54. *Decay in the Seismic Vibrations of a Simple or Tall Structure by Dissipation of their Energy into the Ground.*[1]

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In the absence of some form of damping resistance, almost any structure would suffer serious damage in an earthquake, owing to resonance. The question of air resistance and damping in the material used in the structure as well as in the foundation have been studied by some investigators,[2] but the dissipation of vibrational energy in the form of seismic waves transmitted into the ground, which seems to be the most important part of the problem, has not yet received their attention. Although we stated in a previous paper[3] that the smallness of the amplitudes of a structure in seismic vibrations is due mainly to the radiation or dissipation of vibrational energy into the ground, owing to certain mathematical difficulties, the problem was not followed up until recently when a reinvestigation was made. From the result of our recent calculations with respect to the decaying of seismic vibrations of various structures, the cases of vertical, shearing, and flexural vibrations of a simple as well as of a tall framed structure with floors either rigid or flexible seem to be the problems that we should investigate.

The prevalent belief that, without damping force, the displacements of an oscillating body become infinitely large under resonance conditions, appears to be based on the conception that part of its boundary to which periodic external forces are being applied, cannot be a node of the vibrations in that body. As a matter of fact, the part in question becomes a loop, a node, or an intermediate phase according to the frequency of the vibrations. At the frequency that

1) Preliminary report published in *Proc. Imp. Acad.*, **11** (1935), 174.

2) The part, "Waves propagated in Beam...," which was appended at the end of the paper "On the Decay of Waves in Visco-Elastic Solid Bodies", *Bull. Earthq. Res. Inst.*, **3** (1927), 50, and similar papers published since then by a number of writers.

3) § 10 of our paper, "Some New Problems of Forced Vibrations of a Structure", *Bull. Earthq. Res. Inst.*, **12** (1934), 845.

synchronizes with the natural frequency of the body, the part under consideration becomes a node, with the result that the amplitudes of vibration even under resonance conditions do not exceed a certain limit, which fact will be readily understood if we consider the dissipation of energy. Naturally the same applies to vibrations of a structure subjected to seismic movements of the ground.

Seismic waves that cause structural damage are sometimes bodily waves and at other times surface waves. The bodily waves are incident to the earth's surface practically normally. The scattered waves that dissipate from the structure into the ground seem to be chiefly bodily waves, including their harmonic waves, associated with surface waves. However, in order to avoid mathematical difficulties, we shall take only the principal terms of scattered waves in the present calculation and ignore the effects from the formation of surface waves.

1. A *Simple structure subjected to incident longitudinal waves.*

The structure is assumed to be a uniform circular cylinder of radius ε. Let the incident longitudinal waves with their displacement orientated vertically be

$$u_0 = e^{ih(V_1 t + x)}\,, \tag{1}$$

where $h^2 = \rho p^2/(\lambda+2\mu)$, $V_1 = \sqrt{(\lambda+2\mu)/\rho}$. Assuming then for simplicity that the scattered waves are radiated from a semispherical surface, $r=\varepsilon$, shown in Fig. 1, the expressions for the scattered longitudinal and transverse waves become[4)]

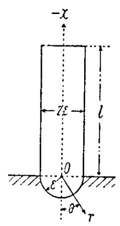

Fig. 1.

$$\Delta = \alpha\cos\theta\frac{e^{i(pt-hr)}}{r}\,, \tag{2'}$$

$$u_1 = -\frac{\alpha}{h^2}\cos\theta\frac{d}{dr}\frac{e^{i(pt-hr)}}{r}\,, \quad v_1 = \frac{\alpha}{h^2}\sin\theta\frac{e^{i(pt-hr)}}{r^2}\,, \tag{2}$$

$$u_2 = -\frac{4\beta}{k^2}\cos\theta\frac{e^{i(pt-kr)}}{r^2}\,, \quad v_2 = \frac{2\beta}{k^2}\sin\theta\frac{1}{r}\frac{d}{dr}e^{i(pt-kr)}\,, \tag{3}$$

where u_1, v_1 and u_2, v_2 satisfy the equations of motion for longitudinal and transverse waves respectively, Δ is the dilatation, α, β are

4) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **2** (1927), 13~20.

constants determined by boundary conditions, and $h^2=\rho p^2/(\lambda+2\mu)$, $k^2=\rho p^2/\mu$, ρ, λ, μ being the density and elastic constants of the earth.

The reasonable way would be to assume that the waves scatter directly from the bottom end of the structure, $x=0$, but, in order to avoid the mathematical difficulties thus introduced, we assumed the semispherical surface to have the same diameter as that of the structure. Such an assumption does not seriously affect the result of the problem, provided the radius, ε, of the sphere is not large compared with the length of the incident waves. Since from this condition it follows that the spherical surface in question always remains spherical with constant radius, we have at $r=\varepsilon$,

$$-(u_1+u_2)_{max.}=(v_1+v_2)_{max.}, \tag{4}$$

from which we get

$$\frac{\alpha}{\beta}=\frac{2h^2\varepsilon^2\{(4+hk\varepsilon^2)+2i(k-h)\varepsilon\}e^{i(h-k)\varepsilon}}{k^2\varepsilon^2(4+h^2\varepsilon^2)}. \tag{5}$$

The vibratory motion in the structure is expressed by

$$u'=Be^{ih'(V_1't+x)}+Ce^{ih'(V_1't-x)}, \tag{6}$$

where $h'^2=\rho'p^2/E$, $V_1'=\sqrt{E/\rho'}$, ρ', E being density and Young's modulus respectively.

