28. *Transient Motions of a Pendulum Caused by an External Vibration with Sudden or Gradual Commencement.*

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The problem regarding the initial motion of an earthquake as recorded by a seismograph has been acquiring wide interests among seismologists. While a seismograph is a correct indicator of the sense of the initial motion of an earthquake which it records, it is not as regards the amplitude and the time with and at which the maximum of the motion takes place. This circumstance is due to the fact that during such an early stage of an earthquake, the motion of the pendulum system in the seismograph is not yet stationary but is transient even if the earthquake vibration is assumed to be purely harmonic. As the well-known magnification diagram for the seismograph only applies to the case in which the vibration of the seismograph pendulum is already stationary, we must prepare another that applies to the case in which the vibration of the pendulum is still transient for the purpose of correctly interpreting the initial part of the seismograph record.

In the present paper, the motion will be calculated that a pendulum makes when it is excited from its original state of rest by an external vibration which will be assumed either sudden or gradual in its commencement. Seeing that, in spite of its importance, no complete result of the numerical calculation of this kind has been published except some by H. P. Berlage, S. T. Nakamura and T. Suzuki,[1] the writer presents here the result of his calculation in the hope that this may prove serviceable to seismologists.

Thanks are due to Mr. A. Zitukawa for his assistance in the

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1) H. P. BERLAGE, Handb. d. Geophys., Bd. IV, Lief. 2 (1930), 269.

S. T. NAKAMURA, *Journ. Met. Soc. Japan*, [ii], 2 (1924), 109; *Proc. Imp. Acad.*, 3 (1927), 33.

T. SUZUKI, *Bull. Earthq. Res. Inst.*, 12 (1934), 15 and 155.

calculation which although involved no theoretical difficulty, was a quite tedious work to carry out.

Although it is probable that at the instant of the commencement of an earthquake neither the position, the velocity nor the acceleration of the motion of the ground is discontinuous, we will at first not take such things into consideration and will simply assume that the external vibration $f(t)$ is expressed as follows:

$$t<0 \qquad f(t)=0,$$
$$t\geqq 0 \qquad f(t)=a\sin\omega t.$$

Using customary notations, the equation of motion of a pendulum subjected to this vibration is

$$\frac{d^2x}{dt^2}+2\varepsilon\frac{dx}{dt}+n^2x=a\omega^2\sin\omega t,$$

with initial conditions

$$t=0 \qquad x=0,$$

and

$$t=0 \qquad \frac{dx}{dt}=-a\omega.$$

The solution of the differential equation which satisfies these conditions is

$$x=\frac{a\omega^2}{\sqrt{(n^2-\omega^2)^2+4\varepsilon^2\omega^2}}\sin\tau e^{-\varepsilon t}\cos\sqrt{n^2-\varepsilon^2}\,t$$
$$+\frac{1}{\sqrt{n^2-\varepsilon^2}}\left\{\frac{a\varepsilon\omega^2}{\sqrt{(n^2-\omega^2)^2+4\varepsilon^2}}\sin\tau-a\omega-\frac{a\omega^3}{\sqrt{(n^2-\omega^2)^2+4\varepsilon^2\omega^2}}\cos\tau\right\}\sin\sqrt{n^2-\varepsilon^2}\,t$$
$$+\frac{a\omega^2}{\sqrt{(n^2-\omega^2)^2+4\varepsilon^2\omega^2}}\sin\omega(t-\tau),$$

where

$$\tan\omega\tau=\frac{\varepsilon\omega^2}{n^2-\omega^2}.$$

Putting $a=1$ and $\omega=1$, numerical values of x were calculated for all possible combinations of

$$\begin{cases} n=0{\cdot}2,\ 0{\cdot}4,\ 0{\cdot}6,\ 0{\cdot}8,\ 1{\cdot}0,\ 2{\cdot}0,\ 3{\cdot}0,\ 4.0,\ 5{\cdot}0, \\ \dfrac{\varepsilon}{n}=0,\ 0{\cdot}2,\ 0{\cdot}4,\ 0{\cdot}6,\ 0{\cdot}8,\ 1{\cdot}0, \end{cases}$$

for every 10° interval of t from $t=0$ to $t=300°$. The x–t–curves which were obtained in this manner for different combinations of n and $\frac{\varepsilon}{n}$

are shown in Figs. 1–9. The amplitudes and the times of the first and second maxima, the time of the first zero, and the ratio of the first and second maxima of x all of which are read on the curves are given in Tables I–VI and are shown graphically in Figs. 10–15.

