1. *Love-waves Generated from a Source of a certain Depth.*

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1. The problem of the generation of Love-waves from a seismic origin has attracted very little attention of investigators until recently and it appears that my previous paper[1] was the first attempt at finding the relation between the depth of the origin and the amplitudes of Love-waves which are generated therefrom. Even though that paper was correct in its result, the method of analysis used in it was not accurate from physical conception in lieu of the fact that I postulated in that paper stationary oscillations in the layer harmonizing with the excitation of shocks from the seismic origin. The present paper has been written with view to establish a theory of the generation of Love-waves which is free from such physical difficulty that was involved in the method in the previous paper and therefore adaptable to practical use with greater certainty.

In the present paper the effect of the free surface as well as the boundary between two media specializing the behaviour of waves in their multiple reflection and refraction at these boundaries was accurately taken into consideration. The present investigation, however, has given rise to results which are very like those of my previous calculation.[2] Among results which came out from my previous calculation the fact that Love-waves may be produced even from a source in the subjacent medium did not convince me in the light of geometrical consideration.[3] But, even from the present calculation it was found that the results of the problem are not much different for both cases at least in relation to the point in question.

2. In the first place the case where a point source lying within the stratum generates distortional waves with amplitudes oriented horizontally in the direction perpendicular to the incident plane will be dealt with, the problem being two-dimensional. Let the thickness of

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1) 2) K. SEZAWA, "Propagation of Love-waves on a Spherical Surface and Allied Problems", *Bull. Earthq. Res. Inst.*, 7 (1929), 437~455.

3) K. SEZAWA, *Publication du Bureau Central Séismologique International*, No. 10 (1934), 115.

the stratum and the depth of the seismic origin from the free surface be η and $\eta-\xi$ respectively and the densities and rigidities of the stratum and those of the subjacent medium ρ, μ, ρ', μ', the axes of x, y being taken as shown in the figure.

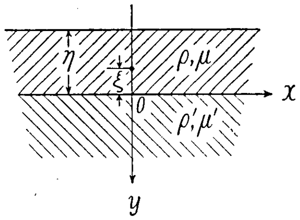

Fig. 1.

The displacement of waves generated from the origin is denoted by

$$v_0 = e^{-ipt} H_0^{(1)}(kr), \tag{1}$$

where $r^2 = x^2 + (y+\xi)^2$ and $k^2 = \rho p^2/\mu$. Now, it is possible to write

$$\left.\begin{aligned} H_0^{(1)}(kr) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{\alpha(y+\xi)+ifx}}{\alpha}\,df, \qquad [y<-\xi] \\ H_0^{(1)}(kr) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-\alpha(y+\xi)+ifx}}{\alpha}\,df, \qquad [y>-\xi] \end{aligned}\right\} \tag{2}$$

provided $\alpha = \sqrt{f^2-k^2}$. From (1) and (2) we have

$$v_0 = \frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty} \frac{e^{ifx+\alpha(y+\xi)}}{\alpha}\,df, \qquad [y<-\xi] \tag{3}$$

$$v_0 = \frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty} \frac{e^{ifx-\alpha(y+\xi)}}{\alpha}\,df. \qquad [y>-\xi] \tag{4}$$

The expression (3) gives the displacement of primary waves propagated upwards and (4) that of waves transmitted downwards. It will be shown that, if waves of the type:

$$V_0 = \phi(ifx + \alpha y - ipt) \tag{5}$$

are transmitted in the stratum and incident on the free surface, the reflected waves are written in the form:

$$V = \phi[ifx - \alpha(y+2\eta) - ipt], \tag{6}$$

and, if the waves of the type:

$$V_{01} = \phi_1(ifx - \alpha y - ipt) \tag{7}$$

are transmitted in the stratum and incident on the lower boundary of that stratum, the reflected waves V_1 as well as refracted waves V_1' are expressed by

$$\left.\begin{aligned} V_1 &= q\phi_1(ifx + \alpha y - ipt), \\ V_1' &= \tau\phi_1(ifx - \beta y - ipt), \end{aligned}\right\} \tag{8}$$

where

$$q = \frac{\mu\alpha - \mu'\beta}{\mu\alpha + \mu'\beta}, \qquad \tau = \frac{2\mu\alpha}{\mu\alpha + \mu'\beta}, \tag{9}$$

in which $\beta = \sqrt{f^2 - k'^2}$, $k'^2 = \rho' p^2/\mu'^2$. Thus, taking account of all kinds of waves successively reflected at two boundaries, we obtain the displacement of the stratum due to primary waves (3) as follows:

$$v=\frac{c^{-ipt}}{\pi}\int_{-\infty}^{\infty} e^{ifx+\alpha\xi}\left\{e^{\alpha y}+e^{-\alpha(y+2\eta)}\right\}\left\{1+qe^{-2\alpha\eta}+q^2e^{-4\alpha\eta}+\cdots\right\}\frac{df}{\alpha},$$

$$[y+\xi<0] \qquad (10)$$

$$v=\frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty} e^{ifx+\alpha\xi}\left\{qe^{\alpha(y-2\eta)}+e^{-\alpha(y+2\eta)}\right\}\left\{1+qe^{-2\alpha\eta}+q^2e^{-4\alpha\eta}+\cdots\right\}\frac{df}{\alpha}.$$

$$[y+\xi>0] \qquad (10')$$

Similarly, the displacement of the stratum due to primary waves (4) is expressed by

$$v=\frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty} e^{ifx-\alpha\xi}q\left\{e^{\alpha\eta}+e^{-\alpha(y+2\eta)}\right\}\left\{1+qe^{-2\alpha\eta}+q^2e^{-4\alpha\eta}+\cdots\right\}\frac{df}{\alpha},$$

$$[y+\xi<0] \qquad (11)$$

$$v=\frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty} e^{ifx-\alpha\xi}\left\{e^{-\alpha y}+qe^{\alpha y}\right\}\left\{1+qe^{-2\alpha\eta}+q^2e^{-4\alpha\eta}+\cdots\right\}\frac{df}{\alpha}.$$

$$[y+\xi>0] \qquad (11')$$

Summing up (10) and (11) and simplifying the resulting expression we have

$$v=\frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty}\frac{e^{ifx}\{(\mu\alpha+\mu'\beta)e^{\alpha\xi}+(\mu\alpha-\mu'\beta)^{-\alpha\xi}\}}{\alpha\{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)e^{-\alpha\eta}\}}\{e^{\alpha(\eta+y)}+e^{-\alpha(\eta+y)}\}df,$$

$$[y+\xi<0] \qquad (12)$$

$$v=\frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty}\frac{c^{ifx}\{(\mu\alpha+\mu'\beta)c^{-\alpha y}+(\mu\alpha-\mu'\beta)^{\alpha y}\}}{\alpha\{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)^{-\alpha\eta}\}}\{e^{\alpha(\eta-\xi)}+e^{-\alpha(\eta-\xi)}\}df.$$

$$[y+\xi>0] \qquad (12')$$

The resulting refracted waves in the subjacent material are expressed by

$$v'=\frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty}\frac{2\mu\, e^{ifx-\beta y}\{e^{\alpha(\eta-\xi)}+e^{-\alpha(\eta-\xi)}\}}{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)e^{-\alpha\eta}}df. \qquad [y>0] \qquad (13)$$

If we put $y=-\xi$, we find that v for $y+\xi<0$ is equal to v for $y+\xi>0$. If we put $y=0$, it follows that $v'=v$ for $y+\xi>0$.