The condition at the upper end of the structure, $x=-l$, is

$$\partial u'/\partial x=0, \tag{7}$$

and the boundary conditions at its lower end are such that the displacements as well as the resultants of vertical stress in the structure are continuous respectively with those in the earth, namely,

$$u'_{x=0}=u_{0x=0}+(u_1+u_2)_{r=\varepsilon,\ \theta=0}, \tag{8}$$

$$E\pi\varepsilon^2\frac{\partial u'}{\partial x_{x=0}}=(\lambda+2\mu)\pi\varepsilon^2\frac{\partial u_0}{\partial x_{x=0}}$$

$$+\int_{\theta=0}^{\frac{\pi}{2}}\int_{\varphi=0}^{2\pi}\left(\lambda\Delta+2\mu\frac{\partial(u_1+u_2)}{\partial r}\right)r^2\cos\theta\sin\theta\,d\theta\,d\varphi_{r=\varepsilon}$$

$$+\int_{\theta=0}^{\frac{\pi}{2}}\int_{\varphi=0}^{2\pi}\mu\left(\frac{\partial(v_1+v_2)}{\partial r}-\frac{(v_1+v_2)}{r}+\frac{1}{r}\frac{\partial(u_1+u_2)}{\partial\theta}\right)r^2\sin^2\theta\,d\theta\,d\varphi_{r=\varepsilon}. \tag{9}$$

Substituting (1), (2), (2'), (3), (5), (6) in (7), (8), (9), and neglecting small quantities higher than the second order, and taking the real part only, we get

$$u' = \frac{\left(2+3\frac{\lambda}{\mu}+4\sqrt{\frac{\lambda}{\mu}+2}\right)\cos h'(l+x)}{\sqrt{\left\{3\sqrt{\frac{\rho' E}{\rho\mu}\left(\frac{\lambda}{\mu}+2\right)}\sin h'l\right\}^2+\left\{4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)\cos h'l\right\}^2}}$$

$$\cdot\cos\left\{pt-\tan^{-1}\frac{3\sqrt{\frac{\rho' E}{\rho\mu}\left(\frac{\lambda}{\mu}+2\right)}\sin h'l}{4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)\cosh'l}\right\}, \tag{10}$$

where $h'=p\sqrt{\rho'/E}$, corresponding to the incident waves

$$u_0=\cos(pt+hx). \tag{1'}$$

The left-hand term under the root sign of the denominator of (10) represents the effect of the dissipation of energy scattered as seis-

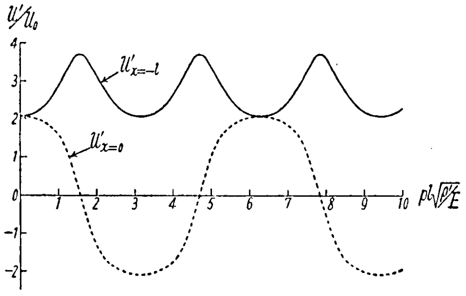

Fig. 2. $\rho'/\rho=1$, $E/\mu=1/10$.

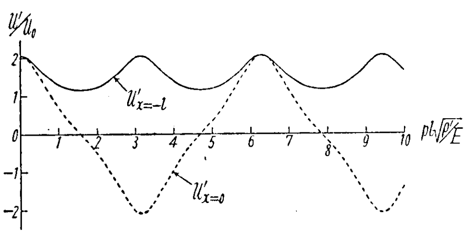

Fig. 3. $\rho'/\rho=1$, $E/\mu=1$.

mic waves. The larger the ratio of E/μ or ρ'/ρ, the larger will be the decrease in the amplitudes of vibration at the periods corresponding to the resonance condition, $\cos h'l=0$, of the case without dissipation of energy. It

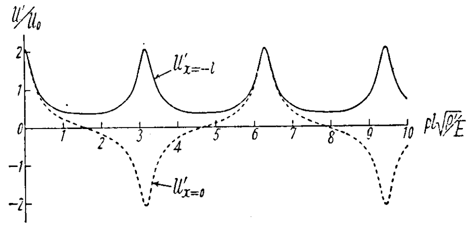

Fig. 4. $\rho'/\rho=1$, $E/\mu=10$.

is also evident that under resonance conditions the displacement at the lower end is always zero. We have calculated the amplitudes at $x=-l$ and those at $x=0$ corresponding to various frequencies for the cases, (i) $\rho'/\rho=1$, $\lambda/\mu=1$, $E/\mu=1/10$, (ii) $\rho'/\rho=1$, $\lambda/\mu=1$, $E/\mu=1$, (iii) $\rho'/\rho=1$, $\lambda/\mu=1$, $E/\mu=10$, all of which are shown in Figs. 2, 3, 4.

2. *Free vertical vibrations of a simple structure.*

In order to find the nature of the free vibrations of the structure, we introduce the original complex form of the solution, analogous to u' in (10), namely

$$u'=Be^{ih'(V_1't+x)}+Ce^{ih'(V_1't-x)}, \tag{6'}$$

where

$$B=\frac{R}{P+Qe^{-2ih'l}}, \quad C=\frac{Re^{-2ih'l}}{P+Qe^{-2ih'l}}, \tag{11}$$

in which

$$\left.\begin{aligned} P&=4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)+3\sqrt{\frac{\rho'E}{\rho\mu}\left(\frac{\lambda}{\mu}+2\right)}, \\ Q&=4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)-3\sqrt{\frac{\rho'E}{\rho\mu}\left(\frac{\lambda}{\mu}+2\right)}, \\ R&=10+3\frac{\lambda}{\mu}+4\sqrt{\frac{\lambda}{\mu}+2}. \end{aligned}\right\} \tag{12}$$

