55. *Energy Dissipation in Seismic Vibrations of a Framed Structure.*

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(Read May 21, 1935.—Received June 20, 1935.)

1. In one[1] of our papers published last year, we determined the distribution of bending moments in the columns of a multi-storied structure under periodic earthquake movements of the ground. Since however the nature of dissipation of vibrational energy was not perfectly understood by us at the time, we discussed the problem in that paper under the assumption that the movement of the lower end or foundation of the structure is the same as that of a free ground surface, that is, one on which no structure is standing. From the results of our foregoing paper[2], it is evident that such an assumption is not permissible unless the effective stiffness or density of the structure differs widely from the rigidity or the density of the ground on which the structure rests. Our present calculation shows that, owing to the dissipation of vibrational energy into the ground, the amplitudes of vibrations of a structure do not exceed a certain limit even under resonance conditions. We also come to a similar conclusion with respect to the bending moments in any member of the structure. This fact appears to answer the question what strength is necessary in the earthquake-proof construction of a structure, that is liable to be subjected to resonating seismic vibrations.

In this paper we investigate three kinds of structures, namely, (i) structures with rigid floors and clamped base, (ii) structures with flexible floors and clamped base, (iii) structures with rigid floors and hinged base. Rigid and flexible floors correspond to the "clamped and supported floors" of the previous paper[3]. By "rigid floor" is meant a floor so rigid as never to bend through horizontal oscillations, as

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1) K. SEZAWA and K. KANAI, "Some New Problems of Forced Vibrations of a Structure", *Bull. Earthq. Res. Inst.*, **12** (1934), 823~853. This paper will hereinafter be referred to as paper (B).

2) K. SEZAWA and K. KANAI, "Decay in the Seismic Vibrations of a (Simple or Tall) Structure by Dissipation of their Energy into the Ground", *Proc. Imp. Acad.*, **11** (1935), 174; *Bull. Earthq. Res. Inst.*, **13** (1935), 681~697.

3) K. SEZAWA and K. KANAI, *ibid.*.

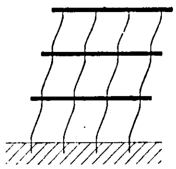
practically obtained in actual buildings; while by "flexible floor" is meant a floor so flexible as to yield to horizontal oscillations, a condition however that hardly obtains in actual construction, excepting in a structure with very long and weakly spanned floors.

As to the number of stories we have treated three cases, namely one-, two-, and three-storied for each of the three types of structure. The waves are assumed to be of the same nature as in the case of a simple structure subjected to horizontal oscillations.

2. *A structure with rigid floors and clamped base.*

Both incident and dissipated waves are assumed to have the same forms as those stated in the preceding paper[4]. The equation of motion of the columns for each floor is the same as that used in last year's paper[5], namely,

$$\frac{\partial^4 y_s}{\partial x_s^4}=0, \qquad (1)$$



its solution being

$$y_s=(A_s+B_s x_s+C_s x_s^2+D_s x_s^3)e^{ipt}, \qquad (2)$$

Fig. 1.

where s signifies the case for columns between the $(s-1)$th and the sth floors, x_s is the coordinate of a current point on the same columns measured negatively from their respective lower ends, and y_s is the horizontal deflection of the point x_s. If l_s be the length of the columns under consideration, the boundary conditions for an n-storied structure are

$$x_n=-l_n; \quad \frac{\partial y_n}{\partial x_n}=0, \quad E_n I_n \frac{\partial^3 y_n}{\partial x_n^3}+m_n\frac{\partial^2 y_n}{\partial t^2}=0, \qquad (3), (4)$$

$$x_s=-l_s, \quad x_{s+1}=0; \quad y_s=y_{s+1}, \quad \frac{\partial y_s}{\partial x_s}=0, \quad \frac{\partial y_{s+1}}{\partial x_{s+1}}=0, \qquad (5), (6), (7)$$

$$E_{s+1}I_{s+1}\frac{\partial^3 y_{s+1}}{\partial x_{s+1}^3}=E_s I_s\frac{\partial^3 y_s}{\partial x_s^3}+m_s\frac{\partial^2 y_s}{\partial t^2}, \qquad (8)$$

$$(s=1, 2, \cdots, n-1.)$$

$$x_1=0; \quad y_{1x_1=0}=u_{0x_1=0}+(u_1+u_2)_{r=\varepsilon,\ \theta=0}, \quad \frac{\partial y_1}{\partial x_{1_{x_1=0}}}=0, \qquad (9), (10)$$

4) Paper (B).

5) K. SEZAWA and K. KANAI, *loc. cit. ante.* in 2).

$$-E_1\pi\varepsilon^2 j^2 \frac{\partial^3 y_1}{\partial x_1^3}_{x_1=0} = \mu\pi\varepsilon^2 \frac{\partial u_0}{\partial x_1}_{x_1=0}$$

$$+\int_{\theta=0}^{\pi}\int_{\varphi=0}^{\pi}\mu\left(\frac{\partial(v_1+v_2)}{\partial r}-\frac{(v_1+v_2)}{r}+\frac{1}{r}\frac{\partial(u_1+u_2)}{\partial\theta}\right)r^2\sin^2\theta d\theta d\varphi_{r=\varepsilon}$$

$$+\int_{\theta=0}^{\pi}\int_{\varphi=0}^{\pi}\left(\lambda\Delta+2\mu\frac{\partial(u_1+u_2)}{\partial r}\right)r^2\sin\theta\cos\theta d\theta d\varphi_{r=\varepsilon}, \tag{11}$$

where m_s is the mass concentrated on the sth floor, and l_s, E_s, I_s are the length, Young's modulus, and the sum of the moments of inertia of cross sections of the columns between the $(s-1)$th and the sth floors, l_1, E_1, $I_1(=\pi\varepsilon^2 j^2)$ being the same for the lowest columns. ρ, λ, μ are again the density and elastic constants of the earth, while we shall soon be using $h^2=\rho p^2/(\lambda+2\mu)$ and $k^2=\rho p^2/\mu$.