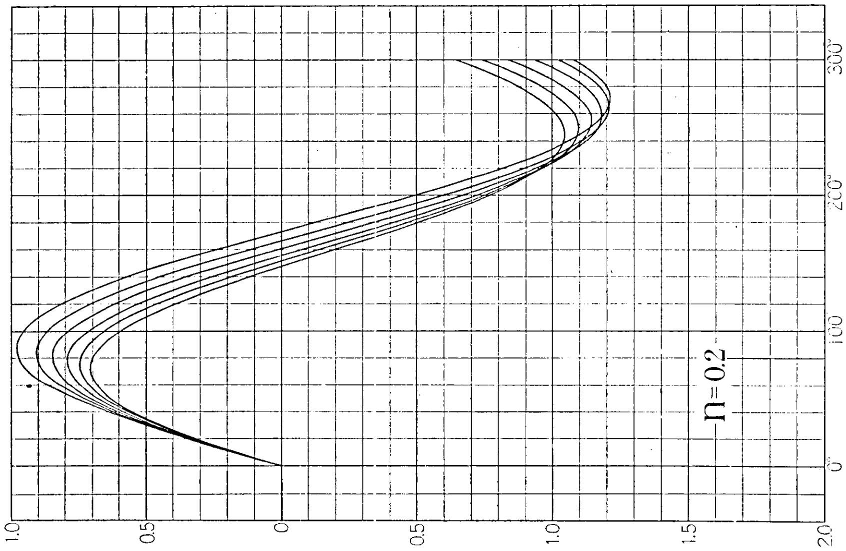

Fig. 1.

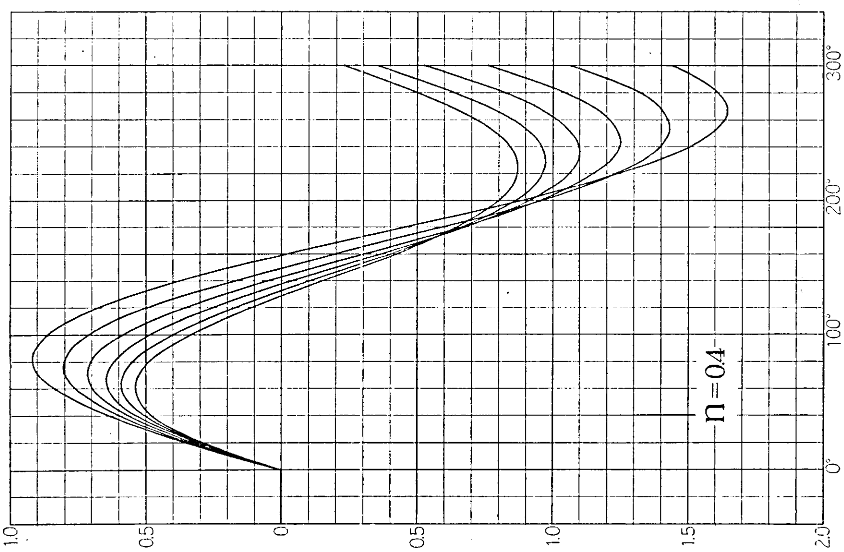

Fig. 2.

