# 18. *Discontinuity in the Dispersion Curves of Rayleigh Waves.*[1]

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Some years ago, one of us in dealing with the problem[2] of the dispersion of Rayleigh waves transmitted along the surface of a stratified body, obtained the corresponding dispersion curves of waves of a few cases, such as in which the ratios of the rigidity of the stratum to that of the subjacent medium are 1/2, 1/3, 1/4, 1/5, the densities of both media being the same and the Poisson's condition of elastic constants being satisfied. The equation necessary for obtaining the velocity of transmission was expressed by a determinant of the form:

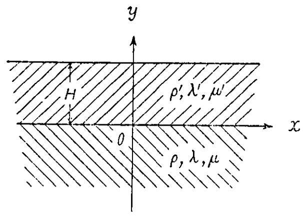


Fig. 1.

$$\begin{vmatrix} -\frac{(f^2-s'^2)}{k'^2}Y_2, & 2\frac{\mu'}{\mu}\frac{ifs'}{k'^2}Y_1, & 0, & -2\frac{\mu'}{\mu}\frac{ifs'}{k'^2}, & 0, & \frac{s'f}{k'^2} \\ -\frac{(f^2-s'^2)}{k'^2}Y_1, & -2\frac{\mu'}{\mu}\frac{ifs'}{k'^2}Y_2, & \frac{\mu'}{\mu}\frac{(f^2-s'^2)}{k'^2}, & 0, & \frac{if^2}{k'^2}, & 0 \\ 2\frac{ifr'}{h'^2}X_1, & \left(2\frac{\mu'}{\mu}\frac{r'^2}{h'^2}-\frac{\lambda'}{\mu}\right)X_2, & -\frac{\mu'}{\mu}\frac{2ifr'}{h'^2}, & 0, & \frac{r'f}{h'^2}, & 0 \\ 2\frac{ifr'}{h'^2}X_2, & \left(2\frac{\mu'}{\mu}\frac{r'^2}{h'^2}-\frac{\lambda'}{\mu}\right)X_1, & 0, & \left(\frac{\lambda'}{\mu}-2\frac{\mu'}{\mu}\frac{r'^2}{h'^2}\right), & 0, & -\frac{if^2}{h'^2} \\ 0, & 0, & -\frac{(f^2+s^2)}{k^2}, & 2\frac{ifs}{k^2}, & -\frac{if^2}{k^2}, & -\frac{sf}{k^2} \\ 0, & 0, & -\frac{2ifr}{h^2}, & \left(2\frac{r^2}{h^2}-\frac{\lambda}{\mu}\right), & -\frac{rf}{h^2}, & \frac{if^2}{h^2} \end{vmatrix}=0, \qquad (1)$$

where $X_1=\cosh r'H$, $X_2=\sinh r'H$, $Y_1=\cos s'H$, $Y_2=\sin s'H$.

1) Preliminary report published in the *Proc. Imp. Acad.*, **11** (1935), 13.
2) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **3** (1937), 1.

Our recent investigation shows that this reduces to

$$\frac{4r's'}{f^2}\left(2-\frac{k'^2}{f^2}\right)\eta-\frac{r's'}{f^2}\left\{4\vartheta+\left(2-\frac{k'^2}{f^2}\right)^2\zeta\right\}\cosh r'H\cos s'H$$

$$+\frac{r'}{f}\varphi\left\{\frac{4rs'^2}{f^3}+\frac{s}{f}\left(2-\frac{k'^2}{f^2}\right)^2\right\}\cosh r'H\sin s'H$$

$$+\frac{s'}{f}\varphi\left\{-\frac{4sr'^2}{f^3}+\frac{r}{f}\left(2-\frac{k'^2}{f^2}\right)^2\right\}\sinh r'H\cos s'H$$

$$+\left\{-\frac{4r'^2s'^2}{f^4}\zeta+\left(2-\frac{k'^2}{f^2}\right)^2\vartheta\right\}\sinh r'H\sin s'H=0, \qquad (2)$$