Solving the problem in the same way as in the previous paper, we get the deflection of the columns, corresponding to the incident waves,

$$u_0=\cos(pt+kx), \tag{12}$$

as follows:

(i) $n=1$,

$$y_1=\frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)\left\{(12-\gamma)+3\gamma\left(\frac{x}{l}\right)^2+2\gamma\left(\frac{x}{l}\right)^3\right\}}{4\sqrt{P^2+Q^2}}\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \tag{13}$$

where

$$P=9\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{Ej^2}{\mu kl^3}\right)\gamma,\quad Q=(12-\gamma)\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right), \tag{14}$$

$$\gamma=\frac{mp^2l^3}{EI}. \tag{15}$$

(ii) $n=2$,

$$y_1=\frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{4\sqrt{P^2+Q^2}}\left[\left\{(12-\gamma_1)(12-\gamma_2)-12\gamma_2\eta\right\}\right.$$

$$\left.+\left\{12\gamma_2\eta+\gamma_1(12-\gamma_2)\right\}\left\{3\left(\frac{x_1}{l_1}\right)^2+2\left(\frac{x_1}{l_1}\right)^3\right\}\right]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \tag{16}$$

$$y_2=\frac{3\left(\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{\sqrt{P^2+Q^2}}\Bigg[(12-\gamma_2)$$

$$+\gamma_2\left\{3\left(\frac{x_2}{l_2}\right)^2+2\left(\frac{x_2}{l_2}\right)^3\right\}\Bigg]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \quad (17)$$

where

$$\left.\begin{aligned} P&=9\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{E_1j_1^2}{\mu k l_1^3}\right)\left\{12\gamma_2\eta+\gamma_1(12-\gamma_2)\right\},\\ Q&=\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)\left\{(12-\gamma_1)(12-\gamma_2)-12\gamma_2\eta\right\},\end{aligned}\right\} \quad (18)$$

$$\gamma_1=\frac{m_1p^2l_1^3}{E_1I_1},\quad \gamma_2=\frac{m_2p^2l_2^3}{E_2I_2},\quad \eta=\frac{E_2I_2l_1^3}{E_1I_1l_2^3}. \quad (19)$$

(iii) $n=3$,

$$y_1=\frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{4\sqrt{P^2+Q^2}}\Bigg\{\Big[12\zeta\gamma_3\{12\eta+(12-\gamma_1)\}$$

$$-(12-\gamma_3)\{(12-\gamma_1)(12-\gamma_2)-12\eta\gamma_3\}\Big]-\Big[12\zeta\gamma_3(12\eta-\gamma_1)+$$

$$(12-\gamma_3)\{\gamma_1(12-\gamma_2)+12\eta\gamma_2\}\Big]\left\{3\left(\frac{x_1}{l_1}\right)^2+2\left(\frac{x_1}{l_1}\right)^3\right\}\Bigg\}\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \quad (20)$$

$$y_2=\frac{3\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{\sqrt{P^2+Q^2}}\Bigg[\{12\zeta\gamma_3-(12-\gamma_2)(12-\gamma_3)\}$$

$$-\{12\zeta\gamma_3+\gamma_2(12-\gamma_3)\}\left\{3\left(\frac{x_2}{l_2}\right)^2+2\left(\frac{x_2}{l_2}\right)^3\right\}\Bigg]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \quad (21)$$

$$y_3=\frac{-36\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{\sqrt{P^2+Q^2}}\Bigg[(12-\gamma_3)$$

$$+\gamma_3\left\{3\left(\frac{x_3}{l_3}\right)^2+2\left(\frac{x_3}{l_3}\right)^3\right\}\Bigg]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \quad (22)$$

where

$$\left.\begin{aligned} P&=-9\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{E_1j^2}{\mu k l_1^3}\right)\Big[12\zeta\gamma_3(\gamma_1-12\eta)-(12-\gamma_3)\{\gamma_1(12-\gamma_2)+12\eta\gamma_2\}\Big],\\ Q&=\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)\Big[12\zeta\gamma_3\{12\eta+(12-\gamma_1)\}\\ &\qquad -(12-\gamma_3)\{(12-\gamma_1)(12-\gamma_2)-12\eta\gamma_2\}\Big],\end{aligned}\right\} \quad (23)$$

$$\gamma_1=\frac{m_1p^2l_1^3}{E_1I_1},\quad \gamma_2=\frac{m_2p^2l_2^3}{E_2I_2},\quad \gamma_3=\frac{m_3p^2l_3^3}{E_3I_3},\quad \eta=\frac{E_2I_2l_1^3}{E_1I_1l_2^3},\quad \zeta=\frac{E_3I_3l_2^3}{E_2I_2l_3^3}. \tag{24}$$

In the special case $E_1=E_2=\cdots=E$, $I_1=I_2=\cdots=I$, $l_1=l_2=\cdots=l$, $m_1=m_2=\cdots=m$, we have

(i) $n=1$, the same as (13), (14), (15).