Generalising (6′) by means of Fourier's integral we get

$$u'=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{Rdh}{P+Qe^{-2ibhl}}\int_{-\infty}^{\infty}f(\sigma)\,e^{ibh\left(V_1't+x+\frac{\sigma}{b}\right)}d\sigma$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{Re^{-2ibhl}dh}{P+Qe^{-2ibhl}}\int_{-\infty}^{\infty}f(\sigma)\,e^{ibh\left(V_1't-x+\frac{\sigma}{b}\right)}d\sigma. \tag{13}$$

where $h'=bh$, corresponding to the incident disturbance $f(x)$ at $t=0$. Since we know that

$$\frac{1}{P+Qe^{-2ibhl}}=\frac{1}{P}\sum_{m=0}^{\infty}(-1)^m\left(\frac{Q}{P}\right)^m e^{-2imbhl}, \tag{14}$$

we have from (13), (14)

$$u'=\frac{1}{2\pi}\left(\frac{R}{P}\right)\sum_{m=0}^{\infty}(-1)^m\left(\frac{Q}{P}\right)^m\Big[F\{b(V_1't+x-2ml)\}$$

$$+F\{b(V_1't-x-2\overline{m+1}l)\}\Big]. \qquad (15)$$

corresponding to the disturbing incident waves in the earth

$$u_0=F(V_1t+x). \qquad (16)$$

The vibrations are thus of a purely exponential type

$$e^{-\varkappa t}, \qquad (17)$$

the logarithmic decrement being therefore

$$\varkappa=\frac{V_1'}{2l}\log_e\left|\frac{Q}{P}\right|. \qquad (18)$$

The greater the difference in the elastic constants or in the densities of the structure and the earth, the more slowly decay the free vibrations. It is not a matter of importance whether the constants of the earth are greater or less than those of the structure.

To serve as examples, we took the disturbance

$$F(V_1t+x)=e^{-\frac{(V_1t+x)^2}{c^2}},$$

and calculated the resulting free vibrations of structures corresponding to three cases (i) $\rho'/\rho=1$, $E/\mu=1/10$, $l/c=0\cdot365$, (ii) $\rho'/\rho=1$, $E/\mu=1$, $l/c=1\cdot155$, (iii) $\rho'/\rho=1$, $E/\mu=10$, $l/c=3\cdot651$, the result being plotted in Figs. 5, 6, 7.

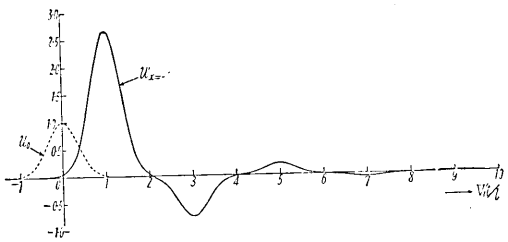

Fig. 5. $\rho'/\rho=1$, $E/\mu=1/10$, $l/c=0\cdot365$.

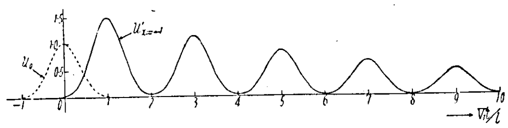

Fig. 6. $\rho'/\rho=1$, $E/\mu=1$, $l/c=1\cdot155$.

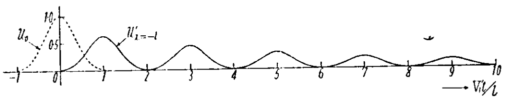

Fig. 7. $\rho'/\rho=1$, $E/\mu=10$, $l/c=3\cdot651$.

3. *A tall structure with rigid floors subjected to incident transverse waves.*

It was shown in a previous paper[5] that the motion of a tall building with rigid floors subjected to horizontal oscillation of the ground is analogous to the case of shearing vibrations of a simple structure. We shall discuss the case of an n-storied structure. Let the axis of x be drawn vertically downwards with its origin at the free surface of the earth, and let ρ, λ, μ; $\rho'(=m/al_1)$, $G(=12{\cdot}4Ej^2/l_1^2)$ be the density and the elastic constants of the earth, and the effective density and the effective rigidity of the structure, where E, j, l_1, a are Young's modulus, radius of gyration of section, length and total sum of the sectional areas of the columns between two adjacent floors respectively. Provided the incident transverse waves in the earth are propagated vertically upwards, the boundary conditions are such that there is no effective shearing stress at the upper end of the structure, while the displacement and the shearing stress at its lower end are continuous respectively with those in the earth resulting from incident transverse waves and scattered waves, both longitudinal and transverse. Let the incident waves orientated horizontally be

$$u_0 = e^{ik(V_2 t + x)}\,, \qquad (19)$$

where $k^2 = \rho p^2/\mu$, $V_2 = \sqrt{\mu/\rho}$ and the scattered waves be of the forms shown in (2′), (2), (3), the angle θ being now taken as shown in Fig. 8. The relation between α and β is again the same as that in (5). The displacement of the structure is also in horizontal sense and expressed by

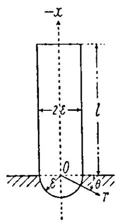

Fig. 8.

