37. *On an Elastic Wave Animated by the Potential Energy of Initial Strain.*

By Hirosi KAWASUMI and Ryôiti YOSIYAMA,

Earthquake Research Institute.

(Read May 21, 1935.—Received June 20, 1935.)

One of the writers estimated the energy carried out by seismic waves in case of a few earthquakes and attributed the energy source to the hypocentral regions. Recently Prof. Ishimoto reminded the writers of the possibility of an alternative explanation that the energy may possibly be the released potential energy which has been stored throughout the medium in a strained state. On his advice we have studied the question theoretically and could prove his opinion actually. Although the case we have treated is a very simple case, the result obtained contains something more than expected, so we will describe it in the following.

I. As a nucleus of strain we have adopted the simplest case of "centre of compression or dilatation". We are now concerned with the case in which the initial strain is suddenly released at a limited portion (say $r=a$) by the disappearance of stress prevailing there. We have therefore to study the transient motion from a statical state to another, and we will employ a harmonic wave function and generalise the time factor so as to satisfy the required conditions.

As is well known, relevant wave function corresponding to the characteristic equation

$$\frac{\partial^2\phi}{\partial t^2}=\frac{\lambda+2\mu}{\rho}\nabla^2\phi \tag{1}$$

is

$$\phi_p=Ar^{-\frac{1}{2}}H^{(1)}_{\frac{1}{2}}(hr)e^{-ipt}+Br^{-\frac{1}{2}}H^{(2)}_{\frac{1}{2}}(hr)e^{ipt}, \tag{2}$$

where ϕ_p is elementary displacement potential and

$$h^2=\rho p^2/(\lambda+2\mu)=(p/V)^2, \tag{3}$$

V being the velocity of longitudinal wave. Now put $A=f(p)dp$ and $B=g(p)dp$ and integrate with regard to p from 0 to ∞, the result

$$\phi=\int_0^\infty f(p)r^{-\frac{1}{2}}H_{\frac{1}{2}}^{(1)}(hr)e^{-ipt}dp+\int_0^\infty g(p)r^{-\frac{1}{2}}H_{\frac{1}{2}}^{(2)}(hr)e^{ipt}dp \qquad (4)$$

is also a solution of the original equation (1) so long as $f(p)$ and $g(p)$ are independent of t and r. Then the displacement and stress corresponding to (4) are

$$u=\frac{\partial\phi}{\partial r}=-\int_0^\infty f(p)hr^{-\frac{1}{2}}H_{\frac{3}{2}}^{(1)}(hr)e^{-ipt}dp-\int_0^\infty g(p)hr^{-\frac{1}{2}}H_{\frac{3}{2}}^{(2)}(hr)e^{ipt}dp, \qquad (5)$$

$$\widehat{rr}=\lambda\Delta+2\mu\frac{\partial u}{\partial r}=\int_0^\infty f(p)\{-(\lambda+2\mu)h^2r^{-\frac{1}{2}}H_{\frac{1}{2}}^{(1)}(hr)+4\mu hr^{-\frac{3}{2}}H_{\frac{3}{2}}^{(1)}(hr)\}e^{-ipt}dp$$

$$+\int_0^\infty g(p)\{-(\lambda+2\mu)h^2r^{-\frac{1}{2}}H_{\frac{1}{2}}^{(2)}(hr)+4\mu hr^{-\frac{3}{2}}H_{\frac{3}{2}}^{(2)}(hr)\}e^{ipt}dp. \qquad (6)$$

We are now to determine $f(p)$ and $g(p)$ from the condition at the boundary, $r=a$, which is expressed by means of Fourier's double integral theorem

$$\widehat{rr}_{r=a}=\varphi(t)=\frac{1}{2\pi}\int_0^\infty dp\int_{-\infty}^\infty \varphi(\omega)e^{ip(t-\omega)}d\omega+\frac{1}{2\pi}\int_0^\infty dp\int_{-\infty}^\infty \varphi(\omega)e^{-ip(t-\omega)}d\omega. \qquad (7)$$

