves from a spherical origin is very simple, and it still gives a general idea of the problem, whence, a full explanation of it will be made in my treatment here of radiation of pure distortional waves. The case in which both kinds of waves are generated will be dealt with mainly from the theoretical standpoint, argumented with a few numerical examples.

## 2. *Pure distortional waves generated from a spherical origin.*

The problem was shown in my previous paper[5], the solution being

$$w_n = A_n \frac{H^{(2)}_{n+\frac{1}{2}}(kr)}{\sqrt{kr}} \frac{dP_n(\cos\theta)}{d\theta} e^{ipt}, \tag{1}$$

where $k^2 = \rho p^2/\mu$ and $w_n$ is the displacement directed azimuthally. If the disturbance at the surface of the spherical origin be $S_n dP_n(\cos\theta)/d\theta$, the condition at $r=a$ is

$$\mu \frac{\partial w_n}{\partial r} = S_n \frac{dP_n(\cos\theta)}{d\theta} e^{ipt}, \tag{2}$$

from which

$$A_n = \frac{S_n}{k\mu} \Big/ \left(\frac{d\phi}{d(kr)}\right)_{r=a}, \tag{3}$$

$\phi$ being written for $H^{(2)}_{n+\frac{1}{2}}(kr)/\sqrt{kr}$. If we write $x = ka$, the algebraic expression for $\phi$ for a few cases are as follows:

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4) K. SEZAWA and K. KANAI, "The Polarization of Elastic Waves generated from a Plane Source," *Bull. Earthq. Res. Inst.*, **14** (1936), 489~505.

5) K. SEZAWA, *loc. cit.* 1).

$$n=0;\quad \phi(x)=\sqrt{\frac{2}{\pi}}\frac{i}{x}e^{-ix},$$

$$n=1;\quad \phi(x)=\sqrt{\frac{2}{\pi}}\left(\frac{i}{x^2}-\frac{1}{x}\right)e^{-ix},$$

$$n=2;\quad \phi(x)=\sqrt{\frac{2}{\pi}}\left\{i\left(\frac{3}{x^3}-\frac{1}{x}\right)-\frac{3}{x^2}\right\}e^{-ix},$$

$$n=3;\quad \phi(x)=\sqrt{\frac{2}{\pi}}\left\{i\left(\frac{15}{x^4}-\frac{6}{x^2}\right)-\left(\frac{15}{x^3}-\frac{1}{x}\right)\right\}e^{-ix},$$

$$n=4;\quad \phi(x)=\sqrt{\frac{2}{\pi}}\left\{i\left(\frac{105}{x^5}-\frac{45}{x^3}+\frac{1}{x}\right)-\left(\frac{105}{x^4}-\frac{10}{x^2}\right)\right\}e^{-ix}.$$

The displacement at an infinite distance from the origin then becomes

$$w_n=\frac{(i)^n e^{ik(a-r)}}{k^2\mu r M_n}S_n\frac{dP_n}{d\theta}e^{-ipt}, \tag{4}$$

$M_n$ being

$$n=0;\quad M_0=\left(\frac{1}{x}-\frac{i}{x^2}\right)e^{-ix},$$

$$n=1;\quad M_1=\left\{\frac{2}{x^2}-i\left(\frac{2}{x^3}-\frac{1}{x}\right)\right\}e^{-ix},$$

$$n=2;\quad M_2=\left\{\left(\frac{9}{x^3}-\frac{1}{x}\right)-i\left(\frac{9}{x^4}-\frac{4}{x^2}\right)\right\}e^{-ix},$$

$$n=3;\quad M_3=\left\{\left(\frac{60}{x^4}-\frac{7}{x^2}\right)-i\left(\frac{60}{x^5}-\frac{27}{x^3}+\frac{1}{x}\right)\right\}e^{-ix},$$

$$n=4;\quad M_4=\left\{\left(\frac{525}{x^5}-\frac{65}{x^3}+\frac{1}{x}\right)-i\left(\frac{525}{x^6}-\frac{240}{x^4}+\frac{11}{x^2}\right)\right\}e^{-ix}.$$

If $S_n dP_n(\cos\theta)/d\theta$ be a constant for different $n$, the ratio of $|w_n|/|w_1|$ assumes the values shown in Table I.