If we put $y=-\eta$ in the expression of (12), we get the displacement at the free surface as follows:

$$v=\frac{2e^{-ipt}}{\pi}\int_{-\infty}^{\infty}\frac{e^{ifx}\{(\mu\alpha+\mu'\beta)e^{\alpha\xi}+(\mu\alpha-\mu'\beta)e^{-\alpha\xi}\}}{\alpha\{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)e^{-\alpha\eta}\}}df. \qquad (14)$$

This integral may be transformed to:

$$v=\frac{2e^{-ipt}}{\pi}\int_{0}^{\infty}\frac{(e^{ifx}+e^{-ifx})\{(\mu\alpha+\mu'\beta)e^{\alpha\xi}+(\mu\alpha-\mu'\beta)e^{-\alpha\xi}\}}{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)e^{-\alpha\eta}}\frac{df}{\alpha}. \qquad (14')$$

To evaluate this expression we consider two integrals:

$$\int_c \frac{e^{iZx}\chi(Z)}{F(Z)}\frac{dZ}{\sqrt{Z^2-k^2}}=0, \qquad (15)$$

$$\int_c \frac{e^{-iZx}\chi(Z)}{F(Z)}\frac{dZ}{\sqrt{Z^2-k^2}}=0, \tag{16}$$

in which

$$\chi(Z)=(\mu\sqrt{Z^2-k^2}+\mu'\sqrt{Z^2-k'^2})e^{\sqrt{Z^2-k^2}\xi}$$
$$+(\mu\sqrt{Z^2-k^2}-\mu'\sqrt{Z^2-k'^2})e^{-\sqrt{Z^2-k^2}\xi}, \tag{17}$$
$$F(Z)=(\mu\sqrt{Z^2-k^2}+\mu'\sqrt{Z^2-k'^2})e^{\sqrt{Z^2-k^2}\eta}$$
$$-(\mu\sqrt{Z^2-k^2}-\mu'\sqrt{Z^2-k'^2})e^{-\sqrt{Z^2-k^2}\eta}, \tag{18}$$

taken round contours in the first and fourth quadrants respectively. The integral (15) has branch points at $Z=k$ and $Z=k'$, and poles at $Z=\kappa_1, \kappa_2, \dots,$ where $\kappa_1, \kappa_2, \dots$ are roots of $F(Z)=0$. $F(\kappa_n)=0$ is the characteristic equation of Love-waves and gives the velocity of that kind of waves. The integral (16) has no branch point nor pole. The reason why these singular points reside in the first quadrant but not in the fourth comes from the assumption that there are temporarily small frictional terms in the original equations of motion as follows:

$$\left.\begin{aligned}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}&=\frac{\rho}{\mu}\left(\frac{\partial^2 v}{\partial t^2}+c\,\frac{\partial v}{\partial t}\right),\\ \frac{\partial^2 v'}{\partial x^2}+\frac{\partial^2 v'}{\partial y^2}&=\frac{\rho'}{\mu'}\left(\frac{\partial^2 v'}{\partial t^2}+c'\,\frac{\partial v'}{\partial t}\right),\end{aligned}\right\} \tag{19}$$

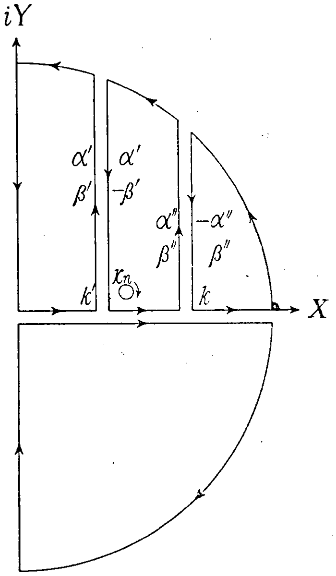

Fig. 2,

where c, c' are very small quantities and may be made zero finally. If the solutions of these equations be expressed by the types:

$$v=e^{-ipt+ifx\pm\alpha y}, \qquad v'=e^{-ipt+ifx\pm\beta y}, \qquad [x>0]$$

we have

$$\alpha=\sqrt{f^2-k^2(1+ik_1/k^2)}, \quad \beta=\sqrt{f^2-k'^2(1+ik'_1/k'^2)}, \tag{20}$$

where $k^2=\rho p^2/\mu^2$, $k'^2=\rho' p^2/\mu'^2$, $k_1=cp$, $k'_1=c'p$. It follows therefore that branch points come into the first quadrant. It is possible to prove that poles $\kappa_1, \kappa_2, \dots$ are in the same quadrant.

Now, the integral (15) may be expressed by

$$\int_0^\infty \frac{e^{ifx}\{(\mu\alpha+\mu'\beta)e^{\alpha\xi}+(\mu\alpha-\mu'\beta)e^{-\alpha\xi}\}}{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)e^{-\alpha\eta}}\frac{df}{\alpha}$$
$$+\int_\infty^0 \frac{e^{-Yx}\{(\mu\sqrt{Y^2+k^2}+\mu'\sqrt{Y^2+k'^2})e^{i\sqrt{Y^2+k^2}\xi}}{\{(\mu\sqrt{Y^2+k^2}+\mu'\sqrt{Y^2+k'^2})e^{i\sqrt{Y^2+k^2}\eta}}$$

$$\frac{+(\mu\sqrt{Y^2+k^2}-\mu'\sqrt{Y^2+k'^2})e^{-i\sqrt{Y^2+k^2}\xi}\}}{-(\mu\sqrt{Y^2+k^2}-\mu'\sqrt{Y^2+k'^2})e^{-i\sqrt{Y^2+k^2}\eta}\}}\frac{dY}{\sqrt{Y^2+k^2}}$$

$$-2\pi i\sum_n e^{i\kappa_n x}\frac{\{(\mu\alpha_1+\mu'\beta_1)e^{\alpha_1\xi}+(\mu\alpha_1-\mu'\beta_1)e^{-\alpha_1\xi}\}}{\alpha_1 F'(\kappa_n)}$$

$$+e^{ik'x}\int_0^\infty\Bigg[\frac{\mu\alpha'\cosh\alpha'\xi+\mu'\beta'\sinh\alpha'\xi}{\mu\alpha'\sinh\alpha'\eta+\mu'\beta'\cosh\alpha'\eta}$$

$$-\frac{\mu\alpha'\cosh\alpha'\xi-\mu'\beta'\sinh\alpha'\xi}{\mu\alpha'\sinh\alpha'\eta-\mu'\beta'\cosh\alpha'\eta}\Bigg]e^{-xY}\frac{idY}{\alpha'}=0, \qquad (21)$$

in which κ_n are roots of $F(Z)=0$ and $\alpha_1=\sqrt{\kappa_n^2-k^2}$, $\beta_1=\sqrt{\kappa_n^2-k'^2}$, while $\alpha'=\sqrt{(k'+iY)^2-k^2}$, $\beta'=\sqrt{(k'+iY)^2-k'^2}$. The integral along the circumference of a circular arc of infinite radius R vanishes in virtue of the factor $\exp.(ixR\cos\theta-xR\sin\theta)$ and of the relation $\eta>\xi$.

The integral (16) is written by

$$\int_0^\infty\frac{e^{ifx}\{(\mu\alpha+\mu'\beta)e^{\alpha\xi}+(\mu\alpha-\mu'\beta)e^{-\alpha\xi}\}}{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)e^{-\alpha\eta}}\frac{df}{\alpha}$$

$$-\int_\infty^0\frac{e^{-Yx}\{(\mu\sqrt{Y^2+k^2}+\mu'\sqrt{Y^2+k'^2})e^{i\sqrt{Y^2+k^2}\xi}}{\{(\mu\sqrt{Y^2+k^2}+\mu'\sqrt{Y^2+k'^2})e^{i\sqrt{Y^2+k^2}\eta}}$$

$$\frac{+(\mu\sqrt{Y^2+k^2}-\mu'\sqrt{Y^2+k'^2})e^{-i\sqrt{Y^2+k^2}\xi}\}}{-(\mu\sqrt{Y^2+k^2}-\mu'\sqrt{Y^2+k'^2})e^{-i\sqrt{Y^2+k^2}\eta}\}}\frac{dY}{\sqrt{Y^2+k^2}}=0. \qquad (22)$$

The integral along the circumference of a circular arc of infinite radius vanishes in virtue of the factor $\exp.(ixR\cos\theta+xR\sin\theta)$ and of the relation $\eta>\xi$.