Comparing (6) and (7) we have

$$f(p)=\frac{1}{2\pi\{-(\lambda+2\mu)h^2a^{-\frac{1}{2}}H_{\frac{1}{2}}^{(1)}(ha)+4\mu ha^{-\frac{3}{2}}H_{\frac{3}{2}}^{(1)}(ha)\}}\int_{-\infty}^\infty \varphi(\omega)e^{ip\omega}d\omega, \qquad (8)$$

$$g(p)=\frac{1}{2\pi\{-(\lambda+2\mu)h^2a^{-\frac{1}{2}}H_{\frac{1}{2}}^{(2)}(ha)+4\mu ha^{-\frac{3}{2}}H_{\frac{3}{2}}^{(2)}(ha)\}}\int_{-\infty}^\infty \varphi(\omega)e^{-ip\omega}d\omega. \qquad (9)$$

Substituting these values into (4), and after a little transformation by means of the relations (3) and $H_{n+\frac{1}{2}}^{(1)}(-z)=(-1)^{n+\frac{1}{2}}H_{n+\frac{1}{2}}^{(2)}(z)$ as well as $\sqrt{\frac{1}{2}\pi z}\,H_{\frac{1}{2}}^{(1)}(z)=ie^{-iz}$ and $\sqrt{\frac{1}{2}\pi z}\,H_{\frac{3}{2}}^{(2)}(z)=\left(-1+\frac{i}{z}\right)e^{-iz}$, we have

$$\phi=\frac{-a}{2\pi\rho r}\int_{-\infty}^\infty \frac{e^{ip\left(t-\frac{r-a}{V}\right)}dp}{(p-\alpha)(p-\beta)}\int_{-\infty}^\infty \varphi(\omega)e^{-ip\omega}d\omega, \qquad (10)$$

provided

$$\left.\begin{aligned}\alpha&=i\frac{2\mathfrak{v}^2}{aV}+\frac{2\mathfrak{v}}{a}\sqrt{1-\left(\frac{\mathfrak{v}}{V}\right)^2},\\ \beta&=i\frac{2\mathfrak{v}^2}{aV}-\frac{2\mathfrak{v}}{a}\sqrt{1-\left(\frac{\mathfrak{v}}{V}\right)^2},\end{aligned}\right\} \qquad (11)$$

and

$$\mathfrak{v}=\sqrt{\frac{\mu}{\rho}}. \tag{12}$$

It is a matter of evaluation of definite integral (10) which we are now concerned.

Our first aim is to solve the case when the stress which has been prevailing is suddenly disappeared. It will be of interest to know the reverse case as well. But it is impossible to solve these cases separately, so we will combine these cases and study the case in which the stress is suddenly generated at $t=0$, and disappears at $t=t_1$ where t_1 is to be taken large sufficiently.

$$\varphi(t)=\begin{cases} 0 & \text{when} \quad t<0, \\ -P_0 & \text{"} \quad 0<t<t_1, \\ 0 & \text{"} \quad t_1<t, \end{cases} \tag{13}$$

and

$$\phi=\frac{aP_0}{2\pi\rho r}\int_{-\infty}^{\infty}\frac{e^{ip\tau}dp}{(p-\alpha)(p-\beta)}\int_0^{t_1}e^{-ip\omega}\,d\omega$$

$$=\frac{aP_0}{2\pi i\rho r}\int_{-\infty}^{\infty}\frac{e^{ip\tau}-e^{ip\tau_1}}{p(p-\alpha)(p-\beta)}dp, \tag{14}$$

where $\tau\equiv t-\frac{r-a}{V}$ and $\tau_1\equiv\tau-t_1$.

By means of a contour integral along the real axis with a small indentation at the origin and an infinite semi-circle on the upper side of the real axis when τ is positive, or the lower semi-circle when τ is negative, we can easily obtain

$$\int_{-\infty}^{\infty}\frac{e^{ip\tau}dp}{p(p-\alpha)(p-\beta)}=\begin{cases} 2\pi i\left[\frac{1}{2\alpha\beta}+\frac{1}{\alpha-\beta}\left(\frac{e^{i\alpha\tau}}{\alpha}-\frac{e^{i\beta\tau}}{\beta}\right)\right], & \text{when } \tau>0, \\ -\frac{\pi i}{\alpha\beta}, & \text{when } \tau<0. \end{cases} \tag{15}$$