$$u' = Be^{ik'(V_2' t + x)} + Ce^{ik'(V_2' t - x)}\,, \qquad (20)$$

where $k'^2 = \rho' p^2/G$, $V_2' = \sqrt{G/\rho'}$. The boundary condition at $x = -l(= -nl_1)$ is

$$\partial u'/\partial x = 0\,, \qquad (21)$$

while, at the lower end of that structure, we have the conditions

$$u'_{x=0} = u_{0x=0} + (u_1 + u_2)_{r=\varepsilon,\ \theta=0}\,, \qquad (22)$$

$$G\pi\varepsilon^2 \frac{\partial u'}{\partial x_{x=0}} = \mu\pi\varepsilon^2 \frac{\partial u_0}{\partial x_{x=0}} + \int_{\theta=0}^{\pi}\int_{\varphi=0}^{\pi} \mu\left(\frac{\partial(v_1+v_2)}{\partial r} - \frac{(v_1+v_2)}{r} + \frac{1}{r}\frac{\partial(u_1+u_2)}{\partial\theta}\right) r^2 \sin^2\theta\, d\theta\, d\varphi_{r=\varepsilon}$$

5) K. SEZAWA and K. KANAI, *Bull. Earthq. Res. Inst.*, **12** (1934), 819.

$$+\int_{\theta=0}^{\pi}\int_{\varphi=0}^{\pi}\left(\lambda\Delta+2\mu\frac{\partial(u_1+u_2)}{\partial r}\right)r^2\sin\theta\cos\theta\, d\theta d\varphi_{r=\varepsilon}\,. \tag{23}$$

In this case, also, for reasons already given, the assumption with respect to the semispherical surface holds. Proceeding as in the previous case we finally get

$$u'=\frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)\cos k'(l+x)}{\sqrt{\left\{3\sqrt{\frac{\rho' G}{\rho\mu}}\left(\frac{\lambda}{\mu}+2\right)\sin k'l\right\}^2+\left\{4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)\cos k'l\right\}^2}}$$

$$\cdot\cos\left\{pt-\tan^{-1}\frac{3\sqrt{\frac{\rho' G}{\rho\mu}}\left(\frac{\lambda}{\mu}+2\right)\sin k'l}{4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)\cos k'l}\right\}, \tag{24}$$

in which $k'=p\sqrt{\rho'/G}$, $l=nl_1$, corresponding to the incident waves

$$u_0=\cos(pt+kx). \tag{19'}$$

We calculated the amplitudes at $x=-l$ and those at $x=0$ for various frequencies for the cases, (i) $\rho'/\rho=10$, $\lambda/\mu=1$, $G/\mu=1/500$, (ii) $\rho'/\rho=10$, $\lambda/\mu=1$, $G/\mu=0{\cdot}0318$, (iii) $\rho'/\rho=10$, $\lambda/\mu=1$, $G/\mu=1$ and plotted the results in Figs. 9, 10, 11. They show certain features of vibrations similar to those in the previous case. We specially selected a relatively large ρ'/ρ, with a view to finding the amplitude of an actual tall, framed structure.

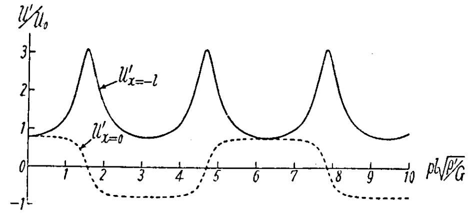

Fig. 9. $\rho'/\rho=10$, $G/\mu=1/500$.

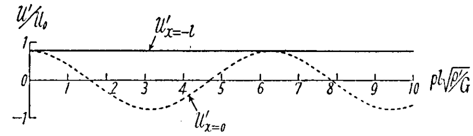

Fig. 10. $\rho'/\rho=10$, $G/\mu=0{\cdot}0318$.

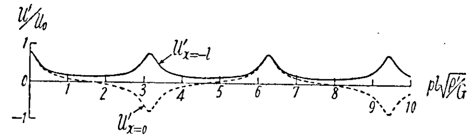

Fig. 11. $\rho'/\rho=10$, $G/\mu=1$.

4. *Free horizontal vibrations of a tall structure with rigid floors.*

With the same mathematics as that used for vertical vibrations, we get the following equation of free vibrations for this case:

$$u'=\frac{1}{2\pi}\frac{R}{P}\sum_{m=0}^{\infty}(-1)^m\left(\frac{Q}{P}\right)^m\Big[F\{b(V_2't+x-2ml)\}$$

$$+F\{b(V_2't-x-\overline{2m+1}l\}\Big], \qquad (25)$$

corresponding to the initial disturbance incident to the earth's surface

$$u_0=F(V_2t+x). \qquad (26)$$

P, Q, R in (25) signify

$$\left.\begin{aligned}
P&=4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)+3\sqrt{\frac{\rho' G}{\rho\mu}\left(\frac{\lambda}{\mu}+2\right)}\,,\\
Q&=4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)-3\sqrt{\frac{\rho' G}{\rho\mu}\left(\frac{\lambda}{\mu}+2\right)}\,,\\
R&=7\sqrt{\frac{\lambda}{\mu}+2}\;-4\,.
\end{aligned}\right\} \qquad (27)$$

It is possible to prove that the logarithmic decrement in this case is expressed by

$$\frac{\sqrt{G/\rho'}}{2l}\log\frac{3\sqrt{(\rho' G/\rho\mu)(\lambda/\mu+2)}+4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)}{3\sqrt{(\rho' G/\rho\mu)(\lambda/\mu+2)}-4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)}\,. \qquad (28)$$

If the type of the initial disturbance be

$$F(V_2t+x)=e^{-\frac{(V_2t+x)^2}{c^2}}\,,$$

the free vibrations of the structure corresponding to the three cases, (i) $\rho'/\rho=10$, $\lambda=\mu$, $G/\mu=1/500$, $l/c=0.02828$, (ii) $\rho'/\rho=10$, $\lambda=\mu$, $G'/\mu=0{\cdot}0318$, $l/c=0{\cdot}1127$, (iii) $\rho'/\rho=10$, $\lambda=\mu$, $G'/\mu=1$, $l/c=0{\cdot}6324$, assume the forms shown in Figs. 12, 13, 14.