Adding (21) and (22) and substituting the resulting expression in (14′), we arrive at

$$v=\sum_n 8ie^{i(\kappa_n x-pt)}\frac{\{\mu\sqrt{k^2-\kappa_n^2}\cos\sqrt{k^2-\kappa_n^2}\xi+\mu'\sqrt{\kappa_n^2-k'^2}\sin\sqrt{k^2-\kappa_n^2}\xi\}}{\sqrt{k^2-\kappa_n^2}F'(\kappa_n)}$$

$$-\frac{2ie^{i(k'x-pt)}}{\pi}\int_0^\infty\frac{\mu\mu'\beta'\cosh\alpha'(\eta-\xi)}{(\mu'\beta')^2\cosh^2\alpha'\eta-(\mu\alpha')^2\sinh^2\alpha'\eta}e^{-xY}dY. \qquad (23)$$

The important part of the integral term of the above expression is in the vicinity of small Y, so that we may put

$$\alpha'=i\sqrt{k^2-k'^2}, \qquad \beta'=\sqrt{2k'Y}\,e^{\frac{i\pi}{4}}, \qquad (24)$$

and therefore the integral in question becomes approximately

$$i^{\frac{1}{2}}\int_0^\infty\frac{\mu\mu'\cos\sqrt{k^2-k'^2}(\eta-\xi)}{i\mu'^2 2k'Y\cos^2\sqrt{k^2-k'^2}\eta-\mu^2(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}e^{-xY}\sqrt{2k'Y}\,dY. \qquad (25)$$

It is well known that

$$\int_0^\infty Y^{\frac{1}{2}}\psi(Y)e^{-Yx}dY=\frac{\Pi(1/2)}{x^{3/2}}\psi(0)+\frac{\Pi(3/2)}{x^{5/2}}\frac{\psi'(0)}{1!}+\frac{\Pi(5/2)}{x^{7/2}}\frac{\psi''(0)}{2!}+\cdots, \tag{26}$$

so that (25) reduces to

$$-i^{\frac{1}{2}}\sqrt{\frac{\pi k'}{2}}\frac{\mu'}{\mu}\frac{\cos\sqrt{k^2-k'^2}(\eta-\xi)}{(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\frac{1}{x^{3/2}}+\cdots. \tag{25'}$$

Substituting (25′) in (23), we arrive at

$$v=\sum_n 8e^{i\left(\kappa_n x-pt+\frac{\pi}{2}\right)}\frac{\{\mu\sqrt{k^2-\kappa_n^2}\cos\sqrt{k^2-\kappa_n^2}\xi+\mu'\sqrt{\kappa_n^2-k'^2}\sin\sqrt{k^2-\kappa_n^2}\xi\}}{\sqrt{k^2-\kappa_n^2}\,F'(\kappa_n)}$$
$$-\sqrt{\frac{2k'}{\pi}}e^{i\left(k'x-pt-\frac{\pi}{4}\right)}\frac{\mu'}{\mu}\frac{\cos\sqrt{k^2-k'^2}(\eta-\xi)}{(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\frac{1}{x^{3/2}}+\cdots. \quad [x>0] \tag{27}$$

The first term of the right-hand side of this equation represents waves transmitted with the velocity of Love-waves, while the second one gives distortional waves propagated with the velocity of transverse waves proper to the subjacent medium. The amplitudes of Love-waves thus formed is not strikingly different whether the seismic origin is deep or not. The amplitudes of pure distortional waves vary inversely as 3/2 th power of the epicentral distance. This is similar to the result of Lamb's problem[4] on Rayleigh-waves in which the amplitudes of dilatational or distortional waves accompanying Rayleigh-waves vary inversely as 3/2 th power of the epicentral distance in a two-dimensional problem and inversely as the square of the same distance in a three-dimensional one. Although similar results were given by a few authors[5] and also involved in some part of our paper[6], it is not certain that the fact is dynamically possible. Such law of decrease of amplitudes in accordance with epicentral distance is the mere result of the treatment of the integral of the type shown in (26) or analogous formulae with the condition that x is large. The true nature may be ascertained by investigating the distribution of displacements within the layer and the subjacent medium.

The distribution of displacements in the stratum is obtained as follows. Taking the expressions (12) and (12′) and considering the contour integrals (15) and (16) with the modification that

$$2\chi(Z)=(\mu\sqrt{Z^2-k^2}+\mu'\sqrt{Z^2-k'^2})\{e^{\sqrt{Z^2-k^2}(\xi+y+\eta)}+e^{\sqrt{Z^2-k^2}(\xi-y-\eta)}\}$$

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4) H. LAMB, *Phil. Trans. Roy. Soc., London*, **203** (1904), 1~42.

5) H. NAKANO, *Geophys. Mag.*, **2** (1930), 189~348; T. SAKAI, *Proc. Phys.-Math. Soc., Japan*, [iii], **15** (1933), 291~327; *Geophys. Mag.*, **3** (1934), 1~72.

6) K. SEZAWA and G. NISHIMURA, "Movement of the Ground due to Atmospheric Disturbance in a Sea Region," *Bull. Earthq. Res. Inst.*, **9** (1931), 291~309.

$$+(\mu\sqrt{Z^2-k^2}-\mu'\sqrt{Z^2-k'^2})\{e^{-\sqrt{Z^2-k^2}(\xi-y-\eta)}+e^{-\sqrt{Z^2-k^2}(\xi+y+\eta)}\},$$

$$[y+\xi<0] \qquad (28)$$

$$2\chi(Z)=(\mu\sqrt{Z^2-k^2}+\mu'\sqrt{Z^2-k'^2})\{e^{-\sqrt{Z^2-k^2}(y-\eta+\xi)}+e^{-\sqrt{Z^2-k^2}(y+\eta-\xi)}\}$$
$$+(\mu\sqrt{Z^2-k^2}-\mu'\sqrt{Z^2-k'^2})\{e^{\sqrt{Z^2-k^2}(y+\eta-\xi)}+e^{\sqrt{Z^2-k^2}(y-\eta+\xi)}\},$$

$$[y+\xi>0] \qquad (28')$$

and proceeding in the same manner of treatment as in the preceding calculation, we arrive at finally

$$v=\sum_n 8e^{i\left(\kappa_n x-pt+\frac{\pi}{2}\right)}\cdot$$
$$\cdot\frac{\{\mu\sqrt{k^2-\kappa_n^2}\cos\sqrt{k^2-\kappa_n^2}\,\xi+\mu'\sqrt{\kappa_n^2-k'^2}\sin\sqrt{k^2-\kappa_n^2}\,\xi\}\cos\sqrt{k^2-\kappa_n^2}(\eta+y)}{\sqrt{k^2-\kappa_n^2}\,F'(\kappa_n)}$$
$$-\sqrt{\frac{2k'}{\pi}}\,e^{i\left(k'x-pt-\frac{\pi}{4}\right)}\,\frac{\mu'}{\mu}\,\frac{\cos\sqrt{k^2-k'^2}(\eta-\xi)\cos\sqrt{k^2-k'^2}(\eta+y)}{(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\,\eta}\,\frac{1}{x^{3/2}}+\cdots,$$