where

$$\left.\begin{aligned}
&\varphi=\frac{\mu'k^2k'^2}{\mu f^4},\quad \zeta=\frac{4rs}{f^2}\left(\frac{\mu'}{\mu}-1\right)^2-\alpha^2,\quad \eta=\frac{2rs}{f^2}\left(\frac{\mu'}{\mu}-1\right)\beta-\alpha\gamma,\\
&\vartheta=\frac{rs}{f^2}\beta^2-\gamma^2,\ \alpha=\frac{2\mu'}{\mu}-\left(2-\frac{k^2}{f^2}\right),\quad \beta=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-2,\\
&\gamma=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-\left(2-\frac{k^2}{f^2}\right),\\
&r^2=f^2-h^2,\quad s^2=f^2-k^2,\quad r'^2=f^2-h'^2,\quad s'^2=k'^2-f^2,\\
&h^2=\rho p^2/(\lambda+2\mu),\ h'^2=\rho'p^2/(\lambda'+2\mu'),\ k^2=\rho p^2/\mu,\ k'^2=\rho'p^2/\mu',
\end{aligned}\right\} \qquad (3)$$

in which $H$ is the thickness of the stratum, $2\pi/f$ the wave length, $\rho$, $\lambda$, $\mu$, $\rho'$, $\lambda'$, $\mu'$ the densities and elastic constants of the stratum and the subjacent medium respectively. By means of equation (2) we determined the dispersion curves of Rayleigh waves in a body having such small ratios of rigidities as 1/20 and $1/\infty$. We further reexamined the case in which the ratio is 1/5, the results of which are given in Table I and plotted in Figs. 2, 3, 4.

Table I. ($\rho=\rho'$, $\lambda=\mu$, $\lambda'=\mu'$.)

| $\mu/\mu'=5$ | | $\mu/\mu'=20$ | | | | $\mu/\mu'=\infty$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| Part CO′D | | Part AOB | | Part CO′D | | Part AOB | | Part COD | |
| $\frac{2\pi}{fH}$ | $\frac{p}{f}\sqrt{\frac{\rho}{\mu'}}$ | $\frac{2\pi}{fH}$ | $\frac{p}{f}\sqrt{\frac{\rho}{\mu'}}$ | $\frac{2\pi}{fH}$ | $\frac{p}{f}\sqrt{\frac{\rho}{\mu'}}$ | $\frac{2\pi}{fH}$ | $\frac{p}{f}\sqrt{\frac{\rho}{\mu'}}$ | $\frac{2\pi}{fH}$ | $\frac{p}{f}\sqrt{\frac{\rho}{\mu'}}$ |
| 2·36 | 1·645 | 3·227 | 1·4142 | 2·664 | 1·732 | 1·000 | 0·9325 | 2·525 | 1·732 |
| 3·32 | 1·732 | 4·170 | 1·732 | 3·88 | 1·871 | 1·333 | 0·958 | 3·880 | 1·898 |
| 4·060 | 1·816 | 4·60 | 1·871 | 4·72 | 2·000 | 2·000 | 1·078 | 4·000 | 1·914 |
| 4·432 | 1·871 | 5·14 | 2·000 | 5·544 | 2·236 | 2·657 | 1·260 | 4·619 | 2·000 |

*(to be continued.)*

Table I. (*continued.*)

| $\mu/\mu'=5$ | | $\mu/\mu'=20$ | | | | $\mu/\mu'=\infty$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| Part CO′D | | Part AOB | | Part CO′D | | Part AOB | | Part COD | |
| $\frac{2\pi}{fH}$ | $\frac{p}{f}\sqrt{\frac{\rho}{\mu'}}$ | $\frac{2\pi}{fH}$ | $\frac{p}{f}\sqrt{\frac{\rho}{\mu'}}$ | $\frac{2\pi}{fH}$ | $\frac{p}{f}\sqrt{\frac{\rho}{\mu'}}$ | $\frac{2\pi}{fH}$ | $\frac{p}{f}\sqrt{\frac{\rho}{\mu'}}$ | $\frac{2\pi}{fH}$ | $\frac{p}{f}\sqrt{\frac{\rho}{\mu'}}$ |
| 5·219 | 2·000 | 6·70 | 2·236 | 6·192 | 2·449 | 3·200 | 1·446 | 5·333 | 2·300 |
| 6·637 | 2·236 | 11·73 | 3·000 | 7·40 | 2·828 | 4·000 | 1·751 | 8·000 | 3·45 |
| Part AOB from prev. paper. | | 15·97 | 3·464 | 8·00 | 3·000 | 4·619 | 2·000 | 28·25 | 12·25 |
| | | 20·85 | 3·715 | 12·30 | 3·873 | 6·097 | 2·236 | | |
| | | 29·224 | 3·873 | 13·65 | 4·062 | 7·289 | 2·449 | | |
| | | 116·0 | 4·062 | 16·70 | 4·472 | 10·02 | 3·000 | | |
| | | | | | | 20·85 | 5·477 | | |
| | | | | | | 48·50 | 12·250 | | |