(ii) $n=2$,

$$y_1=\frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{4\sqrt{P^2+Q^2}}\left[(\gamma^2-36\gamma+144)\right.$$
$$\left.-\gamma(\gamma-24)\left\{3\left(\frac{x_1}{l}\right)^2+2\left(\frac{x_1}{l}\right)^3\right\}\right]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \tag{16'}$$

$$y_2=\frac{3\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{\sqrt{P^2+Q^2}}\left[-(\gamma-12)\right.$$
$$\left.+\gamma\left\{3\left(\frac{x_2}{l}\right)^2+2\left(\frac{x_2}{l}\right)^3\right\}\right]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \tag{17'}$$

where

$$\left.\begin{aligned}P&=9\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{Ej^2}{\mu kl^3}\right)\gamma(24-\gamma)\,,\\ Q&=\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)(\gamma^2-36\gamma+144)\,,\end{aligned}\right\} \tag{18'}$$

$$\gamma=\frac{mp^2l^3}{EI}. \tag{19'}$$

(iii) $n=3$,

$$y_1=\frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{4\sqrt{P^2+Q^2}}\left[(\gamma^3-60\gamma^2+864\gamma-1728)\right.$$
$$\left.-\gamma(12-\gamma)(36-\gamma)\left\{3\left(\frac{x_1}{l}\right)^2+2\left(\frac{x_1}{l}\right)^3\right\}\right]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \tag{20'}$$

$$y_2=\frac{3\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{\sqrt{P^2+Q^2}}\left[-(\gamma^2-36\gamma+144)\right.$$
$$\left.-\gamma(24-\gamma)\left\{3\left(\frac{x_2}{l}\right)^2+2\left(\frac{x_2}{l}\right)^3\right\}\right]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \tag{21'}$$

$$y_3=\frac{-36\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{\sqrt{P^2+Q^2}}\left[(12-\gamma)\right.$$

$$\left.+\gamma\left\{3\left(\frac{x_3}{l}\right)^2+2\left(\frac{x_3}{l}\right)^3\right\}\right]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \qquad (22')$$

where

$$\left.\begin{aligned}P&=-9\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{Ej^2}{\mu k l^3}\right)\gamma(12-\gamma)(36-\gamma),\\ Q&=\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)(\gamma^3-60\gamma^2+864\gamma-1728),\end{aligned}\right\} \qquad (23')$$

$$\gamma=\frac{mp^2l^3}{EI}. \qquad (24')$$

The maximum values of the bending moments $E_sI_s\ (d^2y_s/dx_s^2)$ at each end of the columns corresponding to the maximum values of accelera-

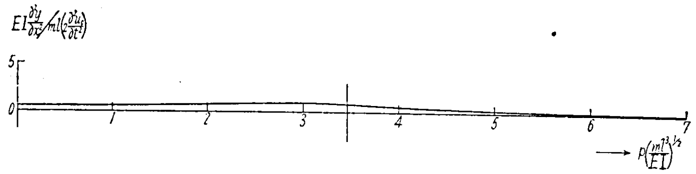


Fig. 2. The case of a singled-storied structure with rigid floor and clamped base.

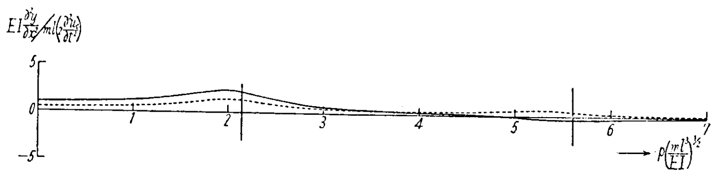


Fig. 3. The case of a two-storied structure with rigid floors and clamped base. Full line: lowest columns. Broken line: second columns.

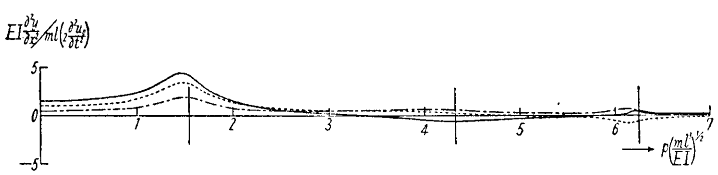


Fig. 4. The case of a three-storied structure with rigid floors and clamped base. Full line: lowest columns. Broken line: second columns. Chain: third columns.

tion of the ground on which no structure rests were determined from the above equations for the special case $E_s=E$, $I_s=I$, $l_s=l$, $m_s=m$, $Ej^2/\mu kl^3=0{\cdot}05$, $\lambda/\mu=14$, and for various values of ml^3/EI; they are diagrammatically shown in Figs. 2, 3, 4. Vertical strips in these figures represent the cases corresponding to resonance conditions. These figures show that the bending moments of the columns in the structure, even under resonance conditions, are not very large—a fact that will be recognized more clearly on comparing the present result with Figs. 3, 4 of paper (B). When writing paper (B), we had hopes that the infinitely large bending moments under resonance conditions, as shown in Figs. 3, 4, ··· of that paper, could be reduced to certain finite values by the introduction of some form of damping resistance, which however is now being realized by more rational methods. It is a matter of importance whether $Ej^2/\mu k l^3=0{\cdot}05$ is probable or not in an actual case. We shall take the case of a concrete structure standing on alluvium ground, when it is then possible to assume that $E/\mu=10$, $j^2/i^2=1/1000\,(l/j=33)$, $1/kl=5\,(l=5\text{ m.}, 2\pi/k=160\text{ m})$, whence it follows that $Ej^2/\mu k l^3=0{\cdot}05$. We find that the assumed value of $Ej^2/\mu k l^3$ does not differ much from an actual case. It is rather probable that in the usual structures,[6] E/μ may take greater values, in which case the dissipation would be more pronounced than in the one we have calculated.

