36. *The Rate of Damping in Seismic Vibrations of a Surface Layer of Varying Density or Elasticity.*

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(Read June 18, 1935.—Received June 20, 1935.)

1. It is established that the vibrations of a surface layer, whether free or forced, must be of such a damping type as shall depend on the ratio of both density and elastic constants of the layer to those of the subjacent medium.[1] The question may arise as to how the vibrations should be damped in the case of a layer with varying density or elastic constants. Although the way of damping in such a case naturally differs greatly according to the distribution of densities or elastic constants in both the layer and in the subjacent medium, it is nevertheless possible to conclude that the damping is due mainly to the dissipation of vibrational energy into the subjacent medium and very little to other causes of damping, such as internal friction in the ground, etc.. It is therefore obvious that the rate of damping is a certain function of density as well as of the elastic constants, and of no other physical constants.

As already stated in our preceding paper,[2] the method of representing damping by the terms at present used in differential equations of motion, and even by the functions in energy equations specifying damping forces, is not dynamically satisfactory. The semi-empirical form of dissipation function due to Lord Rayleigh[3] appears to have been misused by succeeding mathematicians. It will be plain from the nature of the problem that the dissipation problem of a vibrating body should be attacked by solving the usual simultaneous differential equations of motion for the body under consideration as well as for the neighbouring media through which the energy is dissipated.

2. Let the axis of x be drawn vertically downwards (or upwards) from a certain point 0, and let u, u', ρ, ρ', λ, μ, λ', μ' be the displacements, densities, and elastic constants of the subjacent medium and

1), 2) K. SEZAWA and K. KANAI, "Decay Constants of Seismic Vibrations of a Surface Layer", *Bull. Earthq. Res. Inst.*, **13** (1935), 251~264.

3) Lord RAYLEIGH, *Proc. Math. Soc., London*, [i], **4** (1873), 357~368; *Theory of Sound*, **1**, **2**.

the surface layer respectively, H being the thickness of that layer. In the case of distortional waves transmitted vertically, the equations of motion of the two media are expressed by

$$\rho\frac{\partial^2 u}{\partial t^2}=\mu\frac{\partial^2 u}{\partial x^2}, \qquad (1)$$

$$\rho'\frac{\partial^2 u'}{\partial t^2}=\frac{\partial}{\partial x}\left(\mu'\frac{\partial u'}{\partial x}\right). \qquad (2)$$

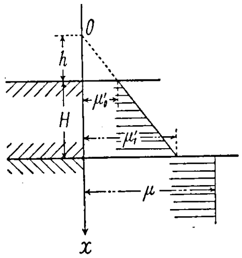

Fig. 1.

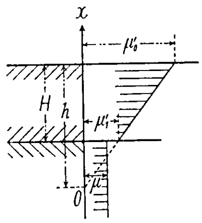

Fig. 2.

In the case of dilatational waves transmitted vertically, it is only necessary to replace μ by $\lambda+2\mu$ and μ' by $\lambda'+2\mu'$, not only in these equations but also in those that will appear hereinafter.

Let us suppose that ρ, μ, ρ' are constants and that μ' varies as

$$\mu'=Ax^n, \qquad (3)$$

when equation (2) then transforms to

$$\frac{d^2u'}{dx^2}+\frac{n}{x}\frac{du'}{dx}+\frac{\rho' p^2}{A}\frac{u'}{x^n}=0, \qquad (4)$$

provided p is the frequency of vibrations of the disturbance, and the time factor is omitted. The solution of (4) is written

$$u'=x^{\frac{1-n}{2}}\left[CH^{(1)}_{\frac{1-n}{2-n}}\left\{\left(\frac{\rho' p^2}{A}\right)^{\frac{1}{2}}\frac{2}{2-n}x^{1-\frac{n}{2}}\right\}\right.$$
$$\left.+DH^{(2)}_{\frac{1-n}{2-n}}\left\{\left(\frac{\rho' p^2}{A}\right)^{\frac{1}{2}}\frac{2}{2-n}x^{1-\frac{n}{2}}\right\}\right]e^{ipt}. \qquad (5)$$

(i) When the elastic constant in the layer increases downwards, the axis of x should be taken also downwards with its origin, 0, at distance h above the free surface, where h is so adjusted that the distribution of elastic constants in the layer is in a specified state. The form of the vibrations in the subjacent medium is found from equation (1), which in the present case is written

$$u=e^{ik(V_1t+x)}+Be^{ik(V_1t-x)}, \qquad (6)$$

in which $V_1=\sqrt{\mu/\rho}$, $k=p/V_1$.