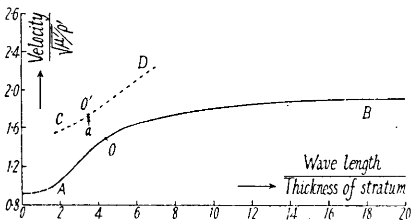


Fig. 2. $\frac{\mu'}{\mu}=\frac{1}{5}$, $\frac{\rho'}{\rho}=1$.

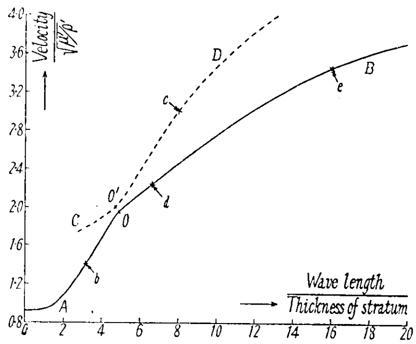


Fig. 3. $\frac{\mu'}{\mu}=\frac{1}{20}$, $\frac{\rho'}{\rho}=1$.

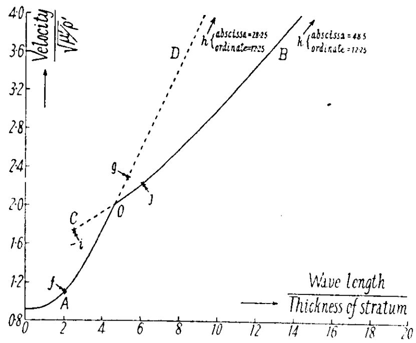


Fig. 4. $\frac{\mu'}{\mu}=\frac{1}{\infty}$, $\frac{\rho'}{\rho}=1$.

It was our belief until recently that only one dispersion curve, excepting those of waves having many horizontal nodal planes in the stratum, is possible in each case of a given ratio of rigidities, but our present calculation shows that there are two dispersion curves of different wave types, even when the ratio is 1/5. The curve AB in Fig. 2 is what we obtained in the previous study, whereas CD is another dispersion curve newly found as the result of a closer examination of the subject. As the ratio decreases, the two curves AB and CD approach each other and tend to bend at O and O′ respectively, while, if the ratio is 1/20, the two curves assume the forms indicated in Fig. 3, fairly sharp discontinuities of inclination appearing at O and O′ in the respective curves. Finally, if the ratio becomes $1/\infty$, part AO of curve AB joins part O′D of curve CD, while on the other hand another part OB of curve AB joins another part CO′ of curve CD, so that both the newly formed curves, AOD and COB, are now continuous as shown in Fig. 4.

Next, with a view to knowing, whether or not any discontinuity in wave types exists at points O and O′ of the dispersion curves, we examined the expressions for displacement[3]; its horizontal and vertical components within the stratum and in the subjacent medium being written by

$$
\begin{aligned}
u'_y = -\frac{s'}{f}\Big[ & -\Big(\frac{k'^2}{f^2}-2\Big)\eta\sin ir'(H-y) - \frac{2ir's'}{f^2}\eta\sin s'(H-y) \\
& + \frac{ir's}{f^2}\varphi\Big\{\Big(\frac{k'^2}{f^2}-2\Big)\cos ir'H\cos s'y + 2\cos s'H\cos ir'y\Big\} \\
& + \frac{ir's'}{f^2}\zeta\Big\{\Big(\frac{k'^2}{f^2}-2\Big)\cos ir'H\sin s'y + 2\sin s'H\cos ir'y\Big\} \\
& - \frac{rs'}{f^2}\varphi\Big\{2\sin s'H\sin ir'y + \Big(\frac{k'^2}{f^2}-2\Big)\sin ir'H\sin s'y\Big\} \\
& + \vartheta\Big\{\Big(\frac{k'^2}{f^2}-2\Big)\sin ir'H\cos s'y + 2\cos s'H\sin ir'y\Big\}\Big],
\end{aligned}
$$