3. *Structure with flexible floors and clamped base.*

The difference between this problem and the preceding one is merely with respect to floor conditions. Here we have

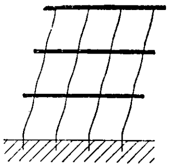


Fig. 5.

$$x_n=-l_n;\ \frac{\partial^2 y_n}{\partial x_n^2}=0\,,\ E_nI_n\frac{\partial^3 y_n}{\partial x_n^3}+m_n\frac{\partial^2 y_n}{\partial t^2}=0\,, \qquad (25)\ (4')$$

$$x_s=l_s,\ x_{s+1}=0;\ y_s=y_{s+1},\ \frac{\partial y_s}{\partial x_s}=\frac{\partial y_{s+1}}{\partial x_{s+1}}, \qquad (5')\ (26)$$

$$E_{s+1}I_{s+1}\frac{\partial^2 y_{s+1}}{\partial x_{s+1}^2}=E_sI_s\frac{\partial^2 y_s}{\partial x_s^2}\,, \qquad (27)$$

$$E_{s+1}I_{s+1}\frac{\partial^3 y_{s+1}}{\partial x_{s+1}^3}=E_sI_s\frac{\partial^3 y_s}{\partial x_s^3}+m_s\frac{\partial^2 y_s}{\partial t^2}\,, \qquad (8')$$

$$(s=1, 2, \cdots, n-1)$$

other conditions being the same as those in the preceding case. The final solutions of the problem corresponding to the incident waves

$$u_0=\cos(pt+kx), \qquad (12')$$

are as follows:

6) K. SEZAWA and K. KANAI's paper which will appear in a forthcoming bulletin of the Institute.

(i) $n=1$,

$$y_1=\frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{2\sqrt{P^2+Q^2}}\left[2(3-\gamma)+\gamma\left\{3\left(\frac{x}{l}\right)^2+\left(\frac{x}{l}\right)^3\right\}\right]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \quad (28)$$

where $$P=9\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{Ej^2}{\mu kl^3}\right)\gamma, \quad Q=4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)(3-\gamma), \quad (29)$$

$$\gamma=\frac{mp^2l^3}{EI}. \quad (30)$$

(ii) $n=2$,

$$y_1=\frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{4\sqrt{P^2+Q^2}}\Big\{[-3\eta\gamma_2\{\xi^2(12-\gamma_1)+12\xi+4\}+4(3-\gamma_1)(3-\gamma_2)]$$
$$+3\{3\eta\gamma_2(-\xi^2\gamma_1+2\xi+2)+2\gamma_1(3-\gamma_2)\}\left(\frac{x_1}{l_1}\right)^2$$
$$+2\{3\eta\gamma_2(-\xi^2\gamma_1+1)+\gamma_1(3-\gamma_2)\}\left(\frac{x_1}{l_1}\right)^3\Big\}\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \quad (31)$$

$$y_2=\frac{3\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{4\sqrt{P^2+Q^2}}\Big[-2\{3\xi\eta\gamma_2(2\xi+1)-2(3-\gamma_2)\}$$
$$-2\xi\{3\eta\gamma_2(2\xi+1)+\gamma_1(3-\gamma_2)\}\left(\frac{x_2}{l_2}\right)$$
$$+\gamma_2(\xi\gamma_1+2)\left\{3\left(\frac{x_2}{l_2}\right)^2+\left(\frac{x_2}{l_2}\right)^3\right\}\Big]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \quad (32)$$

where

$$\left.\begin{aligned}P&=9\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{E_1j^2}{\mu kl_1^3}\right)\left\{3\eta\gamma_2(-\xi^2\gamma_1+1)+\gamma_1(3-\gamma_2)\right\}\\ Q&=\left[-3\eta\gamma_2\{\xi^2(12-\gamma_1)+(12\xi+4)\}+4(3-\gamma_1)(3-\gamma_2)\right]\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right),\end{aligned}\right\} \quad 33)$$

$$\gamma_1=\frac{m_1p^2l_1^3}{E_1I_1}, \quad \gamma_2=\frac{m_2p^2l_2^3}{E_2I_2}, \quad \xi=\frac{l_2}{l_1}, \quad \eta=\frac{E_2I_2l_1^3}{E_1I_1l_2^3}. \quad (34)$$

(iii) $n=3$. The general case omitted.

In the special case, $E_1=E_2=\cdots=E$, $I_1=I_2=\cdots=I$, $l_1=l_2=\cdots=l$, $m_1=m_2=\cdots=m$, we have

(i) $n=1$, the same as (28), (29), (30).

(ii) $n=2$,

$$y_1=\frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{4\sqrt{P^2+Q^2}}\left[(7\gamma^2-108\gamma+36)-3\gamma(5\gamma-18)\left(\frac{x_1}{l}\right)^2\right.$$

$$\left.-4\gamma(2\gamma-3)\left(\frac{x_1}{l}\right)^3\right]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \qquad (31')$$

$$y_2=\frac{3\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{4\sqrt{P^2+Q^2}}\left[-2(11\gamma-6)+2\gamma(\gamma-12)\left(\frac{x_2}{l}\right)\right.$$

$$\left.+\gamma(\gamma+2)\left\{3\left(\frac{x_2}{l}\right)^2+\left(\frac{x_2}{l}\right)^3\right\}\right]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \qquad (32')$$

where

$$\left.\begin{aligned} P&=18\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{Ej^2}{\mu k l^3}\right)(3-2\gamma)\gamma\,, \\ Q&=\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)(7\gamma^2-108\gamma+36)\,, \end{aligned}\right\} \qquad (33')$$

$$\gamma=\frac{mp^2l^3}{EI}\,. \qquad (34')$$

(iii) $n=3$,

$$y_1=\frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{2\sqrt{P^2+Q^2}}\left\{2(13\gamma^3-393\gamma^2+1296\gamma-108)\right.$$