(ii) When the elastic constant in the layer increases upwards, the axis of x should be taken also upwards with its origin, 0, at distance h below the free surface of the layer. The form of vibrations in the subjacent medium in this case is expressed by

$$u=e^{ik(V_1t-x)}+Be^{ik(V_1t+x)}. \tag{7}$$

The boundary condition at the free surface $x=h$ is

$$\partial u'/\partial x=0, \tag{8}$$

while the conditions at the lower boundary of the layer, (i) $x=h+H$, (ii) $x=h-H$, are

$$u=u', \qquad \mu\frac{\partial u}{\partial x}=\mu'\frac{\partial u'}{\partial x}, \qquad (9),\ (10)$$

in the case of distortional waves. Substituting (5), (6), (7) in (8), (9), (10), we obtain the values of B, C, D.

3. For the case in which the elastic constant in the layer increases linearly downwards with its value Ah at the free surface, we put $n=1$ in (5), and solve (5), (6) so as to satisfy boundary conditions (8), (9), (10). The final solution is expressed by

$$u'=\frac{2}{\sqrt{P^2+Q^2}}\left\{J_0(2k'\sqrt{hx})Y_1(2k'h)-Y_0(2k'\sqrt{hx})J_1(2k'h)\right\}.$$

$$.\cos\left\{pt+k(h+H)-\tan^{-1}\frac{P}{Q}\right\}, \tag{11}$$

in which $k'=p\sqrt{\rho'/Ah}$, and

$$P=\sqrt{\frac{\rho'A(h+H)}{\rho\mu}}\left\{J_1(2k'\sqrt{h(h+H)})Y_1(2k'h)\right.$$

$$\left.-Y_1(2k'\sqrt{h(h+H)})J_1(2k'h)\right\}, \tag{12}$$

$$Q=\{J_0(2k'\sqrt{h(h+H)})Y_1(2k'h)-Y_0(2k'\sqrt{h(h+H)})J_1(2k'h)\}, \tag{13}$$

corresponding to the incident waves

$$u_1=\cos k(V_1t+x). \tag{14}$$

For the case in which the elastic constant in the layer increases linearly upwards with its value Ah at the free surface, we put $n=1$

in (5), and solve (5), (7) so as to satisfy the same boundary conditions, the final solution being

$$u' = \frac{2}{\sqrt{P^2+Q^2}}\left\{-J_0(2k'\sqrt{hx})\,Y_1(2k'h)+Y_0(2k'\sqrt{hx})J_1(2k'h)\right\}.$$

$$.\cos\left\{pt-k(h-H)-\tan^{-1}\frac{P}{Q}\right\}, \quad (15)$$

in which $k'=p\sqrt{\rho'/Ah}$, and

$$P=\sqrt{\frac{\rho' A\,(h-H)}{\rho\mu}}\left\{J_1(2k'\sqrt{h\,(h-H)})\,Y_1(2k'h)\right.$$

$$\left.-Y_1(2k'\sqrt{h\,(h-H)})J_1(2k'h)\right\}, \quad (16)$$

$$Q=\{J_0(2k'\sqrt{h\,(h-H)})\,Y_1(2k'h)-Y_0(2k'\sqrt{h\,(h-H)})J_1(2k'h)\}, \quad (17)$$

corresponding to the incident waves

$$u_1=\cos k(V_1t-x). \quad (18)$$

If the densities or elasticities of the two media were to differ extremely from each other, P would be zero, so that $Q=0$ becomes the frequency equation for the vibrations of the case without dissipation. When the frequency of vibrations becomes relatively large, it is possible to apply asymptotic expansions to the cylindrical functions in (11), (12), (13), (15), (16), (17), so that, under the condition corresponding to resonances at relatively large frequencies, the ratio of the amplitudes at the free surface to those of incident waves becomes

$$2\left(\frac{\rho\mu}{\rho'\mu_1'}\right)^{\frac{1}{2}}\left(\frac{\mu_1'}{\mu_0'}\right)^{\frac{1}{4}}, \quad (19)$$

