2. *Elastic Waves Formed by Local Stress Changes of Different Rapidities.*

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Criterion on Positive and Negative Change in Local Stress.

1. It was shown in a previous paper[1] that, if a stress is applied to a body or released from it, the energy transmitted to infinity depends on the rapidity with which the stress is applied or released. The greater the rapidity with which the stress is applied or released, the greater the energy that is transmitted to infinity, and at the same time the greater the energy that is required at the origin. If the stress were applied or released extremely slowly, no wave energy would be transmitted to infinity, whereas, if done with infinite rapidity, exactly half the energy given at the origin will be transmitted to infinity.

When the previous paper was read, the question arose as to whether the energy transmission due to the release of the stress at a certain rapidity would be equal to that due to the application of the same stress— a rather natural question, in view of the prevalent idea that the large energy of some earthquakes is due to sudden release of the strain energy that was stored in the focal region. Our opinion however is that, even should there be an inequality between two cases, the inequality would be due to an origin concerned with the theory of thermodynamics, namely the problem of compression and that of rarefaction differ entirely from each other, particularly in the nature of the supply and evolution of the heat energy, whence it is of pressing importance to explain first why the two cases, when it comes to the problem of elastic waves, are, from the dynamical point of view, equivalent to each other.

Since however we are here dealing merely with a pure elasticity problem, not involving any transfer of heat energy, we can assume that the change is adiabatic. Again, although there is essential

1) K. SEZAWA, "Elastic Waves Produced by Applying Statical Force to a Body or by Releasing it from a Body," *Bull. Earthq. Res. Inst.*, **13** (1935), 729~739.

difference between compression and rarefaction in the change in density of material arising from a force applied, the amount in actual cases is negligible.

2. Fourier's series and integrals, which play an important part in the solution of the present and similar problems, are used under the assumption that the difference between compression and rarefaction in stress, or that between push and pull in displacement, is concerned merely with the algebraic sign in tensor quantities, superposition or subtraction among these quantities being thus permissible in almost every case. The solution of the problem in which a stress is released from a stressed body, is the same as the one in which stress of the same amount but of opposite signs is added to the same body in a stressed condition, the initial stress being assumed to be statical and permanently induced in the body. Since, mathematically, permanent stress cannot be affected by a fresh disturbance, no energy transfer can ever be expected to occur from the addition of new stresses. This manner of treatment is resorted to in several branches of mathematical physics. Thus, the algebraic superposition of wave forms is not restricted to the case of sinusoidal curves, but generally applied to cases of other complex curve forms.

The most important restriction with respect to the problem under consideration is that the curves to be analysed should finally (not necessarily after a long interval of time or space) return to the same zero line as the one at the beginning of the problem. Symbolically, if u be the amplitude of a curve at the coordinate point x, the above condition may be written

$$\int_{-\infty}^{\infty} u' dx = 0 , \tag{1}$$

or

$$\int_{-\infty}^{\infty} u dx = \text{a finite quantity}, \tag{1'}$$

where x may be used as the coordinate in space or in time.

3. The number of actual examples that could be given in proof of the conception just mentioned is endless. The simplest examples of it in mechanics are perhaps the longitudinal waves along a tense string and the sound waves in a deep sea. If the disturbance were of rarefactional type, it is possible for the initial stress in the transmission media to be neutralized, making the resulting stress zero in some extreme cases. Even in a case of this kind, the manner of transmission of the waves, or of their energy cannot be changed unless every law governing physical quantities, such as Young's modulus,

compressibility, etc., differ with differences in the amplitudes of the transmitted disturbances. There are, a number of similar examples in wave phenomena in every branch of physics. It will be clear from these examples that superposition or subtraction of disturbances, or more generally, addition or removal of disturbances in a body, is related merely to the algebraic signs of the disturbances.

4. In order to obtain a clearer understanding of the problem with respect to the energy supply, which is also one of physical phenomena in wave or vibratory motion, let a stress P be removed from some point of a body, and let the velocity of the corresponding movement of the same point be $\dot{u}_1$. The work applied to the body is then expressed by

$$\int_{-\infty}^{\infty} P\dot{u}dt\,, \qquad (2)$$

where $P=P_0+P_1$, P_0 being the initial stress. This expression can be transformed to

$$\begin{aligned}\int_{-\infty}^{\infty} P\dot{u}dt &= \int_{-\infty}^{\infty}(P_0+P_1)(\dot{u}_0+\dot{u}_1)dt \\ &= P_0\int_{-\infty}^{\infty}\dot{u}dt+\int_{-\infty}^{\infty}P_1\dot{u}dt \qquad (2')\end{aligned}$$

where $\dot{u}_0$ is zero and P_0 is independent of t. From (1) it is obvious that $\int_{-\infty}^{\infty}\dot{u}dt=0$, it being well known that $\int_{-\infty}^{\infty}P_1\dot{u}dt$ is always of positive sign, which shows that the work done to the body is the same whether a given stress is added to it or taken from it, provided the transfer of heat is not taken into account. It is to be remembered that in certain problems, the condition, $\int_{-\infty}^{\infty}udt=$ a finite quantity, may be used in lieu of $\int_{-\infty}^{\infty}\dot{u}dt=0$.