$$+3\gamma(-19\gamma^2+294\gamma-216)\left(\frac{x_1}{l}\right)^2$$

$$\left.+\gamma(-31\gamma^2+372\gamma-108)\left(\frac{x_1}{l}\right)^3\right\}\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \qquad (35)$$

$$y_2=\frac{3\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{2\sqrt{P^2+Q^2}}\left\{4(-23\gamma^2+171\gamma-18)\right.$$

$$+\gamma(7\gamma^2-216\gamma+324)\left(\frac{x_2}{l}\right)+6\gamma(2\gamma^2-13\gamma-18)\left(\frac{x_2}{l}\right)^2$$

$$\left.+\gamma(5\gamma^2+10\gamma-24)\left(\frac{x_2}{l}\right)^3\right\}\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \qquad (36)$$

$$y_3 = \frac{3\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{2\sqrt{P^2+Q^2}}\left[12(3\gamma^2+23\gamma-6)+2\gamma(-\gamma^2-15\gamma+234)\left(\frac{x_3}{l}\right)\right.$$
$$\left.-\gamma(\gamma^2+36\gamma+12)\left\{3\left(\frac{x_3}{l}\right)^2+\left(\frac{x_3}{l}\right)^3\right\}\right]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \quad (37)$$

where
$$\left.\begin{aligned} P &= 9\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{Ej^2}{\mu k l^3}\right)\gamma(-31\gamma^2+372\gamma-108), \\ Q &= 4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)(13\gamma^3-393\gamma^2+1296\gamma-108), \end{aligned}\right\} \quad (38)$$

$$\gamma = \frac{mp^2l^3}{EI}. \quad (39)$$

The maximum values of bending moments $E_sI_s(\partial^2 y_s/\partial x_s^2)$ at each end of the columns corresponding to the maximum values of acceleration of the ground on which no structure stands were determined from the above equations for the special case $E_s=E$, $I_s=I$, $l_s=l$, $m_s=m$, $Ej^2/\mu kl^3 = 0{\cdot}05$, $\lambda/\mu=14$, and for various values of ml^3/EI. They are diagrammatically shown in Figs. 6, 7, 8. Although the bending moments of the present case under the first resonance condition are excessively large compared with the one of the previous case, namely the structure with rigid floors, it is still

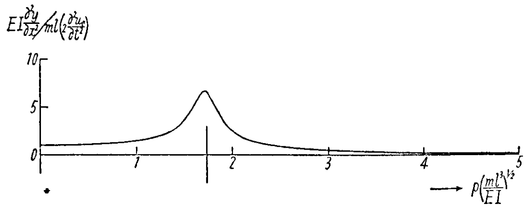


Fig. 6. The case of a singled-storied structure with flexible floor and clamped base.

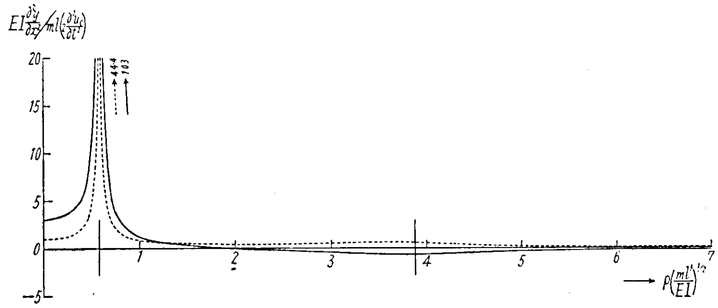


Fig. 7. The case of a two-storied structure with flexible floors and clamped base. Full line: lowest columns. Broken line: second columns.

certain that the moments under higher resonance conditions are exceedingly small. Nevertheless, it will be found, upon comparing our present result with those shown in Figs. 10, 11 of paper (B), that the respective

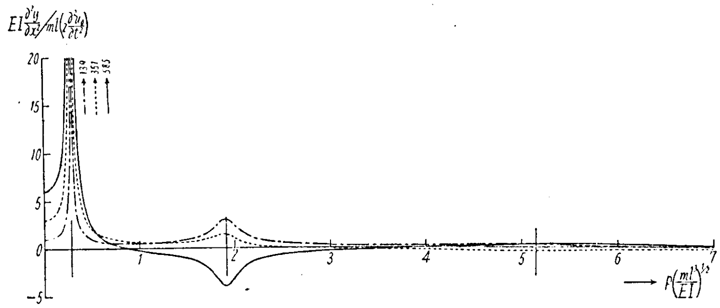


Fig. 8. The case of a three-storied structure with flexible floors and clamped base. Full line: lowest columns. Broken line: second columns. Chain: third columns.

moments under frequencies out of resonance are approximately equal for both this and the last cases, notwithstanding the great difference between them under every resonance condition.

4. *A structure with rigid floors and hinged base.*

The boundary conditions of this case are

$$x_n=-l_n;\ \frac{\partial y_n}{\partial x_n}=0\,,\ E_nI_n\frac{\partial^3 y_n}{\partial x_n^3}+m_n\frac{\partial^2 y_n}{\partial t^2}=0\,, \qquad (3')\ (4'')$$

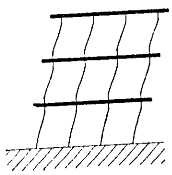


Fig. 9.