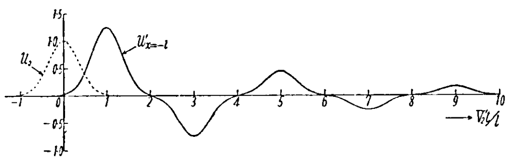

Fig. 12. $\rho'/\rho=10$, $G/\mu=1/500$, $l/c=0{\cdot}02828$.

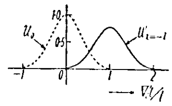

Fig. 13. $\rho'/\rho=10$, $G/\mu=0{\cdot}0318$, $l/c=0{\cdot}1127$.

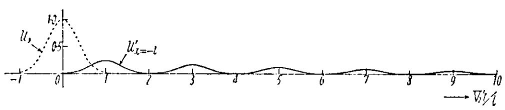

Fig. 14. $\rho'/\rho=10$, $G/\mu=1$, $l/c=0{\cdot}6324$.

5. *The problem of shearing vibrations of a simple structure.*

The problem of horizontal vibrations of a tall structure with rigid floors, studied in the preceding two sections, was nothing more than a modification of the case of a simple structure under horizontal shearing vibrations. The problem in its original form may therefore be at the same time one for the shearing vibrations of a simple structure. Solutions (24), (25), (27) may straightway be converted into solutions of the present case with however the proviso that ρ', G, l are respectively the density, rigidity, and the height of the simple structure under consideration.

6. *A tall structure with flexible floors subjected to incident transverse waves.*

In a previous paper[6] we stated that the motion of a tall building with flexible floors subjected to horizontal movement of the ground is analogous to the transverse vibrations of an elastic rod. We shall discuss an n-storied structure. Referring to Fig. 8, let ρ, λ, μ; $\rho'(=m/al_1)$, j be the density and elastic constants of the earth, the effective density of the structure and the radius of gyration of a section of a column, where m, l_1, a are masses concentrated on every floor, and the length and total sum of the sectional areas of columns between two adjacent floors respectively.

The equation of vibratory motion of the structure is now written

$$\frac{\partial^2 y}{\partial t^2}+\frac{Ej^2}{\rho'}\frac{\partial^4 y}{\partial x^4}=0, \tag{29}$$

its solution being

$$y=e^{ipt}[Ae^{i\sqrt{p}cx}+Be^{-i\sqrt{p}cx}+Ce^{\sqrt{p}cx}+De^{-\sqrt{p}cx}], \tag{30}$$

in which $c=(\rho'/Ej^2)^{1/4}$. The forms of incident waves, u_0, and scattered waves, (u_1, v_1), (u_2, v_2), are respectively the same as those in (19), (2), (3). The relation between α and β is also the same as in (5).

The conditions at the effective upper end, $x=-l\{=(n+1/2)l_1\}$[7] are

$$\frac{\partial^2 y}{\partial x^2}=0\ , \qquad \frac{\partial^3 y}{\partial x^3}=0\ , \tag{31}$$

and, if its lower end be clamped, that is, the structure stands normal

6) K. SEZAWA and K. KANAI, *Bull. Earthq. Res. Inst.*, **12** (1934), 818.

7) K. SEZAWA and K. KANAI, *ibid.*.

to the earth's surface at its lower end, we have

$$y_{x=0}=u_{0x=0}+(u_1+u_2)_{r=\varepsilon\ x=0}, \tag{32}$$

$$\partial y/\partial x_{x=0}=0, \tag{33}$$

$$-E\pi\varepsilon^2 j^2\frac{\partial^3 y}{\partial x^3}_{x=0}=\mu\pi\varepsilon^2\frac{\partial u}{\partial x}_{x=0}$$

$$+\int_{\theta=0}^{\pi}\int_{\varphi=0}^{\pi}\mu\left(\frac{\partial(v_1+v_2)}{\partial r}-\frac{(v_1+v_2)}{r}+\frac{1}{r}\frac{\partial(u_1+u_2)}{\partial\theta}\right)r^2\sin^2\theta\, d\theta\, d\varphi_{r=\varepsilon}$$

$$+\int_{\theta=0}^{\pi}\int_{\varphi=0}^{\pi}\left(\lambda\Delta+2\mu\frac{\partial(u_1+u_2)}{\partial r}\right)r^2\sin\theta\cos\theta\, d\theta\, d\varphi_{r=\varepsilon}, \tag{34}$$

the assumption with respect to ε being the same as in the preceding case. After a prolonged investigation we obtained

$$\left.\begin{matrix}A\\B\end{matrix}\right\}=\frac{4e^{-ik\varepsilon}}{\Theta k^2\varepsilon^2(h^2\varepsilon^2+4)}\left\{\left[-\frac{4}{3}(k\varepsilon-h\varepsilon)\left\{2(h^2\varepsilon^2+2)+\frac{\lambda}{\mu}h^2\varepsilon^2\right\}-k\varepsilon(h\varepsilon k\varepsilon+4)\right]\right.$$

$$+i\left[-\frac{4}{3}(k\varepsilon-h\varepsilon)(h^2\varepsilon^2k\varepsilon+2h\varepsilon+4k\varepsilon)+\frac{2}{3}\frac{\lambda}{\mu}h^2\varepsilon^2(h\varepsilon k\varepsilon+4)\right.$$