$$
\begin{aligned}
v'_y = -i\Big[ & \frac{ir's'}{f^2}\eta\Big\{\Big(\frac{k'^2}{f^2}-2\Big)\cos ir'(H-y) - 2\cos s'(H-y)\Big\} \\
& + \frac{s'}{f}\varphi\Big\{\frac{r}{f}\Big(\frac{k'^2}{f^2}-2\Big)\sin ir'H\cos s'y + \frac{2r'^2 s}{f^3}\cos s'H\sin ir'y\Big\} \\
& + \Big\{\frac{2r'^2 s'^2}{f^4}\zeta\sin s'H\sin ir'y + \Big(\frac{k'^2}{f^2}-2\Big)\vartheta\sin ir'H\sin s'y\Big\} \\
& + \frac{ir's'}{f^2}\Big\{2\vartheta\cos s'H\cos ir'y - \Big(\frac{k'^2}{f^2}-2\Big)\zeta\cos ir'H\cos s'y\Big\} \\
& - \frac{ir'}{f}\varphi\Big\{\frac{2rs'^2}{f^3}\sin s'H\cos ir'y - \frac{s}{f}\Big(\frac{k'^2}{f^2}-2\Big)\cos ir'H\sin s'y\Big\}\Big], \quad (4)
\end{aligned}
$$

3) K. SEZAWA and K. KANAI, *Bull. Earthq. Res. Inst.*, **12** (1934), 641.

$$\left.\begin{aligned} u&=\frac{i\mu' s' k'^2}{\mu f^3}\left[Pe^{ry}+\frac{s}{f}Qe^{sy}\right],\\ v&=-\frac{\mu' s' k'^2}{\mu f^3}\left[\frac{r}{f}Pe^{ry}+Qe^{sy}\right],\end{aligned}\right\} \tag{5}$$

where

$$\left.\begin{aligned} P=&\frac{2sr'}{f^2}\beta\cos s'H+\frac{2r's'}{f^2}\alpha\sin s'H\\ &-\frac{2sr'}{f^2}\left(2-\frac{k'^2}{f^2}\right)\left(\frac{\mu'}{\mu}-1\right)\cosh r'H+\left(2-\frac{k'^2}{f^2}\right)\gamma\sinh r'H,\\ Q=&-\frac{2r'}{f}\gamma\cos s'H-4\frac{rr's'}{f^3}\left(\frac{\mu'}{\mu}-1\right)\sin s'H\\ &+\frac{r'}{f}\left(2-\frac{k'^2}{f^2}\right)\alpha\cosh r'H-\frac{r}{f}\left(2-\frac{k'^2}{f^2}\right)\beta\sinh r'H.\end{aligned}\right\} \tag{6}$$

The ratio of horizontal and vertical displacements at the surface obtained from these equations is plotted in Figs. 5, 6, 7. It will be seen that waves corresponding to part AO of one dispersion curve have generally a large vertical component of surface displacement, whereas those of part OB of the same dispersion curve

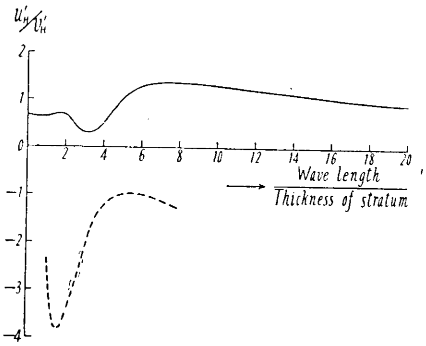


Fig. 5. $\frac{\mu'}{\mu}=\frac{1}{5}$, $\frac{\rho'}{\rho}=1$.