$$[x>0,\ y+\xi<0] \qquad (29)$$

$$v=\sum_n 8e^{i\left(\kappa_n x-pt+\frac{\pi}{2}\right)}$$
$$\cdot\frac{\{\mu\sqrt{k^2-\kappa_n^2}\cos\sqrt{k^2-\kappa_n^2}\,y-\mu'\sqrt{\kappa_n^2-k'^2}\sin\sqrt{k^2-\kappa_n^2}\,y\}\cos\sqrt{k^2-\kappa_n^2}(\eta-\xi)}{\sqrt{k^2-\kappa_n^2}\,F'(\kappa_n)}$$
$$-\sqrt{\frac{2k'}{\pi}}\,e^{i\left(k'x-pt-\frac{\pi}{4}\right)}\,\frac{\mu'}{\mu}\,\frac{\cos\sqrt{k^2-k'^2}(\eta-\xi)\cos\sqrt{k^2-k'^2}(\eta+y)}{(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\,\eta}\,\frac{1}{x^{3/2}}+\cdots.$$

$$[x>0,\ y+\xi>0] \qquad (29')$$

The distribution of displacements in the subjacent medium will be considered. The integral (13) is transformed to

$$v'=\frac{2\mu e^{-ipt}}{\pi}\int_0^\infty\frac{(e^{ifx}+e^{-ifx})e^{-\beta y}\{e^{\alpha(\eta-\xi)}+e^{-\alpha(\eta-\xi)}\}}{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)e^{-\alpha\eta}}\,df. \qquad (30)$$

To evaluate this expression we consider two integrals of the types (15), (16), where

$$\chi(Z)=\{e^{\sqrt{Z^2-k^2}(\eta-\xi)}+e^{-\sqrt{Z^2-k^2}(\eta-\xi)}\}e^{-\sqrt{Z^2-k'^2}\,y} \qquad (31)$$

in the present case, taken round two contours shown in Fig. 2. The integral of the type (15) may be expressed by

$$\int_0^\infty\frac{e^{ifx}\{e^{\alpha(\eta-\xi)}+e^{-\alpha(\eta-\xi)}\}e^{-\beta y}}{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)e^{-\alpha\eta}}\,df$$
$$+\int_\infty^0\frac{e^{-Yx}\{e^{i\sqrt{Y^2+k^2}(\eta-\xi)}+e^{-i\sqrt{Y^2+k^2}(\eta-\xi)}\}e^{-i\sqrt{Y^2+k'^2}\,y}\,dY}{(\mu\sqrt{Y^2+x^2}+\mu'\sqrt{Y^2+k'^2})e^{i\sqrt{Y^2+k^2}(\eta-\xi)}-(\mu\sqrt{Y^2+k^2}+\mu'\sqrt{Y^2+k'^2})e^{-i\sqrt{Y^2+k^2}(\eta-\xi)}}$$

$$-2\pi i\sum_n \frac{e^{i\kappa_n x}\{e^{\alpha_1(\eta-\xi)}+e^{-\alpha_1(\eta-\xi)}\}e^{-\beta_1 y}}{F'(\kappa_n)}$$

$$+e^{ik'x}\int_0^\infty\Big[\frac{\cosh\alpha'(\eta-\xi)e^{-\beta' y}}{\mu\alpha'\sinh\alpha'\eta+\mu'\beta'\cosh\alpha'\eta}$$

$$-\frac{\cosh\alpha'(\eta-\xi)e^{\beta' y}}{\mu\alpha'\sinh\alpha'\eta-\mu'\beta'\cosh\alpha'\eta}\Big]e^{-xY}idY=0, \qquad (32)$$

in which κ_n, α_1, β_1, α', β' have the same meanings as those in (21). The integral of the type (16) becomes

$$\int_0^\infty \frac{e^{-ifx}\{e^{\alpha(\eta-\xi)}+e^{-\alpha(\eta-\xi)}\}e^{-\beta y}}{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)e^{-\alpha\eta}}df$$

$$-\int_{-\infty}^0 \frac{e^{-Yx}\{e^{i\sqrt{Y^2+k^2}(\eta-\xi)}}{(\mu\sqrt{Y^2+k^2}+\mu'\sqrt{Y^2+k'^2})e^{i\sqrt{Y^2+k^2}(\eta-\xi)}}$$

$$\frac{+e^{-i\sqrt{Y^2+k^2}(\eta-\xi)}\}e^{-i\sqrt{Y^2+k'^2}y}dY}{-(\mu\sqrt{Y^2+k^2}+\mu'\sqrt{Y^2+k'^2})e^{-i\sqrt{Y^2+k^2}(\eta-\xi)}}=0. \qquad (33)$$

Adding (32) and (33) and remembering (30), we get

$$v'=\sum_n 8\mu i e^{i(\kappa_n x-pt)}\frac{\cos\sqrt{k^2-\kappa_n^2}(\eta-\xi)e^{-\sqrt{\kappa_n^2-k'^2}y}}{F'(\kappa_n)}$$

$$-\frac{2\mu i e^{i(k'x-pt)}}{\pi}\int_0^\infty \frac{\{\mu\alpha'\sinh\alpha'\eta\sinh\beta'\eta}{(\mu'\beta')^2\cosh^2\alpha'\eta}$$

$$\frac{+\mu'\beta'\cosh\alpha'\eta\cosh\beta' y\}\cosh\alpha'(\eta-\xi)}{-(\mu\alpha')^2\sinh^2\alpha'\eta}e^{-xY}dY. \qquad (34)$$

To get the approximate integral of the second term of this expression we use the relation (24) under the condition that x is large and y is small. Then the integral reduces to

$$i^{\frac{1}{2}}\int_0^\infty \frac{(\mu'\cos\sqrt{k^2-k'^2}\eta}{i\mu'^2 2k'Y\cos^2\sqrt{k^2-k'^2}\eta}$$

$$\frac{-\mu\sqrt{k^2-k'^2}y\sin\sqrt{k^2-k'^2}\eta)\cos\sqrt{k^2-k'^2}(\eta-\xi)}{-\mu^2(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}e^{-xY}\sqrt{2k'Y}\,dY. \qquad (35)$$

This becomes in virtue of (26)

$$i^{\frac{1}{2}}\sqrt{\frac{\pi k'}{2}}\frac{(\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}y\sin\sqrt{k^2-k'^2}\eta)\cos\sqrt{k^2-k'^2}\eta}{-\mu^2(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\frac{1}{x^{3/2}}+\cdots\cdot. \qquad (35')$$

Substituting (35′) in (34) we arrive at

$$v'=\sum_n 8e^{i(\kappa_n x-pt+\frac{\pi}{2})}\frac{\mu\cos\sqrt{k^2-\kappa_n^2}(\eta-\xi)e^{-\sqrt{\kappa_n^2-k'^2}y}}{F'(\kappa_n)}$$

$$-\sqrt{\frac{2k'}{\pi}}e^{i(k'x-pt-\frac{\pi}{4})}.$$

$$\cdot\frac{(\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}y\sin\sqrt{k^2-k'^2}\eta)\cos\sqrt{k^2-k'^2}(\eta-\xi)}{\mu(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\frac{1}{x^{3/2}}+\cdots\cdot \qquad [x>0] \qquad (36)$$

