21. *On the Propagation of Waves along a Surface Stratum of the Earth.*

By Katsutada SEZAWA and Kiyoshi KANAI,

Earthquake Research Institute.

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Several years ago one[1] of us studied the problem of propagation of Rayleigh-waves on the surface of a stratified body, and found the relation between wave-velocity and ratio of wave-length to thickness of a layer for different elastic constants of the media. The investigation of a special case, however, where the lower medium is extremely rigid, was not given in spite of its importance in relation to the free oscillations of a soil. It has recently occurred to us that the last case is more significant from a point of view of the velocity of transmission of waves in the stratum. It may be questioned that waves in the stratum are still dispersive even if the bottom medium is immovable owing to its perfectly rigid property. The following calculation may throw some light on such a question and will furthermore give us some knowledge on the forms of vibrations of the stratum for different wave lengths.

The equations of motion of the elastic medium of the stratum are as usually written in forms:

$$\left.\begin{aligned}\rho\frac{\partial^2\Delta}{\partial t^2}&=(\lambda+2\mu)\left(\frac{\partial^2\Delta}{\partial x^2}+\frac{\partial^2\Delta}{\partial y^2}\right),\\ \rho\frac{\partial^2\varpi}{\partial t^2}&=\mu\left(\frac{\partial^2\varpi}{\partial x^2}+\frac{\partial^2\varpi}{\partial y^2}\right),\end{aligned}\right\}\cdots\cdots(1)$$

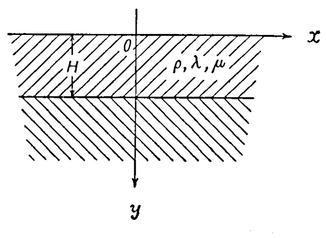

Fig. 1.

where $\Delta, \varpi, \rho, \lambda, \mu$ have the same meanings as what we have usually employed and the axes of x, y are taken as shown in the sketch. The solutions of these equations are expressed by

$$\left.\begin{aligned}\Delta&=(C\cosh ry+D\sinh ry)\,e^{i(pt-fx)},\\ 2\varpi&=(E\cosh sy+F\sinh sy)\,e^{i(pt-fx)},\end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots(2)$$

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1) K. SEZAWA, *Bull. Earthq. Res. Inst.*, 3 (1927), 1.

$$\left.\begin{aligned} u_1 &= \frac{if}{h^2}(C\cosh ry + D\sinh ry)\,e^{i(pt-fx)}, \\ v_1 &= -\frac{r}{h^2}(C\sinh ry + D\cosh ry)\,e^{i(pt-fx)}, \end{aligned}\right\} \quad \ldots\ldots\ldots (3)$$

$$\left.\begin{aligned} u_2 &= \frac{s}{k^2}(E\sinh sy + F\cosh sy)\,e^{i(pt-fx)}, \\ v_2 &= \frac{if}{k^2}(E\cosh sy + F\sinh sy)\,e^{i(pt-fx)}, \end{aligned}\right\} \quad \ldots\ldots\ldots (4)$$

where
$$\Delta = \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}, \quad 2\varpi = \frac{\partial v_2}{\partial x} - \frac{\partial u_2}{\partial y}, \quad \ldots\ldots\ldots (5)$$

$$h^2 = \frac{\rho p^2}{\lambda + 2\mu}, \quad k^2 = \frac{\rho p^2}{\mu}, \quad r^2 = f^2 - h^2, \quad s^2 = f^2 - k^2. \quad \ldots (6)$$

It may be noted that the boundary conditions of the problem are expressed by

$$y=0; \quad \lambda\Delta + 2\mu\frac{\partial}{\partial y}(v_1 + v_2) = 0, \quad \frac{\partial}{\partial y}(u_1 + u_2) + \frac{\partial}{\partial x}(v_1 + v_2) = 0, \quad (7)$$

$$y=H; \quad u_1 + u_2 = 0, \quad v_1 + v_2 = 0, \quad \ldots\ldots \quad \ldots\ldots (8)$$

even though what kinds of waves, namely dilatational or distortional or other waves, may be transmitted along the stratum under the restriction that the movements take place in the plane of xy.