$$\left.\left.+k\varepsilon(h^2\varepsilon^2k\varepsilon+2h\varepsilon+2k\varepsilon)\right]\right\}$$

$$\cdot\left\{\pm(\cos\sqrt{p}cl\sinh\sqrt{p}cl+\sin\sqrt{p}cl\cosh\sqrt{p}cl)\right.$$

$$\left.+i(-\cos\sqrt{p}cl\cosh\sqrt{p}cl+\sin\sqrt{p}cl\sinh\sqrt{p}cl-1)\right\}, \tag{35}$$

$$\left.\begin{matrix}C\\D\end{matrix}\right\}=\frac{-4e^{-ik\varepsilon}}{\Theta k^2\varepsilon^2(h^2\varepsilon^2+4)}\left\{\left[\frac{4}{3}(k\varepsilon-h\varepsilon)(h^2\varepsilon^2k\varepsilon+2h\varepsilon+4k\varepsilon)\right.\right.$$

$$\left.-\frac{2}{3}\frac{\lambda}{\mu}h^2\varepsilon^2(h\varepsilon k\varepsilon+4)-k\varepsilon(h^2\varepsilon^2k\varepsilon+2h\varepsilon+2k\varepsilon)\right]$$

$$\left.+i\left[-\frac{4}{3}(k\varepsilon-h\varepsilon)\left\{2(h^2\varepsilon^2+2)+\frac{\lambda}{\mu}h^2\varepsilon^2\right\}-k\varepsilon(h\varepsilon k\varepsilon+4)\right]\right\}$$

$$\cdot\left\{e^{\pm\sqrt{p}cl}(\cos\sqrt{p}cl\pm\sin\sqrt{p}cl)+1\right\}, \tag{36}$$

$$\beta=\frac{-8}{\Theta}\Big[\frac{E}{\mu}\varepsilon j^2p^{3/2}c^3(\cos\sqrt{pc}l\sinh\sqrt{pc}l+\sin\sqrt{pc}l\cosh\sqrt{pc}l)$$

$$+ik\varepsilon(\cos\sqrt{pc}l\cosh\sqrt{pc}l+1)\Big], \qquad (37)$$

where

$$\Theta=\frac{-16e^{-ik\varepsilon}}{k^2\varepsilon^2(h^2\varepsilon^2+4)}\Bigg\{\Big[\frac{2}{3}(\cos\sqrt{pc}l\cosh\sqrt{pc}l+1)\Big\{2(k\varepsilon-h\varepsilon)(h^2\varepsilon^2k\varepsilon$$

$$+2h\varepsilon+4k\varepsilon)-\frac{\lambda}{\mu}h^2\varepsilon^2(h\varepsilon k\varepsilon+4)\Big\}$$

$$+\frac{E}{\mu}\varepsilon j^2p^{3/2}c^3(\cos\sqrt{pc}l\sinh\sqrt{pc}l+\sin\sqrt{pc}l\cosh\sqrt{pc}l)(h\varepsilon k\varepsilon+4)\Big]$$

$$-i\Big[\frac{4}{3}(\cos\sqrt{pc}l\cosh\sqrt{pc}l+1)(k\varepsilon-h\varepsilon)\Big\{2(h^2\varepsilon^2+2)+\frac{\lambda}{\mu}h^2\varepsilon^2\Big\}$$

$$+\frac{E}{\mu}\varepsilon j^2p^{3/2}c^3(\cos\sqrt{pc}l\sinh\sqrt{pc}l+\sin\sqrt{pc}l\cosh\sqrt{pc}l)(h^2\varepsilon^2k\varepsilon$$

$$+2h\varepsilon+2k\varepsilon)\Big]\Bigg\}. \qquad (38)$$

Substituting these values in the original equations, neglecting small quantities higher than the second order, and taking the real part only, we finally get

$$y=\Big\{(\cos\sqrt{pc}l\cosh\sqrt{pc}l+1)(\cosh\sqrt{pc}x+\cos\sqrt{pc}x)$$

$$+\sin\sqrt{pc}l\sinh\sqrt{pc}l(\cosh\sqrt{pc}x-\cos\sqrt{pc}x)$$

$$+(\cos\sqrt{pc}l\sinh\sqrt{pc}l+\sin\sqrt{pc}l\cosh\sqrt{pc}l)(\sinh\sqrt{pc}x-\sin\sqrt{pc}x)\Big\}\Phi,$$

$$(39)$$

where

$$\Phi=\frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)\cos\left(pt-\tan^{-1}\Psi/\chi\right)}{2\sqrt{\Psi^2+\chi^2}}, \quad C=\left(\frac{\rho'}{Ej^2}\right)^{1/4}=\left(\frac{m}{l_1Eaj^2}\right)^{1/4},$$

$$(40), (41)$$

in which

$$\Psi=3\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{Ej^2}{\mu k l^3}\right)(\sqrt{pc}l)^3(\cos\sqrt{pc}l\sinh\sqrt{pc}l+\sin\sqrt{pc}l\cosh\sqrt{pc}l), \quad (42)$$

$$\chi=4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)(\cos\sqrt{pc}l\cosh\sqrt{pc}l+1), \quad (43)$$

$Ej^2(\sqrt{pc}l)^3/\mu k l^3$ in (42) being equivalent to $\sqrt{E^{1/2}j\rho'^{3/2}p/\mu\rho}$. The displacement (39) corresponds to the incident transverse waves

$$u_0=\cos(yt+kx). \quad (44)$$

It is worth noticing that $\chi=0$ is the frequeney equation of vibrations of the structure which is free at its upper end and absolutely clamped at its lower end, while $\Psi=0$ is the frequency equation of vibrations of a free-free bar whose length is twice the height of the structure.