It is a suitable occasion to compare the expression (27), (29), (29′), (36) giving displacements v at the surface, v for $y+\xi<0$, v for $y+\xi>0$, and v' in some particular cases of coordinates. Let us denote v at the surface by $v_{0l}+v_{0s}$, v at $y+\xi<0$ by $v_{1l}+v_{1s}$, v at $y+\xi>0$ by $v_{2l}+v_{2s}$, where v_{0l}, v_{1l}, v_{2l} correspond to displacements of Love-waves and v_{0s}, v_{1s}, v_{2s} those of pure distortional waves. Similarly, v_l' and v_s' are to be displacements of Love-waves and pure distortional waves at $y>0$. The comparison between these displacements gives us

$$y=-\eta\,; \quad v_{0l}=v_{1l}, \quad v_{0s}=v_{1s}, \quad \frac{\partial v_{0l}}{\partial y}=\frac{\partial v_{1l}}{\partial y}, \quad \frac{\partial v_{0s}}{\partial y}=\frac{\partial v_{1s}}{\partial y}\,;$$

$$y=-\xi\,; \quad v_{1l}=v_{2l}, \quad v_{1s}=v_{2s}, \quad \frac{\partial v_{1l}}{\partial y}=\frac{\partial v_{2l}}{\partial y}, \quad \frac{\partial v_{1s}}{\partial y}=\frac{\partial v_{2s}}{\partial y}\,;$$

$$y=0\,; \quad v_{2l}=v_l', \quad v_{2s}=v_s', \quad \mu\frac{\partial v_{2l}}{\partial y}=\mu'\frac{\partial v_l'}{\partial y}, \quad \mu\frac{\partial v_{2s}}{\partial y}=\mu'\frac{\partial v_s'}{\partial y}.$$

It should be borne in mind that the condition, $y=-\xi$; $\partial v_{1l}/\partial y=\partial v_{2l}/\partial y$, is obtained by the relation that

$$\frac{\mu'\sqrt{\kappa_n^2-k'^2}}{\mu\sqrt{k^2-\kappa_n^2}}=\frac{\sin\sqrt{k^2-\kappa_n^2}\,\eta}{\cos\sqrt{k^2-\kappa_n^2}\,\eta},$$

which is the characteristic equation of Love-waves to determine κ_n.

It will be confirmed from each first term of the expressions (29), (29′), (36) that the displacements of the motion with the velocity of transmission of pure distortional waves are not of the type of bodily waves but resembles superficial waves with amplitude which is maximum at the free surface and decreases in the interior of the body at least within the stratum. This is apparently curious in the case of bodily waves. It is still impossible to explain the fact that the amplitudes of the same waves decrease inversely as 3/2 th power of the epicentral distance even at a range where the energy of transmission of Love-waves becomes stationary. The equation (36) indicates that the bodily waves of v' having the velocity of distortional waves does not satisfy the equation of motion, so that it is not certain that the waves in question exist even from mathematical point of view. The similar uncertainty seems to exist in bodily waves accompanying Rayleigh-waves.[7)]

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7) H. Lamb, H. Nakano, T. Sakai, K. Sezawa, G. Nishimura, *loc. cit.* in foot-notes 4), 5), 6).

3. The case where a point source generating distortional waves lies within the subjacent medium will now be studied. Let the depth of the seismic origin from the free surface be $\eta+\xi$. The displacement of the waves generated from the origin is expressed by

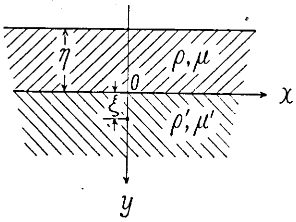

Fig. 3.

$$v_0=e^{-ipt}H_0^{(1)}(k'r), \tag{37}$$

where $r^2=x^2+(y-\xi)^2$ and $k'^2=\rho'p^2/\mu'$. It is possible to write

$$\left.\begin{aligned} H_0^{(1)}(k'r)&=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{\beta(y-\xi)+ifx}}{\beta}\,df, \qquad [y<\xi] \\ H_0^{(1)}(k'r)&=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-\beta(y-\xi)+ifx}}{\beta}\,df, \qquad [y>\xi] \end{aligned}\right\} \tag{38}$$

provided $\beta=\sqrt{f^2-k'^2}$. From (1) and (2) we find

$$v_0=\frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty}\frac{e^{ifx+\beta(y-\xi)}}{\beta}\,df, \qquad [y<\xi] \tag{39}$$

$$v_0=\frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty}\frac{e^{ifx+\beta(y-\xi)}}{\beta}\,df. \qquad [y>\xi] \tag{40}$$

The expression (39) gives the displacement of primary wave propagated upwards and (40) that of waves transmitted downwards. It is shown that, if waves of the type:

$$V_0=\phi(ifx+\beta y-ipt) \tag{41}$$

are transmitted in the lower medium and incident on the boundary surface between the lower medium and the stratum, the reflected waves V from this surface and the refracted waves V_1 in the stratum are expressed by

$$\left.\begin{aligned} V&=q'\phi(ifx-\beta y-ipt), \\ V_1&=\tau'\phi(ify+\alpha y-ipt), \end{aligned}\right\} \tag{42}$$

where

$$q'=\frac{\mu'\beta-\mu\alpha}{\mu'\beta+\mu\alpha}, \qquad \tau'=\frac{2\mu'\beta}{\mu'\beta+\mu\alpha}, \tag{43}$$

in which $\alpha=\sqrt{f^2-k^2}$, $k^2=\rho p^2/\mu$. The refracted waves in the stratum make multiple reflection and refraction at two boundaries of the stratum in the similar manner as those in the preceding section. Thus, the resulting displacement of the stratum becomes

$$v=\frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty}\frac{\tau'}{\beta}e^{ifx-\beta\xi}\{e^{\alpha y}+e^{-\alpha(y+2\eta)}\}\{1+qe^{-2\alpha\eta}+q^2e^{-4\alpha\eta}+\cdots\}\,df, \tag{44}$$

where $q=(\mu\alpha-\mu'\beta)/(\mu\alpha+\mu'\beta)$. Rewriting this we have

$$v=\frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty}\frac{2\mu' e^{ifx-\beta\xi}\{e^{\alpha(\eta+y)}+e^{-\alpha(\eta+y)}\}}{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)e^{-\alpha\eta}}df. \tag{45}$$

Similarly the displacement in the lower medium is expressed by

$$v'=\frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty}\frac{e^{ifx-\beta\xi}}{\beta}\left[\frac{(\mu\alpha+\mu'\beta)\{e^{\alpha\eta+\beta y}+e^{-(\alpha\eta+\beta y)}\}-(\mu\alpha-\mu'\beta)\{e^{\alpha\eta-\beta y}+e^{-(\alpha\eta-\beta y)}\}}{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)e^{-\alpha\eta}}\right]df, \quad [y<\xi] \tag{46}$$

$$v'=\frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty}\frac{e^{ifx-\beta y}}{\beta}\left[\frac{(\mu\alpha+\mu'\beta)\{e^{\alpha\eta+\beta\xi}+e^{-(\alpha\eta+\beta\xi)}\}-(\mu\alpha-\mu'\beta)\{e^{\alpha\eta-\beta\xi}+e^{-(\alpha\eta-\beta\xi)}\}}{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)e^{-\alpha\eta}}\right]df. \quad [y>\xi] \tag{46'}$$

The expression (45) has a form similar to that of (13), so that the integral of (45) is readly written by

$$\begin{aligned}v=&\sum_n 8e^{i\left(\kappa_n x-pt+\frac{\pi}{2}\right)}\frac{\mu'\cos\sqrt{k^2-\kappa_n^2}(\eta-y)e^{-\sqrt{\kappa_n^2-k'^2}\,\xi}}{F'(\kappa_n)}\\&-\sqrt{\frac{2k'}{\pi}}\,e^{i\left(k'x-pt-\frac{\pi}{4}\right)}\frac{\mu'}{\mu}\cdot\\&\cdot\frac{(\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}\sin\sqrt{k^2-k'^2}\eta)\cos\sqrt{k^2-k'^2}(\eta+y)}{\mu(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\frac{1}{x^{3/2}}+\cdots\cdot\end{aligned}$$

$$[x>0] \tag{47}$$

When we put $y=-\eta$ in this expression, we get the displacement at the surface as follows:

$$\begin{aligned}v=&\sum_n 8e^{i\left(\kappa_n x-pt+\frac{\pi}{2}\right)}\frac{\mu' e^{-\sqrt{\kappa_n^2-k'^2}\,\xi}}{F'(\kappa_n)}\\&-\sqrt{\frac{2k'}{\pi}}\,e^{i\left(k'x-pt-\frac{\pi}{4}\right)}\frac{\mu'}{\mu}\frac{(\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}\xi\sin\sqrt{k^2-k'^2}\eta)}{\mu(k^2-k'^2)\sin^2\sqrt{k^3-k'^2}\eta}\frac{1}{x^{3/2}}+\cdots\cdot\end{aligned}$$

$$[x>0] \tag{48}$$

The expression (46) may be transformed to

$$\begin{aligned}v'=\frac{e^{-ipt}}{\pi}\int_{-\infty}^{\infty}e^{ifx-\beta\xi}\Bigg[&\frac{\mu\alpha}{\beta}\frac{\{e^{\alpha\eta+\beta y}+e^{-(\alpha\eta+\beta y)}-e^{\alpha\eta-\beta y}-e^{-(\alpha\eta-\beta y)}\}}{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)e^{-\alpha\eta}}\\&+\frac{\mu'\{e^{\alpha\eta+\beta y}+e^{-(\alpha\eta+\beta y)}+e^{\alpha\eta-\beta y}+e^{-(\alpha\eta-\beta y)}\}}{(\mu\alpha+\mu'\beta)e^{\alpha\eta}-(\mu\alpha-\mu'\beta)e^{-\alpha\eta}}\Bigg]df. \quad [y<\xi]\end{aligned} \tag{49}$$

To evaluate this expression we consider two integrals (15) and (16), where

$$\begin{aligned}\chi(Z)=2\mu e^{-\sqrt{Z^2-k'^2}}\frac{\sqrt{Z^2-k^2}}{\sqrt{Z^2-k'^2}}\{&\cosh(\sqrt{Z^2-k^2}\eta+\sqrt{Z^2-k'^2}y)\\&-\cosh(\sqrt{Z^2-k^2}\eta-\sqrt{Z^2-k'^2}y)\}\end{aligned}$$

$$+2\mu' e^{-\sqrt{Z^2-k'^2}}\{\cosh(\sqrt{Z^2-k^2}\eta+\sqrt{Z^2-k'^2}y)$$
$$+\cosh(\sqrt{Z^2-k^2}\eta-\sqrt{Z^2-k'^2}y)\}, \qquad (50)$$

taken round contours in Fig. 2. Proceeding in the same manner as in preceding cases, we find

$$v'=\sum_n 8e^{i(\kappa_n x-pt+\frac{\pi}{2})}\frac{\mu'\cos\sqrt{k^2-\kappa_n^2}\eta\,(\cos\sqrt{\kappa_n^2-k'^2}y-\sin\sqrt{\kappa_n^2-k'^2}y)e^{-\sqrt{\kappa_n^2-k'^2}\xi}}{F'(\kappa_n)}$$
$$-\sqrt{\frac{2k'}{\pi}}e^{i(k'x-pt-\frac{\pi}{4})}\cdot$$
$$\cdot\left\{\frac{\mu'}{\mu}\frac{(\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}\xi\sin\sqrt{k^2-k'^2}\eta)\cos\sqrt{k^2-k'^2}\eta}{\mu(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\right.$$
$$\left.-\frac{\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}\xi\sin\sqrt{k^2-k'^2}\eta}{\mu\sqrt{k^2-k'^2}\sin\sqrt{k^2-k'^2}\eta}y\right\}\frac{1}{x^{3/2}}+\cdots.$$
$$[x>0,\ y<\xi] \qquad (51)$$

In the same manner of mathematical treatment we obtain the displacement for $y>\xi$ as follows:

$$v'=\sum_n 8e^{i(\kappa_n x-pt+\frac{\pi}{2})}\frac{\mu'\cos\sqrt{k^2-\kappa_n^2}\eta\,(\cos\sqrt{\kappa_n^2-k'^2}\xi-\sin\sqrt{\kappa_n^2-k'^2}\xi)e^{-\sqrt{\kappa_n^2-k'^2}\eta}}{F'(\kappa_n)}$$
$$-\sqrt{\frac{2k'}{\pi}}e^{i(k'x-pt-\frac{\pi}{4})}\left\{\frac{\mu'}{\mu}\frac{(\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}y\sin\sqrt{k_2-k'^2}\eta)\cos\sqrt{k^2-k'^2}\eta}{\mu(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\right.$$
$$\left.-\frac{\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}y\sin\sqrt{k^2-k'^2}\eta}{\mu\sqrt{k^2-k'^2}\sin\sqrt{k^2-k'^2}\eta}\xi\right\}\frac{1}{x^{3/2}}+\cdots.$$
$$[x>0,\ y>\xi] \qquad (51')$$

Comparison between (47), (51), (51′) indicates the existence of the relations:

$$y=-\eta;\quad \frac{\partial v_{0l}}{\partial y}=0,\quad \frac{\partial v_{0s}}{\partial y}=0;$$
$$y=0;\quad v_l=v_{1l}',\quad v_s=v_{1s}',\quad \mu\frac{\partial v_l}{\partial y}=\mu'\frac{\partial v_{1l}'}{\partial y},\quad \mu\frac{\partial v_s}{\partial y}=\mu'\frac{\partial v_{1s}'}{\partial y},$$
$$y=\xi;\quad v_{1l}'=v_{2l}',\ v_{1s}'=v_{2s}',\quad \frac{\partial v_{1l}'}{\partial y}=\frac{\partial v_{2l}'}{\partial y},\quad \frac{\partial v_{1s}'}{\partial y}=\frac{\partial v_{2s}'}{\partial y},$$

where v_{1l}', v_{2l}', v_{1s}', v_{2s}' denote displacements due to Love-waves and pure bodily waves for $y<\xi$ and $y>\xi$ respectively.

4. When the seismic origin comes to a point on the boundary surface between the stratum and the subjacent medium, it is able to put $\xi=0$. Then, from (29), (29′), (36) we get

$$v=\sum_n 8e^{i(\kappa_n x-pt+\frac{\pi}{2})}\frac{\mu\cos\sqrt{k^2-\kappa_n^2}(\eta+y)}{F'(\kappa_n)}$$

$$-\sqrt{\frac{2k'}{\pi}}\,e^{i\left(k'x-pt-\frac{\pi}{4}\right)}\,\frac{\mu'}{\mu}\,\frac{\cos\sqrt{k^2-k'^2}\eta\,\cos\sqrt{k^2-k'^2}(\eta+y)}{(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\,\frac{1}{x^{3/2}}+\cdots,$$

$$[x>0,\ y<0] \qquad (52)$$

$$v'=\sum_n 8e^{i\left(\kappa_n x-pt+\frac{\pi}{2}\right)}\,\frac{\mu\cos\sqrt{k^2-k'^2}\eta\; e^{-\sqrt{\kappa_n^2-\kappa'^2}y}}{F'(\kappa_n)}$$

$$-\sqrt{\frac{2k'}{\pi}}\,e^{i\left(k'x-pt+\frac{\pi}{4}\right)}\cdot$$

$$\cdot\frac{(\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}y\sin\sqrt{k^2-k'^2}\eta)\cos\sqrt{k^2-k'^2}\eta}{\mu(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\,\frac{1}{x^{3/2}}+\cdots.$$

$$[x>0,\ y>0] \qquad (53)$$

From (47), (51), (51′) we find

$$v=\frac{\mu'}{\mu}\sum_n 8e^{i\left(\kappa_n x-pt+\frac{\pi}{2}\right)}\,\frac{\mu\cos\sqrt{k^2-\kappa_n^2}(\eta+y)}{F'(\kappa_n)}$$