Table I.

| $p\sqrt{\frac{\rho}{\mu}}a$ | 10 | 2 | 1 | 0·5 | 0·1 |
|---|---|---|---|---|---|
| $\lvert w_2\rvert/\lvert w_1\rvert$ | 1·0 | 0·734 | 0·184 | 0·152 | 0·0213 |
| $\lvert w_3\rvert/\lvert w_1\rvert$ | 1·035 | 0·322 | 0·0276 | 0·00830 | $0{\cdot}0^3334$ |
| $\lvert w_4\rvert/\lvert w_1\rvert$ | 1·05 | 0·0578 | 0·00316 | $0{\cdot}0^3471$ | $0{\cdot}0^4384$ |

Let the period of the waves and the circumference of the spherical origin be $T=1$ sec and $2\pi a=1$ km respectively. If $|w_1|$ for the case $\sqrt{\mu/\rho}=1$ km/s is assumed to be unity, then the values of $|w_n|$ for different $\sqrt{\mu/\rho}$'s under the assumption that $S_n|dP_n/d\theta|$ as well as $\rho$ always remain constants, are shown in Table II. Since the respective maximum values of $|dP_1/d\theta|$, $|dP_2/d\theta|$, $|dP_3/d\theta|$,……are 1, 3/2, $8/\sqrt{15}$,…, the order of each value in this table would not change even were $S_n$ assumed to be constant in lieu of $S_n|dP_n/d\theta|$.

Table II.

| $\sqrt{\mu/\rho}$ km/s | 0·1 | 0·5 | 1 | 2 | 10 |
|---|---|---|---|---|---|
| $\lvert w_1\rvert$ | 17·3 | 3·10 | 1 | 0·108 | $0{\cdot}0^3866$ |
| $\lvert w_2\rvert$ | 17·3 | 2·27 | 0·184 | 0·0164 | $0{\cdot}0^4185$ |
| $\lvert w_3\rvert$ | 17·9 | 1·00 | 0·0276 | $0{\cdot}0^3896$ | $0{\cdot}0^6296$ |
| $\lvert w_4\rvert$ | 18·2 | 0·179 | 0·00316 | $0{\cdot}0^4509$ | $0{\cdot}0^7333$ |

It will be seen from the foregoing table that the higher the rigidity of the ground through which the waves are transmitted, the more neutralized is the distribution of the displacement in a wave front. If, on the other hand, the ground were fairly soft, the distribution under consideration would be pronounced even at a great distance from the origin. For example, if the amplitude $w_1$ is unity for any $\sqrt{\mu/\rho}$, the maxima of the amplitudes $w_2$ for the cases of $\sqrt{\mu/\rho}=0{\cdot}1$ km s, 0·5 km s, 1 km s, 2 km s, 10 km s assume the values 1, 0·734, 0·184, 0·152, 0·0213 respectively, the relation $S_2|dP_2/d\theta|=S_1|dP_1/d\theta|$ being assumed to exist in every case.

**3.** *Both dilatational and distortional waves radiated from a spherical origin.*

The solution for the present case will be found in the previous papers[6], the expressions being

6) K. SEZAWA, *loc. cit.* 1), 4).