where $\mu_0'(=Ah)$, $\mu_1'(=A\overline{h+H}$ or $=A\overline{h-H})$, μ are elastic constants at the free as well as the lower surfaces of that layer and in the subjacent medium, whereas ρ', ρ are the respective densities of the two media. Relation (19) arises from the condition, $Q=0$, which should exist at resonance frequencies. Since, again, at frequencies that are relatively large the factor consisting of cylindrical functions in (12) or (16) vanishes periodically with respect to k', P vanishes at such values of k', when the denominator corresponding to the root sign in the expression of u' in (11) or (15) contains only Q, whence it follows that the ratio of amplitudes at the free surface to those of the incident waves assumes the value

$$2\left(\frac{\mu_1'}{\mu_0'}\right)^{\frac{1}{4}}. \tag{20}$$

When $\rho\mu=\rho'\mu_1'$, the ratio of amplitudes at frequencies, relatively large and corresponding to the resonance, becomes

$$2\left(\frac{\mu_1'}{\mu_0}\right)^{\frac{1}{4}}. \tag{21}$$

It appears therefore from (20) and (21), that, when $\rho\mu=\rho'\mu_1'$, the ratio of the amplitudes of vibrations at the free surface to those of the incident waves tends to take a constant value as shown in (20) and (21). The amplitudes at the lower boundary of the layer change periodically (not sinusoidally periodically) by values twice those of the incident waves. At frequencies corresponding to resonance without dissipation, the amplitudes of vibration at that boundary are necessarily zero.

4. In order to confirm the nature of the vibrations more thoroughly, we selected five cases, namely (i) $\mu_0'/\mu_1'=1/2$, $\mu_1'/\mu=1$; (ii $\mu_0/\mu_1'=1/16$, $\mu_1'/\mu=1$; (iii) $\mu_0'/\mu_1'=1/16$, $\mu_1'/\mu=1/4$; (iv) $\mu_0'/\mu_1'=16$, $\mu_1'/\mu=1$; and calculated the amplitudes of vibrations at the free surface, u_0', and those at the lower boundary, u_H', of a surface layer for different lengths of incident waves; they are plotted in Figs. 3∼6. In these figures,

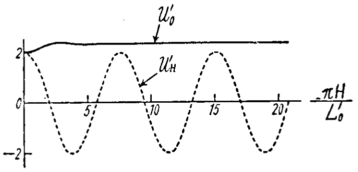

Fig. 3. $\rho'/\rho=1$, $\mu_0'/\mu_1'=1/2$, $\mu_1'/\mu=1$.

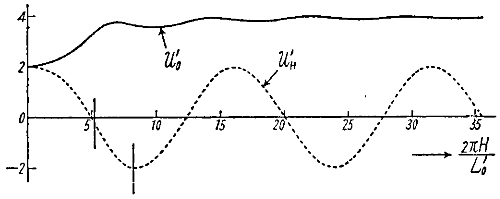

Fig. 4. $\rho'/\rho=1$, $\mu_0'/\mu_1'=1/16$, $\mu_1'/\mu=1$.

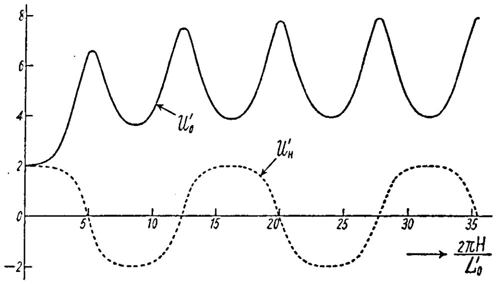

Fig. 5. $\rho'/\rho=1$, $\mu_0'/\mu_1'=1/16$, $\mu_1'/\mu=1/4$.

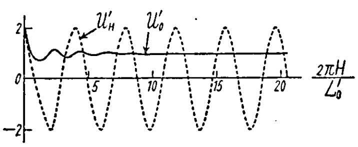

Fig. 6. $\rho'/\rho=1$, $\mu_0'/\mu_1'=16$, $\mu_1'/\mu=1$.

L_0' is the length of the incident waves in the subjacent layer, the amplitudes of which are assumed to be unity.