The nature of the problem for the case under resonance conditions is similar to that of the preceding cases, except with the difference that, in the present problem, however small may be the ratio $Ej^2/\mu k l^3$, the amplitudes of vibrations under higher resonance conditions become rather less than those at frequencies out of resonance. This may also be seen from Figs. 15, 16, 17, which show the amplitudes at $x=-l$

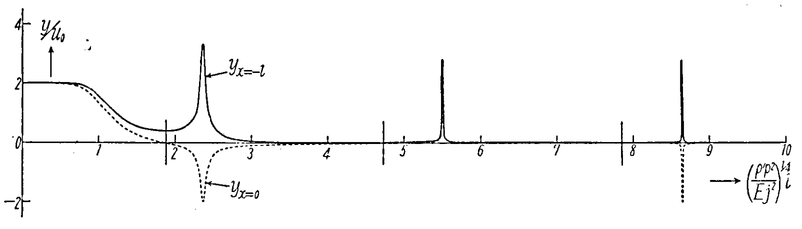

Fig. 15. $\lambda/\mu=14$, $\frac{Ej^2}{\mu k l^3}=16\cdot38$.

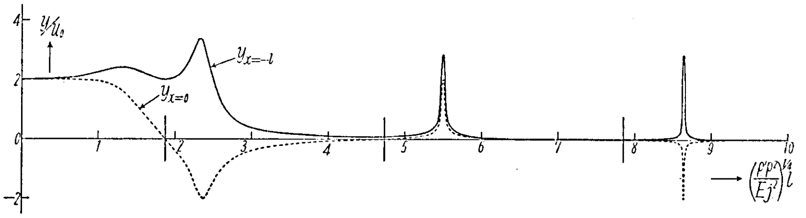

Fig. 16. $\lambda/\mu=14$, $\frac{Ej^2}{\mu k l^3}=3\cdot305$.

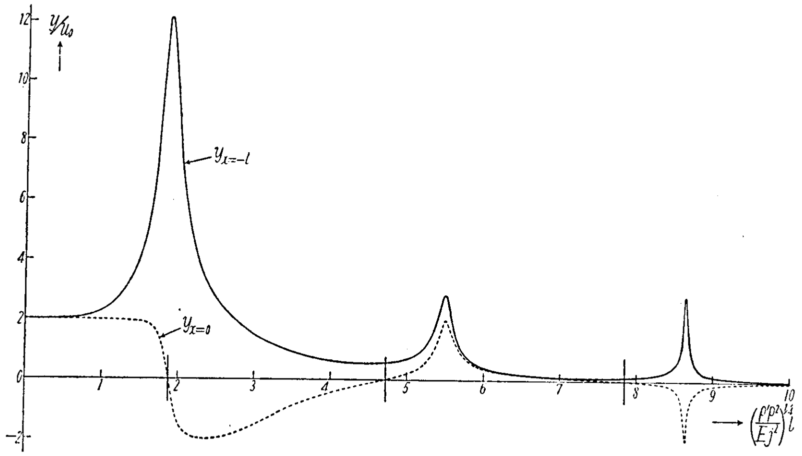

Fig. 17. $\lambda/\mu=14$, $\frac{Ej^2}{\mu k l^3}=0\cdot5464$.

and at $x=0$ of the cases, $Ej^2/\mu kl^3=16\cdot38$, $3\cdot305$, $0\cdot5462$, at different $\rho'(=m/al_1)$ corresponding to the incident waves of unit amplitudes transmitted through the earth, in which $\lambda=14\,\mu$. The short vertical strips represent the frequencies with which the resonance should take place in the usual sense.

7. *Distribution of bending moments of a simple structure in flexural vibrations.*

The result of the foregoing section may be applied to the problem of a simple flexible structure whose vibrations decay owing to the energy dissipated into the ground. For this purpose it is only necessary to suppose that ρ', E, j in solution (35) are the density, Young's modulus, and radius of gyration of the section of the present simple structure.

It is of some interest to investigate the relation between the bending moments distributed, as the result of incident and reflected waves, along the structure and the acceleration of the surface of the earth, on which no structure is standing. Since the bending moment of the structure is expressed by

$$Eaj^2\frac{d^2y}{dx^2}\,, \tag{45}$$

where y is given by equation (39) and the acceleration of the ground surface by

$$\left(2\frac{\partial^2 u}{\partial t^2}\right)_{x=0}. \tag{46}$$

It is possible to make the ratio of these quantities dimensionless by multiplying (46) by $\rho' a l^2$, the ratio thus modified being

$$Eaj^2\frac{\partial^2 y}{\partial x^2} \Bigg/ \rho' a l^2\left(2\frac{\partial^2 u_0}{\partial t^2}\right). \tag{47}$$

Figs. 18∼24 show such values for the cases $(\rho' p^2/Ej^2)^{1/4} l=1$, 1·875, 2·1, 2·365, 3·5, 4·73, 5·498, besides the condition that $Ej^2/\mu k l^3=0{\cdot}5462$ remains constant. These figures tell us that, given the acceleration