Substituting (2), (3), (4) in (7), (8), we get

$$\left.\begin{aligned} -\frac{f^2+s^2}{2h^2}C + \frac{ifs}{k^2}F &= 0, \\ \frac{2ifr}{h^2}D + \frac{f^2+s^2}{k^2}E &= 0, \\ \frac{if}{h^2}(C\cosh rH + D\sinh rH) + \frac{s}{k^2}(E\sinh sH + F\cosh sH) &= 0, \\ -\frac{r}{h^2}(C\sinh rH + D\cosh rH) + \frac{if}{k^2}(E\cosh sH + F\sinh sH) &= 0, \end{aligned}\right\} \quad (9)$$

Eliminating C, D, E, F between four equations of (9), we obtain

$$\frac{\cosh rH - \dfrac{f^2+s^2}{2f^2}\cosh sH}{\sinh rH - \dfrac{f^2+s^3}{2rs}\sinh sH} = \frac{\sinh rH - \dfrac{2rs}{f^2+s^2}\sinh sH}{\cosh rH - \dfrac{2f^2}{f^2+s^2}\cosh sH}. \quad (10)$$

Solving this equation for the case, $\lambda = \mu$, we find the relation between L/H, where $L = 2\pi/f$ = wave length, and the velocity of propagation of waves along the stratum, the result being shown in the annexed figure and Table I. For the sake of comparison we have added the similar relations when the lower medium is not extremely rigid.

From this result it will be seen that the waves transmitted along a stratum are dispersive. The greater the ratio of L/H, the more increases the velocity of propagation tending to take an infinitely great value when L/H becomes infinite. There is little movement in the lower medium, so that the waves may be considered as propagated only in the stratum, but not in the lower medium, and therefore the fact, that the transmission is greatly affected by the existence of the lower medium, is apparently curious. The thought, however, that the movements of the stratum are infinitely great in comparison with those of lower medium and energy of waves is transmitted in the stratum as well as in the lower medium, enables us to know the origin of the dispersive nature of waves.

There are no pure dilatational waves travelling along a stratum. We have only waves of the type as indicated above. As the actual velocity of transmission of disper-