From these figures, one can readily ascertain the nature of the vibrations that we have described in connection with equations (19), (20), (21), $\cdots\cdot$. Should in the case, $\rho\mu=\rho'\mu'_1$, the thickness of the stratum exceed by a few times the length of the incident waves, the amplitudes of vibrations at the free surface will be practically constant for any frequency of vibrations. This shows that the relatively short waves are totally reflected, even when the elastic constant in the layer varies very sharply. Examination of the dynamical theory of the flow of vibrational energy through the layer with respect to the ratio of the amplitudes at the surface to those at the upper boundary of the subjacent medium, also shows that the incident waves are totally reflected.

It is a suitable occasion to add that the action of a seismograph of stiff type like acceleration seismographs resembles the nature of the vibrations of a layer shown in the example (iv). Amplitudes recorded on such seismographs that were set on a soft ground are relatively diminished, particularly in oscillation with periods synchronizing with those of the seismographs. The reverse is the case in similar seismographs set on a rigid ground.

5. In order to find the nature of the free vibrations of the layer, a generalisation of solution (5) by means of Fourier's integral was resorted to under the assumption that the asymptotic expansion is permissible for any wave length. In the case of increasing elasticity with depth, we obtain

$$u'=\frac{1}{\pi}\left(\frac{h+H}{x}\right)^{\frac{n}{4}}\sum_{m=0}^{\infty}(-1)^m\frac{(1-\alpha)^m}{(1+\alpha)^{m+1}}\left[F\left\{\beta\left(V_2t+\psi_2-(2m+1)\psi_1+\frac{h+H}{\beta}\right)\right\}\right.$$
$$\left.+F\left\{\beta\left(V_2t-\psi_2-(2m+1)\psi_1+\frac{h+H}{\beta}\right)\right\}\right], \qquad (22)$$

where

$$\left.\begin{aligned}&\psi_2=\frac{2h}{2-n}\left\{\left(\frac{x}{h}\right)^{\frac{2-n}{2}}-1\right\},\quad \psi_1=\frac{2h}{2-n}\left\{\left(\frac{h+H}{h}\right)^{\frac{2-n}{2}}-1\right\},\\ &\alpha=\sqrt{\frac{\rho' A(h+H)^n}{\rho\mu}},\quad \beta=\sqrt{\frac{\rho'\mu}{\rho A h^n}},\quad V_2=\sqrt{\frac{Ax^n}{\rho'}},\end{aligned}\right\} \qquad (23)$$

corresponding to incident disturbance

$$u_1=F(V_1t+x). \qquad (24)$$

The vibrations are therefore of the exponentially damping type. The ratio of vibration amplitudes of successive similar phases is constant and expressed by

$$\left(\frac{1-\alpha}{1+\alpha}\right)^2. \tag{25}$$

When the elastic constants of both media are continuous at the lower boundary of the layer, we have $\alpha=1$, so that the ratio in (25) is zero and no oscillatory motion is possible, which shows that the waves that come upwards are totally reflected in the neighbourhood of the layer. But this conclusion is based on the assumption that cylindrical functions can be expanded asymptotically for any wave length, which however is not generally valid, so that what we have just stated is only probable for the case of very sharp initial disturbance.

It is possible to get similar expressions for the case in which the elasticity decreases with depth. In the special case, for example, in which the decrement is linear, we have

$$u' = \frac{1}{\pi}\left(\frac{h-H}{x}\right)^{\frac{1}{4}}\sum_{m=0}^{\infty}(-1)^m\frac{(1-\alpha)^m}{(1+\alpha)^{m+1}}\left[F\left\{\beta\left(V_2t-\psi_2+(2m+1)\psi_1-\frac{h-H}{\beta}\right)\right\}\right.$$
$$\left.+F\left\{\beta\left(V_2t+\psi_2+(2m+1)\psi_1-\frac{h-H}{\beta}\right)\right\}\right], \tag{26}$$

where

$$\psi_1=2h\left(\sqrt{\frac{h-H}{h}}-1\right),\quad \psi_2=2h\left(\sqrt{\frac{x}{h}}-1\right),\quad \alpha=\sqrt{\frac{\rho' A(h-H)}{\rho\mu}},$$
$$\beta=\sqrt{\frac{\rho'\mu}{\rho A h}},\quad V_2=\sqrt{\frac{Ax}{\rho}}, \tag{27}$$

corresponding to incident waves

$$u_1=F(V_1t-x). \tag{28}$$

Since in this case, too, the ratio of amplitudes of successive similar phases is the same as in the preceding one, it is also improbable for the layer to oscillate repeatedly, so long as the elastic constants of both media are continuous at the lower boundary of that layer.