$$-\frac{\mu'}{\mu}\sqrt{\frac{2k'}{\pi}}\,e^{i\left(\kappa'x-pt-\frac{\pi}{4}\right)}\,\frac{\mu'}{\mu}\,\frac{\cos\sqrt{k^2-k'^2}\eta\,\cos\sqrt{k^2-k'^2}(\eta+y)}{(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\,\frac{1}{x^{3/2}}+\cdots,$$

$$[x>0,\ y<0] \qquad (54)$$

$$v'=\frac{\mu'}{\mu}\sum_n 8e^{i\left(\kappa_n x-pt+\frac{\pi}{2}\right)}\,\frac{\mu\cos\sqrt{k^2-\kappa_n^2}\eta\; e^{-\sqrt{\kappa_n^2-k'^2}y}}{F'(\kappa_n)}$$

$$-\frac{\mu'}{\mu}\sqrt{\frac{2k'}{\pi}}\,e^{i\left(k'x-pt-\frac{\pi}{4}\right)}\cdot$$

$$\cdot\frac{(\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}y\sin\sqrt{k^2-k'^2}\eta)\cos\sqrt{k^2-k'^2}\eta}{\mu(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\,\frac{1}{x^{3/2}}+\cdots.$$

$$[x>0,\ y>0] \qquad (55)$$

It will be seen from (52), (53), (54), (55) that v, v' due to seismic disturbance from the source in the subjacent medium are μ'/μ times those due to the one from the origin in the stratum. As the movements at the origin have been assumed equal for both cases, the stresses due to the disturbance of the former case should be μ'/μ times those due to the one of the latter case.

5. We are now to solve a three-dimensional problem where a point source lying within the stratum generates distortional waves with amplitudes oriented horizontally perpendicular to the direction of propagation. Let the axes of r, z be taken as shown in the sketch. The displacement of waves generated from the origin is written by

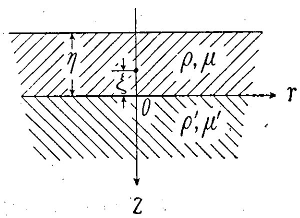

Fig. 4.

$$v_0 = e^{-ipt}\frac{\partial}{\partial r}\frac{e^{ikR}}{R}, \tag{56}$$

where $R^2 = r^2 + (z+\xi)^2$ and $k^2 = \rho p^2/\mu$. It is possible to write

$$\left.\begin{aligned} \frac{e^{ikR}}{R} &= \int_0^\infty \frac{e^{\alpha(z+\xi)}}{\alpha} J_0(fr) f\,df, \qquad [z < -\xi] \\ \frac{e^{ikR}}{R} &= \int_0^\infty \frac{e^{-\alpha(z+\xi)}}{\alpha} J_0(fr) f\,df, \qquad [z > -\xi] \end{aligned}\right\} \tag{57}$$

provided $\alpha = \sqrt{f^2 - k^2}$. From (56) and (57) we obtain

$$v_0 = e^{-ipt}\int_0^\infty \frac{e^{\alpha(z+\xi)}}{\alpha}\frac{\partial J_0(fr)}{\partial r} f\,df, \qquad [z < -\xi] \tag{58}$$

$$v_0 = e^{-ipt}\int_0^\infty \frac{e^{-\alpha(z+\xi)}}{\alpha}\frac{\partial J_0(fr)}{\partial r} f\,df. \qquad [z > -\xi] \tag{59}$$

Now, we know that

$$\begin{aligned} J_0(fr) &= \frac{2}{\pi}\int_0^\infty \sin(fr\cosh j)\,dj \\ &= \frac{1}{i\pi}\int_0^\infty e^{ifr\cosh j}dj - \frac{1}{i\pi}\int_0^\infty e^{-ifr\cosh j}\,dj, \end{aligned} \tag{60}$$

so that

$$v_0 = \frac{\partial}{\partial r}\frac{e^{-ipt}}{\pi i}\int_0^\infty dj\int_{-\infty}^\infty \frac{e^{\alpha(z+\xi)}}{\alpha} e^{ifr\cosh j} f\,df, \qquad [z < -\xi] \tag{61}$$

$$v_0 = \frac{\partial}{\partial r}\frac{e^{-ipt}}{\pi i}\int_0^\infty dj\int_{-\infty}^\infty \frac{e^{-\alpha(z+\xi)}}{\alpha} e^{ifr\cosh j} f\,df. \qquad [z > -\xi] \tag{62}$$

These equations are equal to the respective products of (3) as well as (4) by the factor

$$\frac{\partial}{\partial r}\frac{1}{i}\int_0^\infty dj\cdot f, \tag{63}$$

and therefore the solutions of all waves propagated in the stratum and the subjacent medium are readily obtained by putting $f\cdot df$ in place of df in (3) and (4) and afterwards by multiplying the resulting solutions by $\frac{\partial}{\partial r}\frac{1}{i}\int_0^\infty dj$. In the evaluation of integral use has been made of

$$\left.\begin{aligned} &\frac{1}{i}\int_0^\infty e^{i\xi\cosh j}\,dj = \frac{\pi}{2}H_0^{(1)}(\xi), \\ &\frac{1}{i}\int_0^\infty \frac{e^{i\xi\cosh j}}{(\cosh j)^{3/2}}\,dj = e^{-i\left(\xi+\frac{3\pi}{2}\right)}\sqrt{\frac{\pi}{2\xi}}\left(1 + \frac{7}{4}\,i\,\frac{1}{2\xi} - \frac{15}{8}\frac{1.3}{4\xi^2} + \cdots\right). \end{aligned}\right\} \tag{64}$$

Thus, we get finally

$$v = \sum_n 4\pi\kappa_n \frac{\partial H_0^{(1)}(\kappa_n r)}{\partial r}\, e^{-i\left(pt - \frac{\pi}{2}\right)}.$$

$$\cdot\frac{\{\mu\sqrt{k^2-\kappa_n^2}\cos\sqrt{k^2-\kappa_n^2}\xi+\mu'\sqrt{\kappa_n^2-k^2}\sin\sqrt{k^2-\kappa_n^2}\xi\}\cos\sqrt{k^2-\kappa_n^2}(\eta+z)}{\sqrt{k^2-\kappa_n^2}\,F'(\kappa_n)}$$

$$+k'^2e^{i\left(k'r-pt+\frac{\pi}{2}\right)}\,\frac{\mu'}{\mu}\,\frac{\cos\sqrt{k^2-k'^2}(\eta-\xi)\cos\sqrt{k^2-k'^2}(\eta+z)}{(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\,\frac{1}{r^2}+\cdots,$$

$$[y+z<0] \qquad (65)$$

$$v=\sum_n 4\pi\kappa_n\frac{\partial H_0^{(1)}(\kappa_n r)}{\partial r}\,e^{-i\left(pt-\frac{\pi}{2}\right)}\,.$$

$$\cdot\frac{\{\mu\sqrt{k^2-\kappa_n^2}\cos\sqrt{k^2-\kappa_n^2}z-\mu'\sqrt{\kappa_n^2-k'^2}\sin\sqrt{k^2-\kappa_n^2}z\}\cos\sqrt{k^2-\kappa_n^2}(\eta-\xi)}{\sqrt{k^2-\kappa_n^2}\,F'(\kappa_n)}$$

$$+k'^2e^{i\left(k'r-pt+\frac{\pi}{2}\right)}\,\frac{\mu'}{\mu}\,\frac{\cos\sqrt{k^2-k'^2}(\eta-\xi)\cos\sqrt{k^2-k'^2}(\eta+z)}{(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\,\frac{1}{r^2}+\cdots,$$

$$[y+z>0] \qquad (65')$$

$$v'=\sum_n 4\pi\kappa_n\frac{\partial H_0^{(1)}(\kappa_n r)}{\partial r}\,e^{-i\left(pt-\frac{\pi}{2}\right)}\,\frac{\mu\sqrt{k^2-\kappa_n^2}\cos\sqrt{k^2-\kappa_n^2}(\eta-\xi)\,e^{-\sqrt{\kappa_n^2-k'^2}z}}{\sqrt{k^2-\kappa_n^2}\,F'(\kappa_n)}$$

$$+k'^2e^{i\left(k'r-pt+\frac{\pi}{2}\right)}\frac{(\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}z\sin\sqrt{k^2-k'^2}\eta)\cos\sqrt{k^2-k'^2}(\eta-\xi)}{\mu(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\,\frac{1}{r^2}$$

$$+\cdots. \qquad (66)$$

The first term of the right-hand side of each expression of (65), (65′), (66) represents Love-waves, while the second one gives pure distortional waves transmitted with the velocity proper to the subjacent medium. The amplitudes of Love-waves vary inversely as the square root of the epicentral distance, while those of bodily waves as the inverse square of the same distance.