$$x_s=-l_s\,,\ x_{s+1}=0;\ y_s=y_{s+1}\,, \qquad (5'),$$

$$\frac{\partial y_s}{\partial x_s}=0\,,\ \frac{\partial y_{s+1}}{\partial x_{s+1}}=0\,,\ E_{s+1}I_{s+1}\frac{\partial^3 y_{s+1}}{\partial x_{s+1}^3}=E_sI_s\frac{\partial^3 y_s}{\partial x_s^3}+m_s\frac{\partial^2 y_s}{\partial t^2}\,,\ (6'),(7'),(8'')$$

$$(s=1,2,\cdots,\ n-1)$$

$$x_1=0;\quad y_{1_{x_1=0}}=u_{0_{x_1=0}}+(u_1+u_2)_{r=\varepsilon}\,,\quad \frac{\partial^2 y_1}{\partial x_{1_{x_1=0}}^2}=0\,, \qquad (9'),\ (40)$$

$$-E_1\pi\varepsilon^2 j^2\frac{\partial^3 y_1}{\partial x_{1_{x_1=0}}^3}=\mu\pi\varepsilon^2\frac{\partial u_0}{\partial x}_{x=0}$$

$$+\int_{\theta=0}^{\pi}\int_{\varphi=0}^{\pi}\mu\left(\frac{\partial(v_1+v_2)}{\partial r}-\frac{(v_1+v_2)}{r}+\frac{1}{r}\frac{\partial(u_1+u_2)}{\partial\theta}\right)r^2\sin^2\theta d\theta d\varphi_{r=\varepsilon}$$

$$+\int_{\theta=0}^{\pi}\int_{\varphi=0}^{\pi}\left(\lambda\Delta+2\mu\frac{\partial(u_1+u_2)}{\partial r}\right)r^2\sin\theta\cos\theta d\theta d\varphi_{r=\varepsilon}\,. \qquad (11')$$

The condition for ε is the same as in the preceding cases. The solutions of the problem corresponding to the incident waves

$$u_0 = \cos(pt + kx), \tag{12''}$$

are as follows:

(i) $n=1$,

$$y = \frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{2\sqrt{P^2+Q^2}}\left\{2(3-\gamma)-3\gamma\left(\frac{x}{l}\right)+\gamma\left(\frac{x}{l}\right)^3\right\}\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \tag{41}$$

where

$$P = 9\gamma\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{Ej^2}{\mu k l^3}\right),\quad Q = 4(3-\gamma)\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right), \tag{42}$$

$$\gamma = \frac{mp^2l^3}{EI}. \tag{43}$$

(ii) $n=2$,

$$y_1 = \frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{2\sqrt{P^2+Q^2}}\Bigg[2\{(3-\gamma_1)(12-\gamma_2)-12\eta\gamma_2\}$$

$$+\{\gamma_1(12-\gamma_2)+12\eta\gamma_2\}\left\{-3\left(\frac{x_1}{l_1}\right)+\left(\frac{x_1}{l_1}\right)^3\right\}\Bigg]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \tag{44}$$

$$y_2 = \frac{3\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{\sqrt{P^2+Q^2}}\Bigg\{(12-\gamma_2)$$

$$+3\gamma_2\left(\frac{x_2}{l_2}\right)^2+2\gamma_2\left(\frac{x_2}{l_2}\right)^3\Bigg\}\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \tag{45}$$

where
$$\left.\begin{aligned} P &= 9\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{E_1j^2}{\mu k l_1^3}\right)\left\{\gamma_1(12-\gamma_2)+12\eta\gamma_2\right\},\\ Q &= 4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)\left\{(3-\gamma_1)(12-\gamma_2)-12\eta\gamma_2\right\},\end{aligned}\right\} \tag{46}$$

$$\gamma_1 = \frac{m_1p^2l_1^3}{E_1I_1},\quad \gamma_2 = \frac{m_2p^2l_2^3}{E_2I_2},\quad \eta = \frac{E_2I_2l_1^3}{E_1I_1l_2^3}. \tag{47}$$

(iii) $n=3$,

$$y_1=\frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{2\sqrt{P^2+Q^2}}\Big\{2[(3-\gamma_1)\{12\zeta\gamma_3-(12-\gamma_2)(12-\gamma_3)\}$$
$$+12\eta\{12\zeta\gamma_3+\gamma_2(12-\gamma_3)\}]+[12\eta\{12\zeta\gamma_3+\gamma_2(12-\gamma_3)\}$$
$$-\gamma_1\{12\zeta\gamma_3-(12-\gamma_2)(12-\gamma_3)\}]\left\{3\left(\frac{x_1}{l_1}\right)-\left(\frac{x_1}{l_1}\right)^3\right\}\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \tag{48}$$

$$y_2=\frac{3\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{\sqrt{P^2+Q^2}}\Big[\{12\zeta\gamma_3-(12-\gamma_2)(12-\gamma_3)\}$$
$$-\{12\zeta\gamma_3+\gamma_2(12-\gamma_3)\}\left\{3\left(\frac{x_2}{l_2}\right)^2+2\left(\frac{x_2}{l_2}\right)^3\right\}\Big]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \tag{49}$$

$$y_3=\frac{-36\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{\sqrt{P^2+Q^2}}\Big[(12-\gamma_3)$$
$$+\gamma_3\left\{3\left(\frac{x_3}{l_3}\right)^2+2\left(\frac{x_3}{l_3}\right)^3\right\}\Big]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \tag{50}$$

where
$$\left.\begin{aligned}P&=9\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{E_1j^2}{\mu k l_1^3}\right)\Big[\gamma_1\{12\zeta\gamma_3-(12-\gamma_2)(12-\gamma_3)\}\\&\qquad-12\eta\{12\zeta\gamma_3+\gamma_2(12-\gamma_3)\}\Big],\\Q&=4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)\Big[(3-\gamma_1)\{12\zeta\gamma_3-(12-\gamma_2)(12-\gamma_3)\}\\&\qquad+12\eta\{12\zeta\gamma_3+\gamma_2(12-\gamma_3)\}\Big],\end{aligned}\right\} \tag{51}$$

$$\gamma_1=\frac{m_1p^2l_1^3}{E_1I_1},\quad \gamma_2=\frac{m_2p^2l_2^3}{E_2I_2},\quad \frac{m_3p^2l_3^3}{E_3I_3},\quad \xi=\frac{l_2}{l_1},\quad \eta=\frac{E_2I_2l_1^3}{E_1I_1l_2^3},\quad \zeta=\frac{E_3I_3l_2^3}{E_2I_2l_3^3}. \tag{52}$$

In the special case where $E_1=E_2=\cdots=E$, $I_1=I_2=\cdots=I$, $l_1=l_2=\cdots=l$, $m_1=m_2=\cdots=m$, we get

(i) $n=1$, omitted.