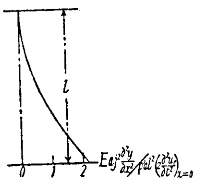

Fig. 18. $\left(\frac{\rho' p^2}{Ej^2}\right)^{\frac{1}{4}} l=1.$

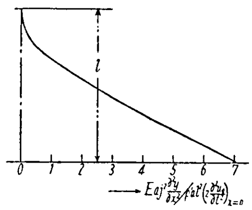

Fig. 19. $\left(\frac{\rho' p^2}{Ej^2}\right)^{\frac{1}{4}} l=1{\cdot}875.$

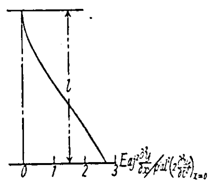

Fig. 20. $\left(\frac{\rho' p^2}{Ej^2}\right)^{\frac{1}{4}} l=2{\cdot}1.$

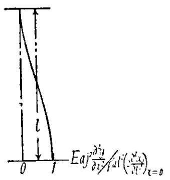

Fig. 21. $\left(\frac{\rho' p^2}{Ej^2}\right)^{\frac{1}{4}} l=2{\cdot}365.$

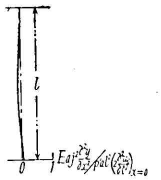

Fig. 22. $\left(\frac{\rho' p^2}{Ej^2}\right)^{\frac{1}{4}} l=3{\cdot}5.$

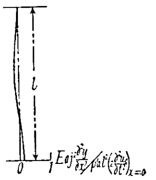

Fig. 23. $\left(\frac{\rho' p^2}{Ej^2}\right)^{\frac{1}{4}} l=4{\cdot}73.$

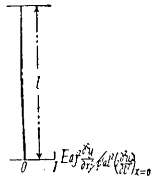

Fig. 24. $\left(\frac{\rho' p^2}{Ej^2}\right)^{\frac{1}{4}} l=5{\cdot}498.$

and frequency of seismic vibrations, the bending moment or bending stress induced at a given part of the structure is determinate. It is evident that, apart from the physical nature of the material used in the structure, acceleration cannot be the only parameter for the failure of the structure, or that even both acceleration and frequency are not sufficient for that purpose.[8] The type of the structure as well as its conditions in relation to that of the ground play an important part in the problem. The ordinates taken horizontally in the curves in

8) K. SEZAWA, *Publ. Bureau Centr. Séism. Int.*, [A], No. 10 (1934), 113.

Figs. 18∼24 indicate how the distribution and magnitudes of bending moments in the structure change with any change in the period, provided however that acceleration in free ground (no structures) is always unity and that the particular type of structure as well as its conditions in relation to that of the ground are specified.

In conclusion we wish to express our thanks to the Council of the Foundation for the Promotion of Scientific and Industrial Research of Japan, with whose aid the progress of the present work was greatly assisted.

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54. 勢力の地下逸散の爲に生ずる高層構造物の震動減衰

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地震波の中にはあらゆる週期の波が存在してゐることは多くの記録を見れば明かである．從て，構造物に何等かの振動減衰性がなければ，如何に强度のある構造物でも共振を起し結局は破壞を免れないであらう．彙報第 12 號で說明したやうに，構造物の破壞は材料の破壞的性質を別問題とすると，(i) 土地の加速度，(ii) 土地の週期，(iii) 構造物の型の三個のもので決定できたのであるが，それでも共振の所では必ず破壞することになつてゐたのである．果してそうであるとすると構造物の設計の基準がなくなることになる．しかし減衰性があれば，共振の所の震動力が一定の大さに止まる譯である．それで構造物の材料の內部抵抗（筆者の一人も七八年前に考へたことがある．力學的に最も初めであつたと記憶する）や空氣抵抗，基礎のプラスチックなどによる減衰性を考へて見ると之等のものではどうしても追付かぬことがわかる．そこで勢力が彈性波として再び地中へ逸散するものとして計算して見ると非常に大きな減衰があり得，且つ如何なる割合に減衰するかといふこともはつきり數量的に算定できるのである．而して之は前掲の三種の性質以外に (iv) 構造物の性質と土地の性質との關係から決定されるのである．構造物と土地との結合狀態や振動の數理等については本文に示してある通りであつて，それ等の Elements はすべて他の多くの實驗結果が基礎となつてゐるから，そこに數學上の勝手な假定などといふものは何もない．寧ろ他の地震現象說明論などで其場あたりの氣分で議論をされるのとは全く譯が違つてゐる．

この論文に於て，單一構造物に上下動や水平動の働く場合や高層構造物に水平動や上下動が働く場合の强制振動及び自由振動が，(i) 種々の週期，(ii) 構造物のないときの土地の種々の加速度，(iii) 種々の構造型，(iv) 土地の性質と構造物の性質との關係によつて數量的に決定できることを示してある．共振に當る强制振動でも共振幅其他が一定のものであるから之等を基準として構造物を設計すればよい譯である．

共振の狀態に於ては構造物又は土地の基礎に當る部分が夫々の振動の節點に近いといふことか

らでも共振に於ても構造物の振幅が大して大きくはないさいふことが了解できるであらう．又，構造物の剛度が土地のそれに比して大きければ大きい程，共振のときの振幅が小さくなることは注目に値するさ思ふ．而も之等の剛度は夫々の彈性力さ密度さの積（場合によつては多少複雜な積）に比例してゐるものである．

構造物の固有振動の減衰の恒數も上述のものの一部分で定まり，固體粘性などは問題にならない．

床が撓み易い高層構造物や煙突などの震動については種々面白いことがあるけれども茲には省略し本文中の説明に讓ることにする．

只今の研究によつてこれまで多くの構造學者（又は力學家）の研究した構造物の振動問題，殊に共振の概念は根本的に正しくなかつたさいふことはいひ得るさ思ふ．