$$\left.\begin{aligned}
\Delta&=A_nH^{(2)}_{n+\frac{1}{2}}(hr)P_n(\cos\theta)e^{ipt},\\
u_1&=-\frac{A_n}{h^2}\frac{d}{dr}\frac{H^{(2)}_{n+\frac{1}{2}}(hr)}{\sqrt{r}}P_n(\cos\theta)e^{ipt},\\
v_1&=-\frac{A_n}{h^2}\frac{H^{(2)}_{n+\frac{1}{2}}(hr)}{r^{\frac{3}{2}}}\frac{dP_n(\cos\theta)}{d\theta}e^{ipt},\\
u_2&=-n(n+1)\frac{B_n}{k^2}\frac{H^{(2)}_{n+\frac{1}{2}}(hr)}{r^{\frac{3}{2}}}P_n(\cos\theta)e^{ipt},\\
v_2&=-\frac{B_n}{k^2}\frac{1}{r}\frac{d}{dr}\left\{\sqrt{r}\,H^{(2)}_{n+\frac{1}{2}}(kr)\right\}\frac{dP_n(\cos\theta)}{d\theta}e^{ipt},
\end{aligned}\right\}\tag{5}$$

where $h^2=\rho p^2/(\lambda+2\mu)$, $k^2=\rho p^2/\mu$, $u_1+u_2=u$, $v_1+v_2=v$. Substituting these solutions in the boundary conditions

$$\left.\begin{aligned}
\widehat{rr}&=\lambda\Delta+2\mu\frac{\partial u}{\partial r}=p_nP_n(\cos\theta)e^{ipt},\\
\widehat{r\theta}&=\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}\right)=0,
\end{aligned}\right\}\tag{6}$$

we obtain at $r=a$

$$\left.\begin{aligned}
&\sqrt{h}A_n\left[\left\{\left(\frac{y}{x}\right)^2-2\right\}\phi(x)-2\frac{d^2\phi(x)}{dx^2}\right]-\sqrt{k}B_n2n(n+1)\frac{d}{dy}\frac{\phi(y)}{y}=\frac{p_n}{\mu},\\
&\frac{\sqrt{k}B_n}{\sqrt{h}A_n}=\frac{-\frac{d}{da}\frac{\phi(x)}{x}\quad\frac{\phi(x)}{x^2}-\frac{1}{x}\frac{d\phi(x)}{dx}}{\frac{d}{dy}\frac{1}{y}\frac{b}{dy}\left\{y\phi(y)\right\}-\frac{1}{y^2}\frac{d}{dy}\left\{y\phi(y)\right\}+n(n+1)\frac{\phi(y)}{y^2}},
\end{aligned}\right\}\tag{7}$$

where $x=hr$, $y=kr$. Substituting (7) in (5) we get the general solutions. In the particular cases $n=0$, 1, 2, 3, 4; the solution of $u_1$ at $r=\infty$ assumes the form

$$u_1(=u_{1n})=-(i)^n\frac{p_n}{\rho h^2r}\frac{e^{ih(a-r)}}{M_n}P_n(\cos\theta)e^{ipt},\tag{8}$$

where $M_n$ for a few cases are of the forms:

$$n=0;\quad M_0=\frac{4}{x^2}-i\left(\frac{4}{x^3}-\frac{c^2}{x}\right),$$

$$n=1;\quad M_1=\left(\frac{12}{x^3}-\frac{c^2}{x}\right)-i\left(\frac{12}{x^4}-\frac{4+c^2}{x^2}\right)$$

$$-4\frac{\sqrt{k}B_1}{\sqrt{h}A_1}\left\{\frac{2}{y^3}-i\left(\frac{3}{y^4}-\frac{1}{y^2}\right)\right\},$$

$$\frac{\sqrt{k}B_1}{\sqrt{h}A_1}=\frac{\frac{6}{x^3}-i\left(\frac{6}{x^4}-\frac{2}{x^2}\right)}{\left(\frac{6}{y^3}-\frac{1}{y}\right)-i\left(\frac{6}{y^4}-\frac{3}{y^2}\right)},$$

$$n=2;\quad M_2=\left(\frac{72}{x^4}-\frac{4+3c^2}{x^2}\right)-i\left(\frac{72}{x^5}-\frac{28+3c^2}{x^3}+\frac{c^2}{x}\right)$$