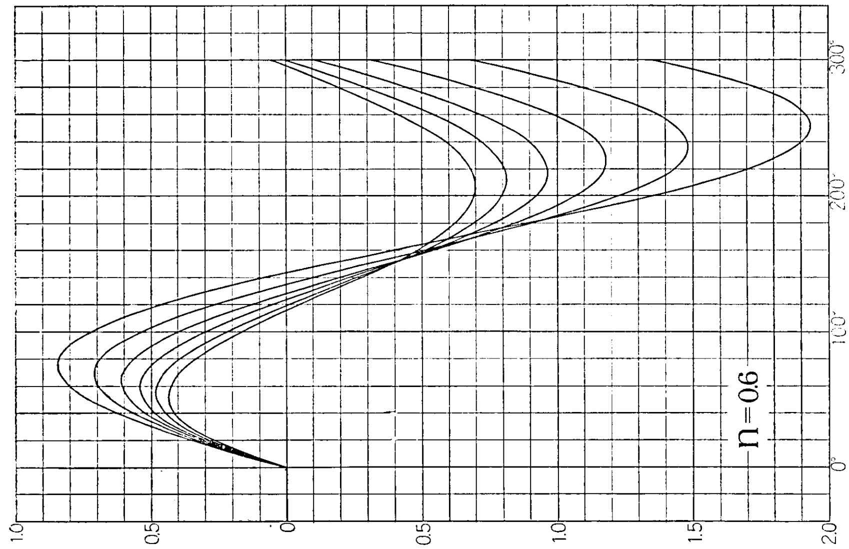
1.0
0.5
0
0.5
1.0
1.5
2.0
0°
100°
200°
300°
n=0.6

Fig. 3.

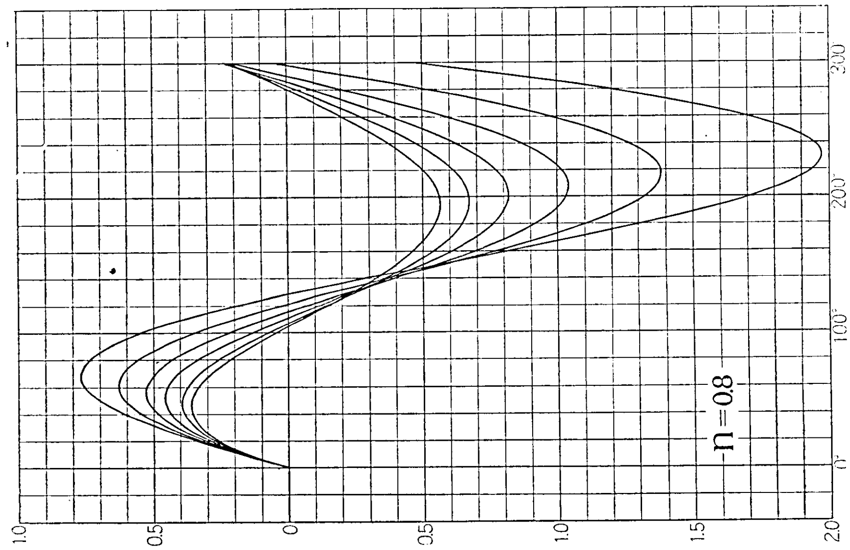
1.0
0.5
0
0.5
1.0
1.5
2.0
0°
100°
200°
300°
n=0.8

Fig. 4.

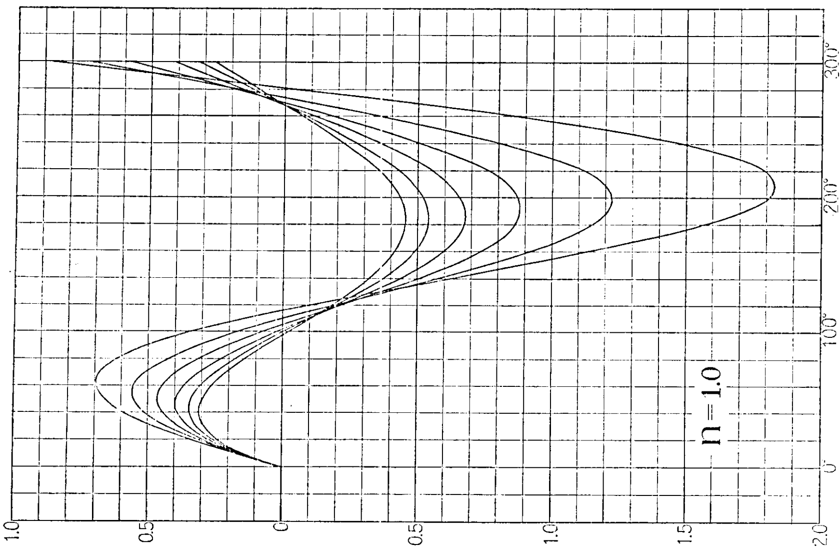
n=1.0
0°
100°
200°
300°
1.0
0.5
0
0.5
1.0
1.5
2.0

Fig. 5.