And we have

a) when $\tau<0$,

$$\phi=0. \tag{16}$$

b) when $0<\tau<t_1$,

$$\phi=\frac{aP_0}{\rho r}\left[\frac{1}{\alpha\beta}+\frac{e^{i\alpha\tau}}{\alpha(\alpha-\beta)}+\frac{e^{i\beta\tau}}{\beta(\beta-\alpha)}\right]$$

$$=\frac{a^3P_0}{4\mu r}\left[\sqrt{\frac{\lambda+2\mu}{\lambda+\mu}}e^{-\frac{2\mathfrak{v}^2}{aV}\tau}\sin\left\{\frac{2\mathfrak{v}}{a}\sqrt{1-\left(\frac{\mathfrak{v}}{V}\right)^2}\tau+\varDelta\right\}-1\right], \tag{17}$$

where

$$\tan\varDelta=\sqrt{\frac{\lambda+2\mu}{\lambda+\mu}}.$$

c) when $0<\tau_1\equiv t-t_1-\dfrac{r-a}{V}$,

$$\phi=\frac{a^3P_0}{4\mu r}\sqrt{\frac{\lambda+2\mu}{\lambda+\mu}}e^{-\frac{2\mathfrak{v}^2}{aV}\tau_1}\left[e^{-\frac{2\mathfrak{v}^2}{aV}t_1}\sin\left\{\frac{2\mathfrak{v}}{a}\sqrt{1-\left(\frac{\mathfrak{v}}{V}\right)^2}\tau+\varDelta\right\}\right.$$

$$\left.-\sin\left\{\frac{2\mathfrak{v}}{a}\sqrt{1-\left(\frac{\mathfrak{v}}{V}\right)^2}\tau_1+\varDelta\right\}\right]$$

$$\approx-\frac{a^3P_0}{4\mu r}\sqrt{\frac{\lambda+2\mu}{\lambda+\mu}}e^{-\frac{2\mathfrak{v}^2}{aV}\tau_1}\sin\left\{\frac{2\mathfrak{v}}{a}\sqrt{1-\left(\frac{\mathfrak{v}}{V}\right)^2}\tau_1+\varDelta\right\}, \tag{18}$$

if $\dfrac{2\mathfrak{v}^2}{aV}t_1\gg1$.

Thus we see from the above displacement potential ϕ that the disturbance travels with the velocity V at a distance and begins suddenly after a travel time $t=\dfrac{r-a}{V}$. The disturbance consists of a statical part $\dfrac{a^3P_0}{4\mu r}$ and a vibratory part with period $\pi a/\mathfrak{v}\sqrt{1-(\frac{\mathfrak{v}}{V})^2}$ which quickly dies away exponentially with damping ratio $e^{\pi\sqrt{\frac{\mu}{\lambda+\mu}}}(=9{\cdot}22$ when $\lambda=\mu)$. And if t_1 is sufficiently large the deformation before arrival of the disturbance which started $r=a$ at $t=t_1$ is practically the statical one, while after the arrival of the latter disturbance $(0<\tau_1)$ the statical deformation disappears and only damped harmonic wave remains which at last dies away.

In the representative case b)

$$u=\frac{\partial\phi}{\partial r}=\frac{a^3P_0}{4\mu}\left[\frac{1}{r^2}\left\{1-\frac{1}{\alpha-\beta}(\beta e^{i\alpha\tau}-\alpha e^{i\beta\tau})\right\}-\frac{i\alpha\beta}{r}(e^{i\alpha\tau}-e^{i\beta\tau})\right]$$

$$=\frac{a^3P_0}{4\mu}\left[\frac{1}{r^2}\left\{1-\sqrt{\frac{\lambda+2\mu}{\lambda+\mu}}e^{-\frac{2\mathfrak{v}^2}{aV}\tau}\sin\left(\frac{2\mathfrak{v}}{a}\sqrt{1-\left(\frac{\mathfrak{v}}{V}\right)^2}\tau+\varDelta\right)\right\}\right.$$

$$\left.-\frac{2}{a}\sqrt{\frac{\lambda+\mu}{\mu}}\frac{1}{r}e^{-\frac{2\mathfrak{v}^2}{aV}\tau}\sin\frac{2\mathfrak{v}}{a}\sqrt{1-\left(\frac{\mathfrak{v}}{V}\right)^2}\tau\right]. \tag{19}$$