$$-12\frac{\sqrt{k}B_2}{\sqrt{h}A_2}\left\{\left(\frac{12}{y^4}-\frac{1}{y^2}\right)-i\left(\frac{12}{y^5}-\frac{5}{y^2}\right)\right\},$$

$$\frac{\sqrt{k}B_2}{\sqrt{h}A_2}=\frac{\left(\frac{24}{x^4}-\frac{2}{x^2}\right)-i\left(\frac{24}{x^5}-\frac{10}{x^3}\right)}{\left(\frac{48}{y^4}-\frac{5}{y^4}\right)-i\left(\frac{48}{y^5}-\frac{21}{y^3}+\frac{1}{y}\right)},$$

$$n=3;\quad M_3=\left(\frac{600}{x^5}-\frac{52+15c^2}{x^3}+\frac{c^2}{x}\right)-i\left(\frac{600}{x^5}-\frac{252+15c^2}{x^4}+\frac{4+6c^2}{x^2}\right)$$

$$-24\frac{\sqrt{k}B_3}{\sqrt{h}A_3}\left\{\left(\frac{75}{y^5}-\frac{8}{y^3}\right)-i\left(\frac{75}{y^6}-\frac{33}{y^4}+\frac{1}{y^2}\right)\right\},$$

$$\frac{\sqrt{k}B_3}{\sqrt{h}A_3}=\frac{\left(\frac{150}{x^5}-\frac{16}{x^3}\right)-i\left(\frac{150}{x^6}-\frac{66}{x^4}+\frac{2}{x^2}\right)}{\left(\frac{450}{y^5}-\frac{51}{y^3}+\frac{1}{y}\right)-i\left(\frac{450}{y^6}-\frac{201}{y^4}+\frac{8}{y^2}\right)},$$

$$n=4;\quad M_4=\left(\frac{6300}{x^6}-\frac{660+105c^2}{x^4}+\frac{4+10c^2}{x^2}\right)-i\left(\frac{6300}{x^7}-\frac{2750+105c^2}{x^5}\right.$$

$$\left.+\frac{84+45c^2}{x^3}-\frac{c^2}{x}\right)-40\frac{\sqrt{k}B_4}{\sqrt{h}A_4}\left\{\left(\frac{630}{y^6}-\frac{95}{y^4}+\frac{1}{y^2}\right)\right.$$

$$\left.-i\left(\frac{630}{y^7}-\frac{285}{y^5}+\frac{12}{y^3}\right)\right\},$$

$$\frac{\sqrt{k}B_4}{\sqrt{h}A_4}=\frac{\left(\frac{1260}{x^6}-\frac{170}{x^4}+\frac{2}{x^2}\right)-i\left(\frac{1260}{x^7}-\frac{570}{x^5}+\frac{24}{x^4}\right)}{\left(\frac{5040}{y^6}-\frac{615}{y^4}+\frac{12}{y^2}\right)-i\left(\frac{5040}{y^7}-\frac{2295}{y^5}+\frac{105}{y^2}-\frac{1}{y}\right)},$$

where $x=ha$, $y=ka$, $c^2=k^2/h^2$.

If $p_nP_n(\cos\theta)$ be a constant for different $n$'s, the ratio of $|u_{1n}|/|u_{10}|$ assumes the values shown in Table III. Mark (R) indicates that resonances of a certain kind are likely to exist for vibrations of type $u_{1n}$.

Table III.

| $p\sqrt{\frac{\rho}{\lambda+2\mu}}a$ | 2 | 1 | 0·5 | 0·1 |
|---|---|---|---|---|
| $\lvert u_{11}\rvert/\lvert u_{10}\rvert$ | 1·35 | 0·235 | 0·650 } (R) | 4·0 (R) |
| $\lvert u_{12}\rvert/\lvert u_{10}\rvert$ | 0·74 | 0·0848 } (R) | 0·310 } (R) | 0·00061 |
| $\lvert u_{13}\rvert/\lvert u_{10}\rvert$ | —— | 0·0490 } (R) | 0·053 | —— |
| $\lvert u_{14}\rvert/\lvert u_{10}\rvert$ | —— | 0·0009 | 0·00004 | —— |

Let the period of the waves and the circumference of the spherical origin be again $T=1$ sec and $2\pi a=1$ km respectively. Assuming that $|u_{10}|$ for $\sqrt{\mu/\rho}=1$ km/s is unity, the values of $|u_{1n}|$ for different $\sqrt{\mu/\rho}$'s under the assumption that $p_nP_n(\cos\theta)$ as well as $\rho$ are invariably constant for any $n$, are shown in Table IV.