6. An approximate method of solving the problem of damped vibrations of a superficial layer due to dissipation of their energy into the subjacent medium, is to use integral equations. Although this method has already been used by some authors[4] in applying integral

4) For instance, T. ITOO, *Journ. Astr. Geophys.*, **12** (1935), 173~218; The problem as he has worked it out may be attacked in the same way as similar problems are done in the usual textbooks and in a number of professional papers; in his case particularly so, owing to the very simple assumption made with respect to boundary conditions. The usual methods of analytical mathematics would suffice to deal with such problems. However, his attempt to solve the problem of vibrations in a superficial layer subjected to periodic force applied to its lower boundary, by assuming it to be extremely rigid in the case of free vibrations, is opposed to the dynamical point of view.

equations to the solution of free or forced vibrations of a surface layer lying on an extremely rigid substratum, the case of the decay of seismic vibrations of a surface layer dissipating its energy into the subjacent medium has not received the attention of investigators, probably because of the difficulty in determining the Green's function that fulfills such conditions. We found that this problem can be solved very simply provided the incident waves are of pure periodic type.

Take the axis of x directed downwards with its origin at the free surface of the earth, and let H, $\rho'(x)$, $\mu'(x)$ be the thickness, density, and rigidity of the stratum. Taking into consideration again the special nature of the integral equation, we shall discuss the problem apparently statically. Let the horizontal displacement of the lower boundary of the stratum due to a unit load applied horizontally at any point in the layer be A, its value being finite and different according to the frequency of vibrations, except in the case of zero frequency in which A may take an infinitely large value. The reason why A takes a finite value in vibration problems is very simple. The nodal surface lying just below the common boundary of two media, if the subjacent medium were replaced by the extended part of the upper one, corresponds to the surface supporting the loads in the stratum. The position of the surface under consideration varies as the difference of vibration frequency, and the value of A also differs correspondingly. The horizontal displacement at x due to a unit load applied at $x=a$ in the layer is then such that

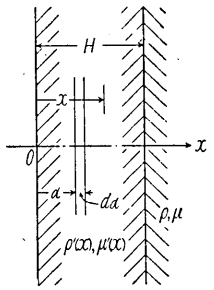

Fig. 7.

$$\left.\begin{aligned} k(x,a) &= \int_x^H \frac{dz}{\mu'(z)} + A, \quad (x>a) \\ k(x,a) &= \int_x^H \frac{dz}{\mu'(z)} + A. \quad (x<a) \end{aligned}\right\} \tag{29}$$

These represent Green's functions for the present problem. In the vibration problem, if the horizontal displacement at $x=a$ be $u'(a)\cos pt$ and the frequency of vibrations be p, then the load due to the inertia force acting on element da at $x=a$, will be

$$p^2\rho'(a)u'(a)\cos pt\,da. \tag{30}$$

The displacement at $x=x$ due to this load is therefore expressed by

$$p^2\rho'(a)u'(a)\cos pt\,da\,k(x, a), \tag{31}$$

so that displacement, $u'(x)\cos pt$, at $x=x$ due to total loads in the layer assumes the form

$$u'(x)=p^2\int_0^H \rho'(a)k(x, a)u'(a)da, \tag{32}$$

the time factor being omitted. The inertia forces in the subjacent medium naturally have no part in the problem; the movement in that medium being taken only in the boundary conditions at $x=H$. We now write

$$\rho'(a)k(x, a)=K(x, a), \tag{33}$$

when (32) reduces to

$$u'(x)=p^2\int_0^H K(x, a)u'(a)da. \tag{34}$$

This is a homogeneous integral equation, $K(x, a)$ being its kernel. The boundary conditions at $x=H$ are such that

$$u'(H)=2\sin(kH+\gamma),^{5)} \tag{35}$$

$$p^2\int_0^H \frac{\partial K(H, a)}{\partial H}u'(a)da=2\left(\frac{\mu}{\mu'}\right)_{x=H}k\cos(kH+\gamma), \tag{36}$$

where μ, $2\pi/k$ are the rigidity and wave length in the subjacent layer, while γ is the phase angle, which may be determined by the conditions of the vibrations. The right-hand sides of (35) and (36) represent the effect of the incident and reflected waves. Now, (35) and (36) are equivalent to the single equation

$$[u'(H)]^2+\left[\frac{p^2}{k}\frac{\mu'}{\mu}\int_0^H \frac{\partial K(H,a)}{\partial H}u'(a)da\right]^2=4. \tag{37}$$

5) The form of this expression represents the resulting motion due to incident and reflected waves in the subjacent medium. The importance of γ (as well as of A in Green's function) will now be understood. Many authors seem to have missed the use of γ. Infinitely large amplitudes in vibrations are the result arising from the calculation under such incorrect assumption.