5. In the previous paper we treated a disturbance of the type

$$\left.\begin{aligned} &P=0, \quad (r=a,\ t<0) \\ &P=-P_0e^{-ct}(1-e^{-gt}), \quad (r=a,\ t>0,\ c\to 0)\end{aligned}\right\}(3)$$

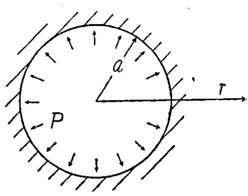

Fig. 1.

transmitted outwards with a velocity peculiar to elastic waves. This form of disturbance indicates that the expressions for disturbance satisfy the condition in (1). Even were we to take the more general state at any t or x, namely,

$$\left.\begin{aligned}&\phi=0, \quad (\tau<0)\\ &\phi=\frac{aP_0}{2\pi i\rho r}\left\{\int_{-\infty}^{\infty}\frac{e^{ip\tau}\,dp}{(p-ic)(p-\alpha)(p-\beta)}-\int_{-\infty}^{\infty}\frac{e^{ip\tau}\,dp}{(p-iq)(p-\alpha)(p-\beta)}\right\},\ (\tau>0)\end{aligned}\right\}\quad(4)$$

in which $\tau=t-(r-a)/v$, condition (1) is also satisfied, representing the transmission of the disturbance outwards with velocity v. Since ϕ indicates the mere radial transmission of waves from the origin, the nature of ϕ would not be changed if there were any preexisting stress in the medium or even if another series of waves were travelling through that medium.

An alternative way of introducing c in the first term on the right-hand side of (4) for satisfying condition (1) is to use

$$\phi=\frac{aP_0}{2\pi i\rho r}\int_{-\infty}^{\infty}\frac{e^{ip\tau}-e^{ip\tau_1}}{p(p-\alpha)(p-\beta)}\,dp\,, \qquad (5)$$

where $\tau_1=t-(r-a)/v-t_1$. But, the nature of the problem with respect to the ineffectiveness of the preexisting stresses or displacements on the disturbance under consideration is the same as in case (4).

6. We shall show what an odd result would be obtained were the initial stress confused with that of a prescribed dynamical disturbance. It is well known that the energy of waves is proportional to the square of their amplitudes. Let the amplitudes of the initial stress be V and that of the prescribed dynamical stress U. Then, on the assumption that the initial stress also participates in wave energy, we would have the condition that the amount of the change in energy due to the disturbing motion is

$$(U+V)^2-V^2(=2UV+V^2),$$

which may be modified in any way by adjusting V for the initial stress. It follows then that, in the case of seismology, were longitudinal waves of compressional type propagated through a deep layer of the earthcrust, a very large energy would be transmitted, whereas, were the same waves of rarefactional type propagated also through a deep layer of the crust, very feeble or sometimes negative energy would be transmitted. It is an established fact in the classical theory of elasticity that the initial stress does not affect the new disturbing stress phenomena, excepting the action due to density change in the material caused by the initial stress.

Elastic Waves Formed by Local Stress Changes of Different Rapidities.

7. The nature of the initial stress, which is essential to the consideration of the elastic waves produced by releasing a stress from a body, has now been clearly ascertained. We shall next calculate the displacement of the medium as well as the amount of the transmitted energy, both at an infinite distance from the origin of the stress change.