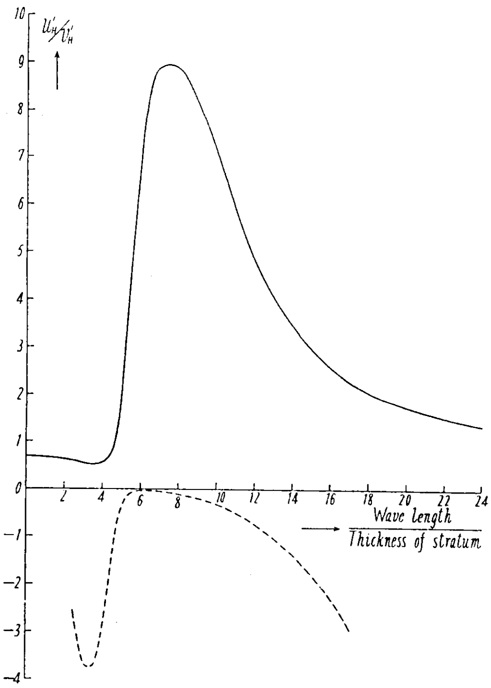


Fig. 6. $\frac{\mu'}{\mu}=\frac{1}{20}$, $\frac{\rho'}{\rho}=1$.

have a large horizontal component. The orbital motion of waves corresponding to AOB is in the same sense as those of the usual Rayleigh waves transmitted on the surface of a semi-infinite body. As regards waves corresponding to curve CO′D, particularly when the ratio of the rigidities of both media is finite, say 5 or 20, although hardly any discontinuous change in the ratio of horizontal to vertical displacements could be found at the point O′, the curve indicates that the ratio becomes stationary at the same point; in other words, the ratio does not change for a small variation of wave length in the immediate vicinity of that point. It is notable, however, that the orbital motion of waves corresponding to CO′D is in the sense opposite to that of the usual Rayleigh waves.

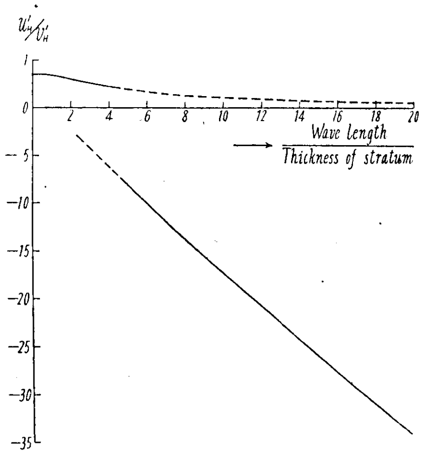


Fig. 7. $\frac{\mu'}{\mu}=\frac{1}{\infty}, \frac{\rho'}{\rho}=1.$

We also studied the distribution of displacements at different depths, the results of calculation of cases corresponding to points a, b, c, d, e, f, g, h, i, j, (indicated in Figs. 2, 3, 4) being shown in Figs. 8, 9, 10, 11, 12. These figures show that the distribution of displacements for waves corresponding to part AO is approximately similar to that for waves corresponding to part O′D, while on the other hand there is another striking similarity between the waves corresponding to part CO′ and the waves corresponding to part OB. Inasmuch as the distribution of displacements for waves of lengths corresponding to AO and O′D differs entirely from that for waves of lengths corresponding to CO′ and OB, it appears that a fairly discontinuous change in type of waves should exist at O or O′ on curve AB or CD. We thus arrive at the conclusion that the importance of points O and O′ consists not only in being discontinuities in dispersion curves, but also in causing a discontinuous change in types of waves of such values of wave lengths as correspond to those points.

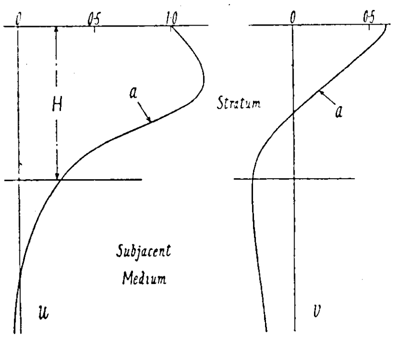


Fig. 8.