It is necessary to insert a certain quantity similar to γ in all like problems. Even in the problem of microseismic vibrations of a surface layer or in that of seiches in a lake, under periodic pulsation in atmosphere, the insertion of a quantity analogous to γ is important, particularly in the discussion of the oscillations under resonance condition.

For simiplicity in evaluating the integral in (34) satisfying condition (37), we used the trapezoidal rule. We divided the layer into ν equal horizontal slices and put $\beta_0=1/2$, $\beta_1=\cdots=\beta_q=\cdots=\beta_{\nu-1}=1$, $\beta_\nu=1/2$, when (34) is then equivalent to $\nu+1$ simultaneous algebraic equations of the forms

$$\frac{p^2H}{\nu}\sum_{q=0}^{\nu}\beta_q K\left(\frac{sH}{\nu},\frac{qH}{\nu}\right)u'\left(\frac{qH}{\nu}\right)=u'\left(\frac{sH}{\nu}\right);\ [s=0,1,2,\cdots,\nu] \tag{38}$$

and (37) is equivalent to

$$[u'(H)]^2+\left[\frac{p^2}{k^2}\frac{\mu'}{\mu}\frac{H}{\nu}\sum_{q=0}^{\nu}\beta_q\frac{\partial K\left(H',\frac{qH}{\nu}\right)}{\partial H'}u'\left(\frac{qH}{\nu}\right)\right]^2=4. \tag{39}$$

Eliminating $\nu+1$ values of u' in (38), it is possible to obtain only one value of A, although it may seem that $\nu+1$ values of A exist. Substituting now the value of A thus obtained in $\nu+1$ equations in (38), we get the ratios of

$$u'(0):u'\left(\frac{H}{\nu}\right):\cdots:u'\left(\frac{sH}{\nu}\right):\cdots:u'(H). \tag{40}$$

Substituting these ratios in (39), we obtain the absolute values of

$$u'(0),\quad u'\left(\frac{H}{\nu}\right),\quad \cdots,\quad u'\left(\frac{sH}{\nu}\right),\quad \cdots,\quad u'(H).$$

The solution of more general cases, namely, the case of forces of irregular periods or the one with certain initial conditions, may be easily obtained by the usual familiar methods of analysis.

With a view to compare the result of the present approximate method with that of an accurate one solved by using differential equations, we selected a case such that

$$\rho'(x)=\rho'_0,\quad \mu'(x)=\mu'_0\frac{h+x}{h}, \tag{41}$$

when Green's function becomes

$$\left.\begin{aligned} k(x,a)&=\frac{h}{\mu'_0}\log\frac{h+H}{h+x}+A, \quad [x>a]\\ k(x,a)&=\frac{h}{\mu'_0}\log\frac{h+H}{h+a}+A. \quad [x<a]\end{aligned}\right\} \tag{42}$$

Substituting these in (38) and (39) for the two cases (i) $\mu'_0/\mu'_1=1/16$,

$\mu_1'/\mu=1$, $\frac{2\pi H}{L'}=k'H=5{\cdot}25(k'h=0{\cdot}35)$, (ii) $\mu_0'/\mu_1'=1/16$, $\mu_1'/\mu=1$, $\frac{2\pi H}{L_0'}=k'H=8{\cdot}25(k'h=0{\cdot}55)$, which correspond to the marks shown by the vertical strips in Fig. 4, we calculated the distribution of displacements in the layer. These two examples indicate cases corresponding approximately to (i) $P=0$, and (ii) $Q=0$ respectively in equation (11). The values of A derived from (38) by putting $\nu=4$ are (i) $-2{\cdot}24h^2/\mu_0'H$, (ii) $17{\cdot}5h^2/\mu_0'H$ respectively for the two case under consideration, while the corresponding distributions of displacements obtained from (38), (39) are shown diagrammatically by the full lines in Figs. 7, 8. The accurate values determined by (11) are shown by broken lines in the same figures. The values of A found above were very roughly obtained. It was found that some differences in the values of A are

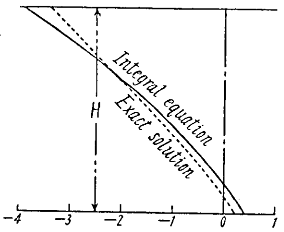

Fig. 8. $\rho'/\rho=1$, $\mu_0'/\mu_1'=1/16$, $\frac{2\pi H}{L_0'}=5{\cdot}25$.