(ii) $n=2$,

$$y_1=\frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{2\sqrt{P^2+Q^2}}\Big[2(\gamma^2-27\gamma+36)$$
$$+\gamma(\gamma-24)\left\{3\left(\frac{x_1}{l}\right)-\left(\frac{x_1}{l}\right)^3\right\}\Big]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \tag{44'}$$

$$y_2=\frac{3\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{\sqrt{P^2+Q^2}}\Bigg[(12-\gamma)$$

$$+\gamma\left\{3\left(\frac{x_2}{l}\right)^2+2\left(\frac{x_2}{l}\right)^3\right\}\Bigg]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \quad (45')$$

where
$$\left.\begin{aligned}P&=9\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{Ej^2}{\mu kl^3}\right)\gamma(24-\gamma)\,,\\ Q&=4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)(\gamma^2-27\gamma+36)\,,\end{aligned}\right\} \quad (46')$$

$$\gamma=\frac{mp^2l^3}{EI}. \quad (47')$$

(iii) $n=3$,

$$y_1=\frac{\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{2\sqrt{P^2+Q^2}}\Bigg[2(\gamma^3-51\gamma^2+540\gamma-432)$$

$$+\gamma(12-\gamma)(36-\gamma)\left\{3\left(\frac{x_1}{l}\right)-\left(\frac{x_1}{l}\right)^3\right\}\Bigg]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \quad (48')$$

$$y_2=\frac{3\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{\sqrt{P^2+Q^2}}\Bigg[-(\gamma^2-36\gamma+144)$$

$$+\gamma(\gamma-24)\left\{3\left(\frac{x_2}{l}\right)^2+2\left(\frac{x_2}{l}\right)^3\right\}\Bigg]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \quad (49')$$

$$y_3=\frac{36\left(7\sqrt{\frac{\lambda}{\mu}+2}-4\right)}{\sqrt{P^2+Q^2}}\Bigg[(\gamma-12)$$

$$-\gamma\left\{3\left(\frac{x_3}{l}\right)^2+2\left(\frac{x_3}{l}\right)^3\right\}\Bigg]\cos\left(pt-\tan^{-1}\frac{P}{Q}\right), \quad (50')$$

where
$$\left.\begin{aligned}P&=9\sqrt{\frac{\lambda}{\mu}+2}\left(\frac{Ej^2}{\mu kl^3}\right)\gamma(\gamma-12)(36-\gamma)\,,\\ Q&=4\left(\sqrt{\frac{\lambda}{\mu}+2}-1\right)(\gamma^3-51\gamma^2+540\gamma-432)\,,\end{aligned}\right\} \quad (51')$$

$$\gamma=\frac{mp^2l^3}{EI}. \quad (52')$$

We have calculated the maximum values of the bending moments $E_sI_s(\partial^2 y/\partial x^2)$ for each end of the columns corresponding to the maximum values of acceleration of the ground free of any structures, for the special case $E_s=E$, $I_s=I$, $l_s=l$, $m_s=m$, $Ej^2/\mu k l^3=0{\cdot}05$, $\lambda/\mu=14$, and for various values of ml^3/EI. They are plotted in Figs. 10, 11, 12. The nature of the problem for frequencies out of resonance is again similar to that without dissipation, but the bending moments of the present case under resonance conditions take certain values between those under the corresponding conditions of the two cases, namely, the one for the structure with rigid floors and the other for the structure with flexible floors, in

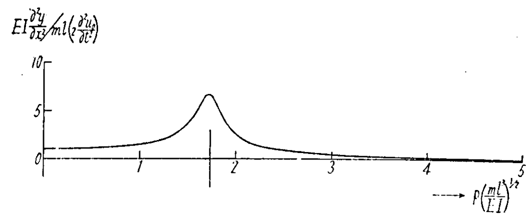


Fig. 10. The case of a singled-storied structure with rigid floor and hinged base.

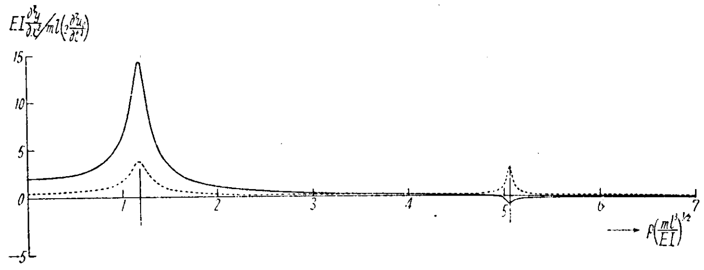


Fig. 11. The case of a two-storied structure with rigid floors and hinged base. Full line: lowest columns. Broken line: second columns.