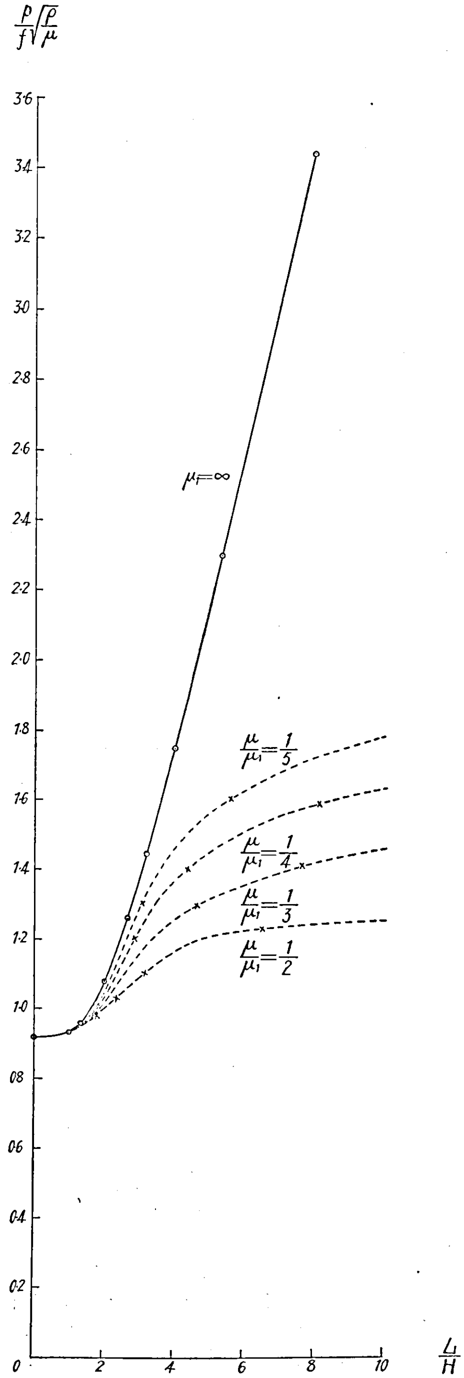


Fig. 2. $(\lambda = \mu)$.

sive waves is represented by the group velocity, some of waves that are usually assumed to be bodily waves in a stratum may in some cases belong to waves of the present type as propagated with group

Table I.

L/H	0	1	1·333	2	2·667	3·2	4	5·333	8
$\frac{p}{f}\sqrt{\frac{\rho}{\mu}}$	0·9194	0·9325	0·958	1·078	1·26	1·446	1·751	2·3	3·45

velocities peculiar to wave-lengths. In the present case the shortest possible waves take a group velocity which is equivalent to that of Rayleigh-waves and longer waves take group velocities of less magnitudes.[2)]

The ratio of horizontal to vertical components of displacement of a point on the free surface is of a great importance. Such ratios corresponding to many cases of Rayleigh-waves on a stratified layer were dealt with by Dr. Suzuki.[3)] We have too calculated the similar ones for the present problem. The ratio of u to v on the free surface is denoted by

$$\frac{u}{v}=\frac{u_1+u_2}{v_1+v_2}=\frac{\frac{if}{h^2}C+\frac{s}{k^2}F}{-\frac{r}{h^2}D+\frac{if}{k^2}E}. \qquad \text{(11)}$$

From (9)

$$\left.\begin{aligned} D&=-\left\{\frac{\cosh rH-\frac{f^2+s^2}{2f^2}\cosh sH}{\sinh rH-\frac{2rs}{f^2+s^2}\sinh sH}\right\}C,\\ E&=\frac{2ik^2fr}{h^2(f^2+s^2)}\left\{\frac{\cosh rH-\frac{f^2+s^2}{2f^2}\cosh sH}{\sinh rH-\frac{2rs}{f^2+s^2}\sinh sH}\right\}C,\\ F&=-\frac{ik^2(f^2+s^2)}{2h^2fs}C. \end{aligned}\right\} \qquad \text{(12)}$$

2) The criterion that waves of infinite length in the present problem should not be propagated with infinite velocity but with a certain velocity corresponding to group velocity was suggested by Professor Terada.

3) T. SUZUKI, *Bull. Earthq. Res. Inst.*, 11 (1933), 187.

Substituting (12) in (11) we find

$$\frac{u}{v} = -\frac{i(f^2+s^2)\left\{\sinh rH - \dfrac{2rs}{f^2+s^2}\sinh sH\right\}}{2fr\left\{\cosh rH - \dfrac{f^2+s^2}{2f^2}\cosh sH\right\}}. \qquad \dots\dots (13)$$