We found in the previous paper the expressions for the displacement of the medium and the transmitted energy. The change in the pressure at $r=a$, namely

$$\left.\begin{aligned} &P=0\,, \quad (r=a, \quad t<0) \\ &P=P_0 e^{-ct}(1-e^{-qt}), \\ &\qquad\qquad (r=a, \quad t>0) \end{aligned}\right\} \quad (3')$$

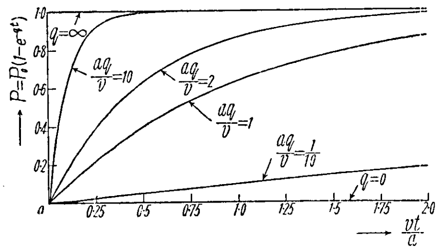

Fig. 2.

in which $c\to 0$ (as will be shown graphically in Fig. 2), and q indicates the rapidity of the stress change. The dispalcement at $r\to\infty$ can be determined from equation (13) of the previous paper, namely, from

$$u=\frac{a^3P_0}{4\mu}\left[\frac{1}{r^2}+\frac{\alpha\beta}{Q}\left(\frac{q}{vr}-\frac{1}{r^2}\right)e^{-q\tau}+\frac{\beta q}{(\alpha-\beta)(\alpha-iq)}\left(\frac{\alpha}{vr}-\frac{i}{r^2}\right)e^{i\alpha\tau}\right.$$
$$\left.-\frac{\alpha q}{(\alpha-\beta)(\beta-iq)}\left(\frac{\beta}{vr}-\frac{i}{r^2}\right)e^{i\beta\tau}\right],$$

writing in a more convenient form,

$$u=\frac{a^3P_0}{4\mu r^2}+\frac{\dfrac{a^3P_0}{4\mu r^2}}{\left\{4-4\left(\dfrac{aq}{v}\right)+\left(\dfrac{v}{\mathfrak{v}}\right)^2\left(\dfrac{aq}{v}\right)^2\right\}}\left\{\left[-4e^{-q\tau}+\left(\frac{aq}{v}\right)e^{-X\tau}\right.\right.$$
$$\cdot\left\{4-\left(\frac{v}{\mathfrak{v}}\right)^2\left(\frac{aq}{v}\right)\cos Y\tau+\frac{\left(4-2\left(\dfrac{v}{\mathfrak{v}}\right)^2-\left(\dfrac{v}{\mathfrak{v}}\right)^2\left(\dfrac{aq}{v}\right)\right)}{\sqrt{\left(\dfrac{v}{\mathfrak{v}}\right)^2-1}}\sin Y\tau\right\}\Bigg]$$
$$\left.+4\left(\frac{r}{a}\right)\left(\frac{aq}{v}\right)\left[e^{-q\tau}-e^{-X\tau}\left\{\cos Y\tau+\frac{\left(2-\left(\dfrac{v}{\mathfrak{v}}\right)^2\left(\dfrac{aq}{v}\right)\right)}{2\sqrt{\left(\dfrac{v}{\mathfrak{v}}\right)^2-1}}\sin Y\tau\right\}\right]\right\}, \quad (6)$$

where $$X=2\frac{v}{a}\left(\frac{\mathfrak{v}}{v}\right)^2, \quad Y=2\frac{\mathfrak{v}}{a}\sqrt{1-\left(\frac{\mathfrak{v}}{v}\right)^2}.$$

The final result for the displacement variation in t as well as in r (in the case $r\rightarrow\infty$) due to stress change at $r=a$ with different rapidities, is shown in Fig. 3.

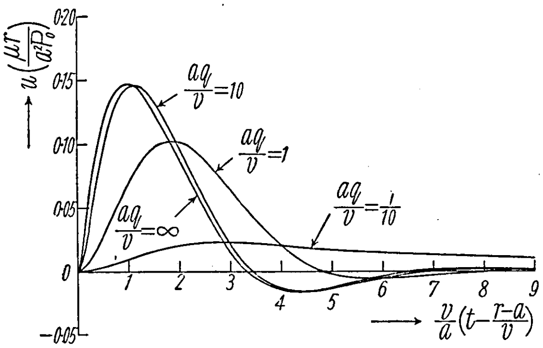

Fig. 3.

The transmission of energy is determined from equation (18) in the previous paper, namely

$$E_{r\infty}=\frac{\pi a^9P_0^2}{\mu r^3}+\frac{\pi a^6P_0^2}{2\mu\left(\frac{4\mu}{\rho a^2}-\frac{4\mu q}{\rho av}+q^2\right)^2}\Bigg[-\left(\frac{4\mu}{\rho a^2}\right)^2\left(\frac{1}{r^3}-\frac{2q}{vr^2}\right)$$

$$+\frac{1}{a^3}\left(q^2+\frac{4\mu q}{\rho av}+\frac{4\mu}{\rho a^2}\right)q^2-\frac{1}{r^3}\left(q^2-\frac{8\mu q}{\rho av}+\left(\frac{4\mu}{\rho av}\right)^2\right)q^2$$

$$-\frac{2}{r}\left(\frac{4\mu}{\rho a^2}\right)^2\left(\frac{1}{v^2}+\frac{1}{4\mathfrak{v}^2}\right)q^2-\frac{8\mu}{\rho a^2r^2v}\left(-\frac{4\mu}{\rho av}+q\right)q^2$$