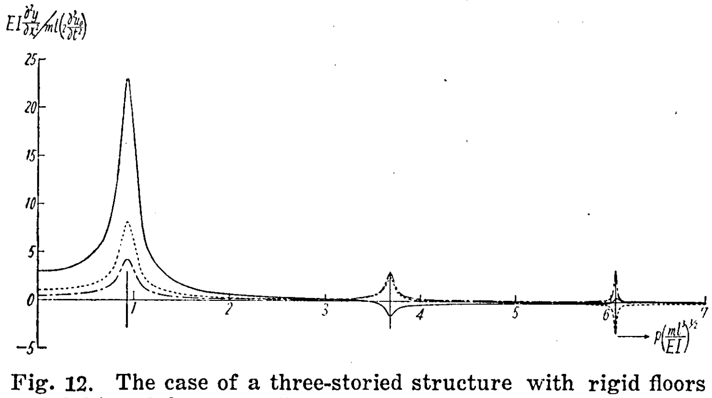


Fig. 12. The case of a three-storied structure with rigid floors and hinged base. Full line: lowest columns. Broken line: second columns. Chain: third columns.

both of which the structures are clamped at the base. Another feature, that can be confirmed from these figures, is that, although the moments at high frequencies are exceedingly small, the greatest one among such moments is not necessarily induced in the lowest columns. This nature, which may also be found even by the examination of preceding cases, appears to answer the question what part of a building would suffer a serious damage during earthquake movements.

5. *Concluding remarks.*

The solutions of the lateral deflection of the respective structures as here elaborated, may be applied to all cases with any value of $Ej^2/\mu kl^3$. The larger the ratio of $Ej^2/\mu kl^3$, the smaller become the amplitudes or bending moments at the periods corresponding to the resonance condition in the usual sense. Since $Ej^2/\mu kl^3$ is the product of E/μ, $(j/l)^2$, $L/2\pi l$, where L is the wave length of the incident waves in the earth, it follows that, the elasticity of the earth being given, the amplitudes or the bending moments under resonance conditions may be made smaller and smaller by increasing the elastic constants or the sectional area of the columns. Curious as it may seem, it is possible to get a similar effect by increasing the mass of the structure. To increase the mass of the structure, keeping the stiffness of columns at a constant value, is dynamically equivalent to increasing the fre- quency of seismic vibrations as revealed from the form of the parameter $p(ml^3/EI)^{1/2}$ that we used in every case. Even under the first resonance condition p may be made smaller in lieu of increa-s ing m, while $L/2\pi l$ becomes accordingly larger. This is analogous to increasing $Ej^2/\mu kl^3$ in the vicinity of the first resonance.

It may be remarked again that, by putting $Ej^2/\mu kl^3=0$ in any of the solutions given in this paper, we would have exactly the same solution as in that of the corresponding case shown in paper (B).

The items determining the feature of seismic vibrations of a structure, besides the nature of materials used in it, are now evident and may be enumerated as: (i) the acceleration of the ground on which no structure rests, (ii) periods of seismic vibrations, (iii) the form of the structure, (iv) the effective stiffness of the structure relative to that of the ground.

The nature of the dissipation of seismic vibrations in actual structures is now being studied, the results of which we hope to be able to publish in a forthcoming publication.[7]

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7) *loc. cit. ante.* in 6).

Our present investigation was greatly assisted by aid received, from the Foundation for the Promotion of Scientific and Industrial Research of Japan, to the Council of which we wish to express our sincere thanks.

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55. 架構構造物の震動に於ける勢力逸散

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彙報第 12 號の論文に於てその中の結果について言及して置いたやうに，共振の所の無限大の振幅は勢力の地下逸散によつて或一定の大さに限定されることが數理的にわかるのである．構造物は一層から三層までのものを夫々について (i) 基礎と床とが夫々固定の狀態のもの，(ii) 基礎が固定され，床が撓み易いもの，(iii) 基礎で鉸脚になり，床が固定のものに分けて考へ，且つ土地に種々の加速度及び週期の地震があるときの振動狀態を研究したものである．之等についても (a) 構造物がないときの土地の加速度，(b) 震動の週期，(c) 構造物の型，(d) 構造物の剛度と土地の剛度との關係，(e) 構造物の材料の性質によつて構造物の震動性はすべて決定できるのである．數量的の結果は本文中に委しく出てゐる．

構造物をできるだけ剛くする方がよいのは，靜力學的の强度の意味や，共振を避けることよりも，寧ろ共振狀態に於ける振幅や柱の屈曲モーメントが小さくなるといふことに於て意味があるのである．又，高次の共振の所程，振幅やモーメントが益々小さくなり，斯る場合の振幅やモーメントは共振を外れた所の振幅やモーメントよりも小さくなることが知られるのである．之等の事實は振動勢力の地下逸散を度外視しては到底得られないのである．

床の撓み方が中位の場合は別に計算をやつて見なければならぬけれども，實際問題としてそのやうな場合は非常に少いから今直ちにその計算をやつて見るには及ばないと考へられる．

尙，柱と土地との結合狀態について，これ迄構造學者がやつたやうに中間位の固定といふ假定などは，振動減衰上には大した意味のないことがわかる．

又，この論文中の結果に於て假りに勢力逸散性を除いて見ると，すべての解は彙報第 12 號に述べたものと夫々一致するのである．

この論文では解の方程式だけは一般にあふやうにやつてあるけれども，數値計算の例としては最も可能的な一二の場合についてのみである．それで今後は實際の建物の寸法について計算を試み，それによつて各建物がたとひ共振を起しても如何なるモーメントしか起り得ないかと云ふ事を推定して置きたい考である．この問題は，逸散のために建物の共振があまり出ないことを意味してゐるから，地震を待つてやり得る實驗の結果もそれ程はつきりした値が出ないことを暗示してゐる．しかし，只今のやうな方法で Prediction をやつて置く事は，其計算の Elements がやはり實驗室で正確に測つた値であるだけそれだけ，信頼して貰つてよいものと思ふ．卽ち，この計算は振動實驗では到底出し得ず而も物理的にも應用的にも大切な問題の解を與へるものである．