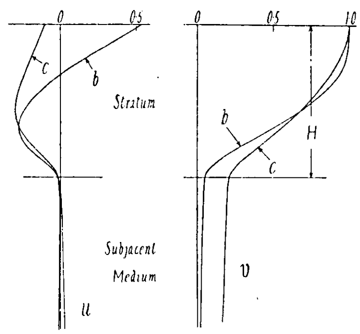


Fig. 9.

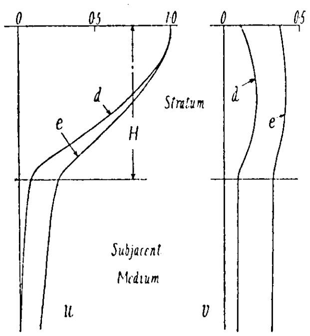


Fig. 10.

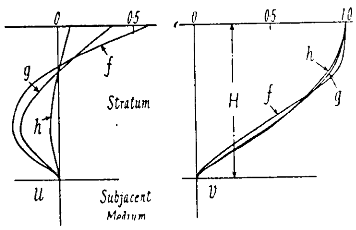


Fig. 11.

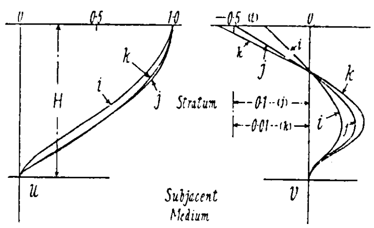


Fig. 12.

## 18. レーレー波の分散曲線に於ける不連續性

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地表層が一つあつてこの層の彈性と下層の彈性との比が 1/2 乃至 1/5 位の場合のレーレー波の分散問題は，我々の一人が 7～8 年前に研究した事がある．最近我々は上述の比が 1/20，1/∞ の場合を委しくしらべ，且つ 1/5 の場合をも再び吟味してみたのである．

地表層が一つある場合にレーレー波の分散曲線は一つしかないと最近まで信じられてをつたのであるが，只今の計算によると別の分散曲線の存在することがわかつた．第 2 圖の AB 曲線は以前に出したものであるが，CD は玆に新しく見出された別の分散曲線を示す．彈性の比が增加すると AB と CD とは接近して行き，彈性比が 1/20 になると第 3 圖の如くなり，且つ O と O′ とに各曲線の傾斜の不連續點が生ずるやうになる．彈性比が 1/∞ になると AB 曲線の AO の部分と CD 曲線の O′D とが結びつき，又 CD 曲線の CO′ と AB 曲線の OB とが連なり，結局第 4 圖に示すやうに新しく作られた AOD 曲線と COB 曲線とが夫々連續性の曲線となるのである．

次に上述の O 及 O′ に於て波型についても何か不連續的の事が存在するかどうか知る爲に振動變位の式を出してそれをしらべてみたのである．AO の部分に相當する波長の波では垂直變位が比較的に大きく，OB の部分に相當する波長の波では水平變位が比較的に大きいことがわかつた．AOB を通じて波動點の軌跡は半無限體を傳播するレーレー波の場合と同樣である．CO′D 曲線に相當する波については，彈性比が 5 とか 20 のやうに有限の場合には O′ に於て垂直變位又は水平變位について著しい不連續性が認められなかつたけれども，O′ に於て曲線が定常性を持つこと，換言すれば，O′ の極く近所で少しの波長變化に對して波動速度が不變になるといふことがわかつた．又，CO′D に相當する波動の表面變位は普通のレーレー波の場合とは逆の軌跡をとることが知られたのである．

我々は又固體中の種々の深さに於ける變位の分布を研究してみた．それによると，AO の部分に相當する場合の分布と O′D の部分に相當する場合のそれとはよく相似を保つてゐるし，CO′ の部分と OB の部分とでは，又別の相似が存在することがわかるのである．然るに，この兩種の分布狀態は著しく違つてゐるから，AOB 曲線と CO′D 曲線上の O と O′ とに於て，夫々分布の急激な變化のあることが知られるのである．果してそうであるとすると，O と O′ とは分散曲線に於て不連續性を與へるのみでなく，變位の分布狀態に於ても不連續性を引き起すといふ意味に於ても非常に重要性があるといふ結論に到達したわけである．