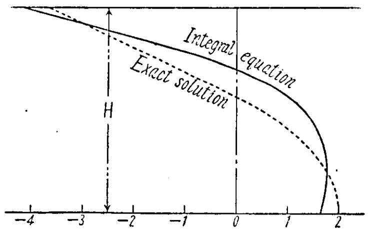

Fig. 9. $\rho'/\rho=1$, $\mu_0'/\mu_1'=1/16$, $\frac{2\pi H}{L_0'}=8{\cdot}25$.

not, in fact, much important in the present calculation of displacement distributions.

It will be seen from the foregoing how the probable values of displacements may be obtained by such a rough method of integral equation as that using merely five trapezoidal ordinates; and also that it is possible to confirm the nature of the dissipation even by the method of integral equations.

36. 不均一彈性又は密度の地表層の震動減衰の割合

地震研究所 {妹澤克惟
金井清}

地表層が均一な場合の性質は既に論じてあるが，不均一な場合にも同じやうな性質があるかどうかをしらべて見たところが，場合によつて特殊な性質があることがわかつた．即ち不均性が可なり著しくても下層との接面のところで連續的になつてをれば，地表層の自由繰返し振動は實際上殆ど起らぬし，又強制振動にしても共振に相當する場合の振幅がそうでない場合と大して變らぬものである．しかしそこに段ができると繰返しの振動が起る．其減衰の係數は兩層の密度及び彈性によつて定まり，內部の摩擦等には關係がなくてもよい．

前記の接面の所で連續的なものでも，低い振動數では共振の所が多少大振幅になるけれども高い振動數では共振の所も共振でない所も振幅に變りがない．而もこの高い振動數の所の振幅の方が低い振動數の所の共振にあたる振幅よりも寧ろ大きいのは變つてゐるといふこともできる．接面の所が不連續な場合には，如何に高い振動數でも夫々の共振の所が大振幅になる．この場合にもやはり低い振動數の場合の共振の大振幅よりも稍々大きい氣味にある．而して之等は大振幅であるとはいへ，彈性と密度の條件によつて或一定値を取ることに變りがない．

地表層の彈性又は密度の值が下層の夫等よりも高い傾向にあるときには，共振にあたる所の振幅が然らざる場合よりも小さくなる．これと同じやうな性質は例へば加速度地震計のやうに剛い地震計の記錄にも現れるやうに思ふ．そのやうな地震計を柔軟な土地の上に置いて測定すると，その地震計の固有週期に近い週期の震動に於ける記錄はその器械の Calibration をやつた場合よりも小なる振幅を與へることになる．逆に地震計の効果的剛度よりも剛い土地の上で用ひると該週期の附近で振幅が寧ろ大きく出る．從て固有週期の極く短い器械で剛い土地の震動を測ると斯る土地では加速度が大きいといふやうな見掛けの結論も出るかも知れぬ．極めて長週期の地震計ではそのやうな變ひは概して少いやうに思はれる．同樣な問題は地上の建物についてもいはれるが，別の機會に讓ることにする．

尙，方法を換へて積分方程式の方法を應用して見た．Green の函數の中に變數を置き且つ上層と下層との接面の條件を正確に取ることによつて，勢力が逸散する場合の解が得られるのであつて，そのやうな方法は勢力の逸散が今まで應用數學者の注意を惹かなかつた爲にそれだけ，まだどこにも解いたのが見當らぬやうである．このやうに物理的に必要な注意を拂つて計算すると，可なり近似的に積分方程式を解いても其結果による變位分布は前の正確な場合の變位分布と非常に近いことがわかるのである．但し注意すべきは，この方法は勢力の逸散はあるにしても，入射波が規則正しい週期を持つ場合に限られてゐることである．