$$-\frac{\frac{4\mu}{\rho a^2}q}{\left(\frac{4\mu}{\rho a^2}+\frac{4\mu q}{\rho av}+q^2\right)}\Bigg\{\frac{\rho}{\mu v}\left[\left(\frac{4\mu}{\rho a^2}\right)^2+\frac{4\mu q^2}{\rho a^2}\right]q^2$$

$$-\frac{2}{r}\left(\frac{1}{r^2}+\frac{1}{4\mathfrak{v}^2}\right)\left[\left(\frac{4\mu}{\rho a^2}\right)^2+\frac{16\mu^2}{\rho^2a^3v}q+\frac{4\mu}{\rho a^2}q^2\right]q$$

$$+\frac{2}{vr^2}\left[\left(\frac{4\mu}{\rho a^2}\right)^2+\frac{32\mu^2}{\rho^2a^3v}q+\left(\frac{4\mu}{\rho av}\right)^2q^2-q^4\right]$$

$$+\frac{2}{r^3}\left[-\frac{16\mu^2}{\rho^2a^3v}+\left(\frac{16\mu^2}{a^2\rho^2v^2}+\frac{4\mu}{\rho a^2}\right)q+q^3\right]\Bigg\}\Bigg].$$

We recently discovered that this equation can be transformed to a very simple form

$$E_{r\infty}=\frac{\pi a^6P_0^2}{2\mu r^3}+\frac{\pi a^3P_0^2}{2\mu a^3}\frac{\left(\frac{aq}{\mathfrak{v}}\right)^2}{\left\{4+4\frac{aq}{v}+\left(\frac{aq}{\mathfrak{v}}\right)^2\right\}}, \tag{7}$$

where $v=\sqrt{(\lambda+2\mu)/\rho}$, $\mathfrak{v}=\sqrt{\mu/\rho}$. We have calculated the energy $E_{\infty\infty}$ transmitted to an infinite distance $r=\infty$ for various rapidities (q) with which the stress at $r=a$ is changed, the result being plotted in Fig. 4. It will be seen that, when $q=0$, no energy is transmitted, whereas, when $q\to\infty$, the energy becomes $E_{\infty\infty}=\pi a^3P_0^2/2\mu$.

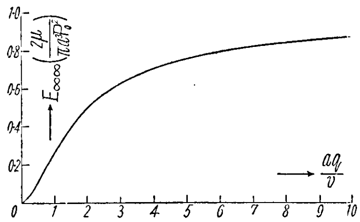

Fig. 4.

From what has already been explained, the result in Fig. 4 is the same whether a stress is applied at $r=a$ or removed from the same origin. It will also be known that, were the energy possible to evolve from the strained part with a certain volume through which elastic waves are transmitted, energy $E_{\infty\infty}$ would not assume zero value even when $q=0$.

Remarks on Compressional and Rarefactional Waves.

8. From the fact that wave phenomena are independent of the initial stress, it is now possible to conclude that the push and pull types of wave motion may be made to coincide respectively with the compressional and rarefactional types of waves.

It is obvious that, if a pressure were applied with some rapidity to a cavity as we have considered, waves of push type, namely, compressional waves, would be observed, whereas, if the pressure were released, waves of pull type, namely, rarefactional waves, would result.

We shall next consider the case in which a large volume of the earth's crust is uniformly compressed, and that a break in the form of a cavity (which actually is improbable) has occurred at some point in the crust. Waves of the pull type, namely, rarefactional waves will then be radiated from this cavity, notwithstanding that the crust is initially compressed. If, on the other hand, a large volume of the crust were pulled uniformly from all directions, the resulting break at any point within that volume would produce waves of the push type, namely, compressional waves, in spite of the fact that the crust is initially in a rarefied state.

2. 局所的内力を種々の速さで變化する場合に生ずる彈性波

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この前の論文で示したところの局所的内力を附加へたり，取除いたりする場合に生ずる彈性波の問題を，内力變化の符號的關係につき一層委しく說明し，次に種々の速さで變化したときに生ずる彈性波の波形，勢力の割合等を算出したのである．又前の論文で非常に複雜な形をなした公式を極く簡單な形に直した．

この研究から，地殼の靜力學的壓力卽ち普通の如く全體的壓力であらうと，又特に張力的狀態にあらうと，地震初動の押しと引きとは必ず壓縮性地震波と膨脹性地震波とに相當するものであるといふ說明を附加へて置いた．