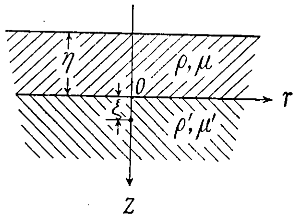

Fig. 5.

6. In a similar manner it is possible to determine the three-dimensional problem of wave motion due to a seismic origin residing in the lower medium. Thus, we obtain

$$v=\sum_n 4\pi\kappa_n\frac{\partial H_0^{(1)}(\kappa_n r)}{\partial r}\,e^{-i\left(pt-\frac{\pi}{2}\right)}\,\frac{\mu'\cos\sqrt{k^2-\kappa_n^2}(\eta+z)e^{-\sqrt{\kappa_n^2-k'^2}\xi}}{F'(\kappa_n)}$$

$$+k'^2e^{-i\left(k'r-pt+\frac{\pi}{2}\right)}\,\frac{\mu'}{\mu}.$$

$$\cdot\frac{(\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}\xi\sin\sqrt{k^2-k'^2}\eta)\cos\sqrt{k^2-k'^2}(\eta+z)}{\mu(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\,\frac{1}{r^2}$$

$$+\cdots, \qquad (67)$$

$$v'=\sum_n 4\pi\kappa_n\frac{\partial H_0^{(1)}(\kappa_n r)}{\partial r}\,e^{-i\left(pt-\frac{\pi}{2}\right)}.$$

$$\cdot\frac{\mu'\cos\sqrt{k^2-\kappa_n^2}\eta\,(\cos\sqrt{\kappa_n^2-k'^2}z-\sin\sqrt{\kappa_n^2-k'^2}z)e^{-\sqrt{\kappa_n^2-k'^2}\xi}}{F'(\kappa_n)}$$

$$+k'^2e^{i\left(k'r-pt+\frac{\pi}{2}\right)}\left\{\frac{\mu'}{\mu}\frac{(\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}\xi\sin\sqrt{k^2-k'^2}\eta)\cos\sqrt{k^2-k'^2}\eta}{\mu(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\right.$$

$$\left.-\frac{\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}\xi\sin\sqrt{k^2-k'^2}\eta}{\mu\sqrt{k^2-k'^2}\sin\sqrt{k^2-k'^2}\eta}z\right\}\frac{1}{r^2}+\cdots\cdots,$$

$$[z<\xi] \qquad (68)$$

$$v'=\sum_n 4\pi\kappa_n\frac{\partial H_0^{(1)}(\kappa_n r)}{\partial r}e^{-i\left(pt-\frac{\pi}{2}\right)}.$$

$$\cdot\frac{\mu'\cos\sqrt{k^2-\kappa_n^2}\eta(\cos\sqrt{\kappa_n^2-k'^2}\xi-\sin\sqrt{\kappa_n^2-k'^2}\xi)e^{-\sqrt{\kappa_n^2-k'^2}z}}{F'(\kappa_n)}$$

$$+k'^2e^{i\left(k'r-pt+\frac{\pi}{2}\right)}\left\{\frac{\mu'}{\mu}\frac{(\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}z\sin\sqrt{k^2-k'^2}\eta)\cos\sqrt{k^2-k'^2}\eta}{\mu(k^2-k'^2)\sin^2\sqrt{k^2-k'^2}\eta}\right.$$

$$\left.-\frac{\mu'\cos\sqrt{k^2-k'^2}\eta-\mu\sqrt{k^2-k'^2}z\sin\sqrt{k^2-k'^2}\eta}{\mu\sqrt{k^2-k'^2}\sin\sqrt{k^2-k'^2}\eta}\xi\right\}\frac{1}{r^2}+\cdots\cdots.$$

$$[z>\xi] \qquad (68')$$

The nature of the problem is the same as that of the two-dimensional one. It is a very difficult question why the energy of bodily waves involved in an annular area decreases with epicentral distance in spite of the existence of the conservation of energy of Love-waves which are progressing outwards. The part of v' corresponding to bodily waves indicates clearly that such waves are not possible to exist owing to the fact that the expression does not satisfy the equation of motion.

7. Although the general results were too well explained in respective parts of the calculation to warrant summarizing here, it is not without significance to pick up some important ones.

As already mentioned, there are two kinds of waves that are generated from a seismic origin having horizontal displacements. One of them corresponds to Love-type waves and the other pure distortional waves having the proper velocity peculiar to the lower medium. The displacement of Love-type waves varies as a certain negative exponential function of the depth when the origin is in the lower medium, but it varies as some harmonic function of the depth when the origin resides in the stratum. The distribution of displacements of pure distortional waves in the body is somewhat similar to that of surface waves, but the variation of amplitudes according to the depth of the origin is different; even when the depth of the seismic origin is very large, the diminution of displacement at the surface is not remarkable. The law of decrease of displacements of this type of waves with the increase of the epicentral distance is the same as that of bodily waves accompanying Rayleigh-waves; the displacement decreases inversely as 3/2th power of the epicentral distance in the two-dimensional problem and as the square of the same distance in the three-dimensional one, but this encounters contradiction with the principle of the conservation of energy of waves owing to the fact that the energy of surface waves is not changed in progressing outwards and also that the distribution of amplitudes of bodily waves for different depth is the same at any epicentral distance. Moreover, the expressions of displacements corresponding to these bodily waves do not satisfy the equations of motion. If this be the case, we are not yet now in position to explain the smallness of amplitudes of bodily waves from the basis of the present result; another paper[8] appears to confirm the true nature of the amplitudes of this kind of waves.

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1. 深さのある震源から發生するラブ波

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表層がある半無限固體中に水平變位だけの震動原點があるとき，それによつて生ずる横波及びラブ波の性質をしらべてみた．波動は波の反射及び屈折を正確に考へて算出したものである．原點が表面層中にあるときは勿論ラブ波を發生するけれども，原點が下層の中にある場合でも僅かであるがラブ波の出ることがわかつた．このラブ波に伴つて遠い震央距離に現れる横波は計算上では震央距離の二乘に反比例して振幅が小さくなることになる．これは常識でわかるやうに距離の一乘に反比例することと矛盾してゐるやうに思はれたので，固體中のすべての深さに於ける振幅の解を出して見たところがその解は彈性體の振動方程式を滿足しないことがわかつた．これは恐らく計算中に現れる函數論の分岐點の應用に不充分な所があつた爲と思はれる．果してそうであるとするとラムがその中年に出したレーレー波に伴ふ固體波の問題にも多少の疑がないとはいはれない．固體波の振幅の問題は次に現れる論文のやうな行方で出るものと思ふ．固體波の問題にはこのやうな疑があるけれどもラブ波については何等の問題はないと思ふ．筆者は數年前に同じ問題を別の解法で出したことがあるが，その結果は只今の結果とよく一致してゐるのである．

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8) K. SEZAWA and K. KANAI, "Periods and Amplitudes of Oscillations in L- and M-phases," *Bull. Earthq. Res. Inst.*, 13 (1935), 18.