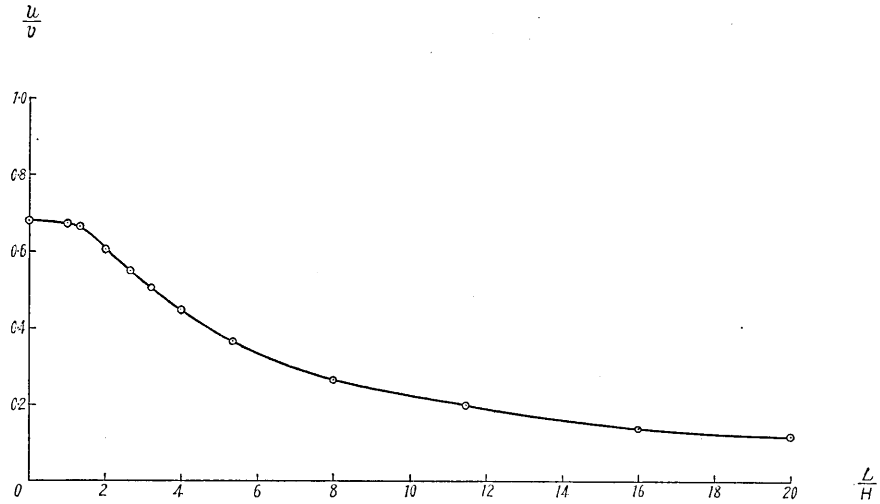


Fig. 3.

The result is shown in Table II and Fig. 3. For shortest possible waves the ratio is the same as that for Rayleigh-waves, while for longer waves the ratio becomes smaller and smaller, the horizontal component tending finally to take zero value in the limiting case.

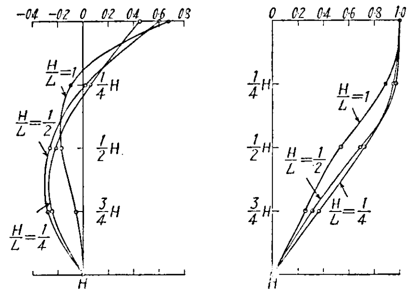
Table II.

L/H	0	1	1·333	2	2·667	3·2	4	5·333	8
u/v	0·681	0·6725	0·6635	0·605	0·5525	0·507	0·449	0·366	0·265

The distribution of displacements in the stratum, when $\lambda=\mu$, is calculated from

$$\left.\begin{aligned} \frac{u_{y=\xi}}{u_{y=0}} &= \frac{i3f(C\cosh r\xi + D\sinh r\xi)+s(E\sinh s\xi + F\cosh s\xi)}{i3fC+sF}, \\ \frac{v_{y=\xi}}{v_{y=0}} &= \frac{-3r(C\sinh r\xi + D\cosh r\xi)+if(E\cosh s\xi + F\sinh s\xi)}{-3rD+ifE}, \end{aligned}\right\} \qquad (14)$$

and these values for $H/L=1$, $1/2$, $1/4$, where $L=2\pi/f$, are shown in Fig. 4. The fact, that the nodes of of the horizontal component of displacement of longer waves are placed deeper and deeper in the stratum in comparison with those of shorter waves and also the amplitudes of waves in deeper parts are diminished as wave-length decreases, is worth noticing.



(a) u. (b) v.

Fig. 4.

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21. 地表層に傳はる波動の性質について

地震研究所 {妹澤克惟 / 金井清}

無限に剛い基底上に任意の彈性を有する一つの層を置き，其の境面は密着してゐるとき，基底の物質は全然動かないものとして，その層に傳はる波動を考へてみた．單純に考へると，基底は全然運動しないから，その層を傳はる縱波又はその面中に運動する横波が特別にあつてその速度もその彈性や密度に特有なものでありそうに見えるけれども，正確に計算をやつてみるとそのやうな波は存在しなくて寧ろレーレー波に近いものゝあり得ることが知られた．しかもその波動速度は波長と共に著しく増加するものであることもわかつた．但しそれは波形の移る速さであつて波群速度は波形の波長の増加と共に寧ろ低下することがわかる．

表面上の水平と垂直の極大變位の割合は波長が極めて短いときにはレーレー波のそれに等いけれども波長が長くなるに從て水平變位が次第に減少して行くことが知られる．又波長が長くなると層中深い所の變位が割合に大きくなり，且つ水平變位の節點が底の方にあることがわかる．

つけ加へたいことは，基底は全然運動に與らないのにも拘らず，換言すれば何等の波動勢力も傳はらないとしたのにも拘らず，波動の速度其他のものが基底の影響を大いに受けてゐる事である．物理的に考へると當然なことかも知れぬが，震波速度の問題に多少の參考になるやうにも思はれたので敢て計算の結果を出して置いたのである．