$$\dot{u}=\frac{a^3P_0}{4\mu}\frac{i\alpha\beta}{\alpha-\beta}\left[\left(\frac{1}{r^2}+\frac{i\alpha}{Vr}\right)e^{i\alpha\tau}-\left(\frac{1}{r^2}+\frac{i\beta}{Vr}\right)e^{i\beta\tau}\right], \tag{20}$$

$$\widehat{rr}=-\frac{a^3P_0}{r^3}-\frac{a^3P_0}{\alpha-\beta}\left\{\left(\frac{1}{r^3}+\frac{\alpha i}{Vr^2}-\frac{\alpha^2}{4\mathfrak{b}^2r}\right)\beta e^{i\alpha\tau}\right.$$
$$\left.-\left(\frac{1}{r^3}+\frac{\beta i}{Vr^2}-\frac{\beta^2}{4\mathfrak{b}^2r}\right)\alpha e^{i\beta t}\right\}. \tag{21}$$

And the total energy flowing out through a spherical surface at r is

$$E_b=\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta d\theta\int_0^{t_1}\{-\widehat{rr}\dot{u}\}r^2d\tau,$$

where t_1 is very large by the assumption and may be replaced by ∞, and

$$E_b=\frac{\pi a^6P_0^2}{\mu r^3}+\frac{\pi a^6P_0^2}{2\mu}\left[\frac{1}{a^3}-\frac{1}{r^3}\right]. \tag{22}$$

The first term in the above equation represents the energy due to the statical deformation and the latter is due to the dynamical part. We can therefore obtain the entities in case c) if we discard the statical part in the equations (19), (20) and (21) and change signs. And the second term in (22) corresponding to the case c)

$$E_c=\frac{\pi a^6P_0^2}{2\mu}\left[\frac{1}{a^3}-\frac{1}{r^3}\right] \tag{23}$$

is zero at $r=a$, and increases with r to the limit $\dfrac{\pi a^3P_0^2}{2\mu}$.

The statical problem is also obtained by putting $\tau=\infty$ in (17), (19), (20) and (21).

$$\left.\begin{aligned}\phi&=-\frac{a^3P_0}{4\mu r},\\ u&=\frac{a^3P_0}{4\mu r^2},\\ \widehat{rr}&=-\frac{a^3P_0}{r^3}.\end{aligned}\right\} \tag{24}$$

The strain energy function for this case becomes

$$W=\frac{3a^6P_0^2}{8\mu r^6}, \tag{25}$$

and since the potential energy is the volume integral of W, the potential energy stored in the medium from $r=a$ to r is

$$E_p=\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta d\theta\int_a^r Wr^2dr=\frac{\pi a^6 P_0^2}{2\mu}\left[\frac{1}{a^3}-\frac{1}{r^3}\right], \tag{26}$$

which is exactly the total energy flux E_c in case c). Thus we could prove that the energy carried out by the elastic wave in case c) is the released potential energy which have been stored in the medium in the strained state.

We cannot therefore overlook the possibility of the above alternative explanation on the origin of seismic energy besides the one emitted from the hypocentral region. But the dependence of E_b and E_c on r is only through the term proportional to $\frac{1}{r^3}$ and they quickly tend to the same limit in both cases. We cannot therefore distinguish between these cases from the observation at large distance from the origin if we do not compare the higher terms in $\frac{1}{r}$ than the accuracy of usual observation can afford. Strictly speaking the energy source in case c) is the whole medium, but from the actual point of view, the most of the energy is confined to comparatively small region near the origin.

It is also to be noted that the displacement is continuous while the velocity of the particle and the stress $\widehat{rr}$ is discontinuous at $\tau=0$ and $\tau_1=0$.

II. Though our first aim has thus been attained, we shall now pay some attention to the appearance of the damped harmonic motion. The period of this wave is dependent on the radius of the boundary sphere a and the velocities of elastic waves. It thus resembles a kind of self-oscillation. The occurrence of this oscillation is due to the singularities in the integrand of the definite integral, and is not merely the result of our assumption of the form $\varphi(t)$. The circumstance bears striking similarity with the forced oscillation of a pendulum. In fact, the forced oscillation of a damped pendulum defined by

$$\ddot{\theta}+2\varepsilon\dot{\theta}+n^2\theta=\varphi(t)$$

is easily proved to be

$$\theta=\frac{-1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ipt}dp}{(p-\alpha)(p-\beta)}\int_{-\infty}^{\infty}\varphi(\omega)e^{-ip\omega}d\omega,$$