Table IV.

| $\sqrt{\mu/\rho}$ km/s | 0·5 | 1 | 2 | 10 |
|---|---|---|---|---|
| $\lvert u_{10}\rvert$ | 2·91 | 1 | 0·134 | 0·00103 |
| $\lvert u_{11}\rvert$ | 3·93 | 0·235 | 0·0870 | 0·00412 |
| $\lvert u_{12}\rvert$ | 2·15 | 0·0848 | 0·0415 | $0{\cdot}0^6629$ |
| $\lvert u_{13}\rvert$ | —— | 0·0490 | 0·0071 | —— |
| $\lvert u_{14}\rvert$ | —— | 0·0009 | $0{\cdot}0^555$ | —— |

These tables show that in this case too, the higher the rigidity of the ground composing the seismic region, the more neutralized is the distribution of the displacements in a wave front. In the case of soft ground the distribution of the displacement under consideration is rather pronounced.

It is not clear to us as why the resonance-like conditions should be present, particularly in this case. As however we are investigating now only the problem of a vibrating sphere radiating dissipation waves, we shall probably be able to find an answer to the question when it has been investigated for its own sake.

In conclusion I wish to express my thanks to Dr. Inouye for his suggestions and to Dr. Kanai for his assistance in obtaining the mathematical formulae.

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## 45. 媒體の異同が地震初動の分布に及ぼす影響

地震研究所 妹澤克惟

震動原の大きに比較して波長が短い場合には原點の震動機構が地震初動の分布に割合によく現れるけれども，波長が長い場合には之が打消されて初動分布の狀態に現れ難いといふやうな事が井上氏によつて面白く發表された．この論文では同じ週期のものが原點に働いても，波動の傳播する媒體が違ふと，初動分布の狀態に著しい影響を與へるといふ事を示したのである．例へば原點が球狀をなしその球周が 1 km であり，週期 1 秒の横波だけが出ると假定する．媒體の$\sqrt{\mu/\rho}=$1 km/s のときに $P_1(\cos\theta)$ 分布の波動振幅が無限大距離で 1 とする．同じ剪應力が働いて $P_1$，$P_2$，$P_3$，$P_4$ 分布の波動が出る場合に，$\sqrt{\mu/\rho}=1$ km/s ではそれ等の無限大遠方に於ける振幅は 1，0·184，0·0276，0·00316 となる．然るに $\sqrt{\mu/\rho}=0{\cdot}1$ km/s になると同じ振幅は 17·3，17·3，17·9，18·2 となり，$\sqrt{\mu/\rho}=10$ km/s になると $0{\cdot}0^3866$，$0{\cdot}0^4185$，$0{\cdot}0^6296$，$0{\cdot}0^7333$ となる．卽ち初動分布から震原機構を推定するには震動週期が違ふと結果が勿論違ふけれども，その媒體の彈性等が異なるものであると又同様に著しく異なる結果が出ることを示すものである．

震原から縦波と横波とが出る場合も計算を試みた．この場合は筆者が 1926 年頃發表した數式の表し方に多少不完全な所があり，これは筆者が前から氣がつきながら明瞭な訂正をしなかつた所が（但し振動學 656 頁には具體的の表し方が記載してある）井上氏から親切に之を誰でも直ちに使用できるやうに正しく記錄して置くやうとの御注意もあり，かたがた數式に特に力を入れて論文に示して置いたのである．

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