which is exactly the same form as (10) provided $\alpha=i\varepsilon+\sqrt{n^2-\varepsilon^2}$, $\beta=i\varepsilon-\sqrt{n^2-\varepsilon^2}$. We have ample knowledge on the motion of a pendulum, and we can demonstrate it by experiment. The displacement potential ϕ for a given pressure $\varphi(t)$ at $r=a$ can be inferred from the motion of a pendulum under the same force if t is replaced by $\tau=t-\frac{r-a}{V}$. We shall not therefore enter into detailed discussion of several other examples we have studied, but we cannot help to state some bearing in the interpretation of seismic motions. So far as the writers are aware, elucidation on the occurrence of oscillatory motions in the earthquake motions hitherto proposed are the explanation by the breaking up of the non-periodic initial wave forms by dispersion, or by the self-oscillation of some part of the crust excited by seismic motions, but nothing is known on the mechanism of the alternative explanation by the intrinsic nature caused at a hypocentre notwithstanding prevalently assumed implicitly. We can now add an explanation for it. As we know the occurrence of self-oscillation in the motion of a pendulum at every discontinuity of disturbance, so we are to expect the occurrence of damped harmonic waves in case of an earthquake.

Another point of interest is the proportionality of the period to the radius of the boundary sphere. We are well acquainted with the fact that the period of seismic wave is the longer the larger the earthquake is. One of the writers pointed out that we have to consider a kind of hypocentral region, and the seismic phenomena which take place at the region cannot be inferred from the observation of elastic waves at large distance from the origin. Such a chaos may possibly play some rolls in causing seismic waves by the traction at a boundary surface. Then the well known fact on the length of period of seismic waves can be easily explained.

Finally we state that in case of such simple cases discussed above, the motions at large distance from the origin are

$$u=\mp\frac{a^2P_0}{2\sqrt{\mu(\lambda+\mu)}}\frac{1}{r}\,e^{-\frac{2\mathfrak{v}^2}{aV}\tau}\sin\frac{2\mathfrak{v}}{a}\sqrt{1-\left(\frac{\mathfrak{v}}{V}\right)^2}\,\tau$$

for each case b) and c), and the damping ratio is as large as 9·2 when $\lambda=\mu$, and the motion is hardly different from that of shock type, and the appearance of such a type of motion in actual earthquake makes us feel some interest.

Concluding Remark. We have proved in a simplest case that elastic wave may be animated by the potential energy of initial strain. We have seen also that a kind of damped harmonic wave resembling self-oscillation is excited when stress on a spherical surface is given. Showing the similarity with the motion of a pendulum under a given force we have proposed one of the possible explanations on the vibratory nature of seismic waves. We should expect that similar treatment is applicable in a more general case when stress or displacement of more general form are given on a spherical surface. Indeed we could also see that similar vibratory waves are excited when displacements (excepting a few cases) and stresses of general forms are given on a spherical surface, and the results will be published shortly.

In conclusion the writers wish to express their cordial thanks to Prof. M. Ishimoto for the precious suggestions.

---

37. 歪の潜勢力によつて起される彈性波

地震研究所 河角 廣
吉山 良一

筆者の1人は 2〜3 の地震の調査に際し地震波によつて運び出される全勢力を推算し，之れを震原域から出たものさ考へたが，最近石本教授は歪の潜勢力も地震波に勢力を供給し得る事を注意せられたので筆者等は共の最も簡單な場合を理論的に計算し，石本教授の説を實證する事が出來た.

或る球面で突然歪力が消滅する場合に起される彈性波を出して見るさ，一種の減衰振動の波で振幅は中心距離に逆比例するもので遠方に於ては靜力學的の變位よりも大きいものであるが，共れによつて運び出される勢力は全く初の歪の潜勢力に等しい事を知つた. 逆に歪力が突然加へられる場合も計算し比較して見た. 彈性の際限內では，何れの場合にも傳播速度に相違は見られない.

次に振動性の波動の起るのは全く振子の自己振動の起るのさ同様である事を示し，共の週期が球の半徑に比例し彈性波の速度に關係する事から一種の自己振動さ見得る事を示し，地震動の性質の1解釋さして此の結果が興味ある點を指摘した. 更に一般的の場合も同様に取扱へるがそれは次回に報告する.