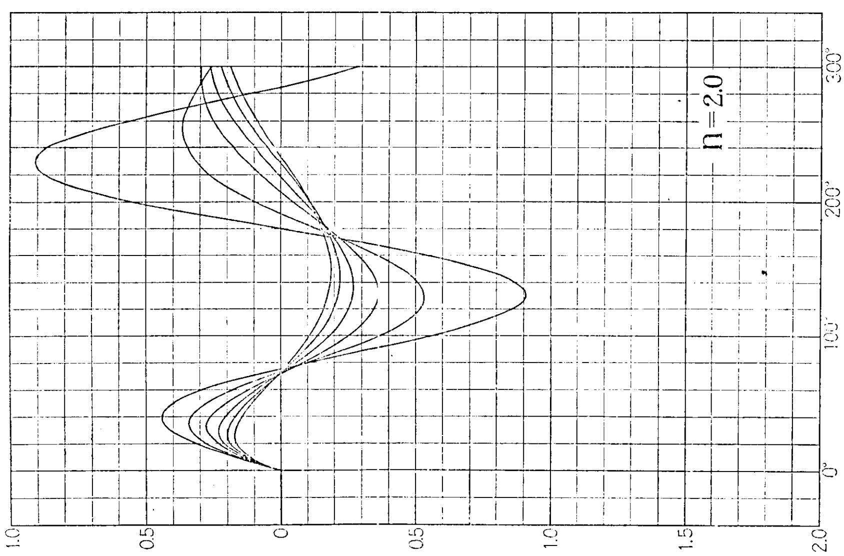
n=2.0
0°
100°
200°
300°
1.0
0.5
0
0.5
1.0
1.5
2.0

Fig. 6.

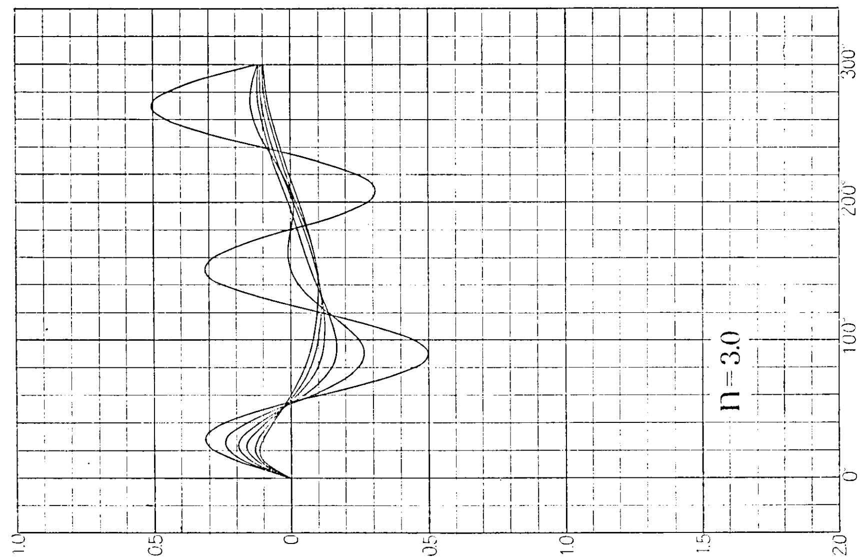

Fig. 7.

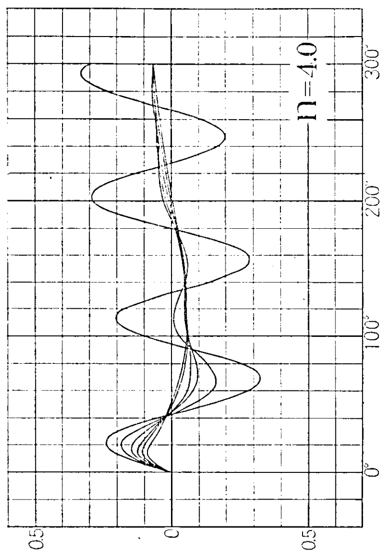

Fig. 8.

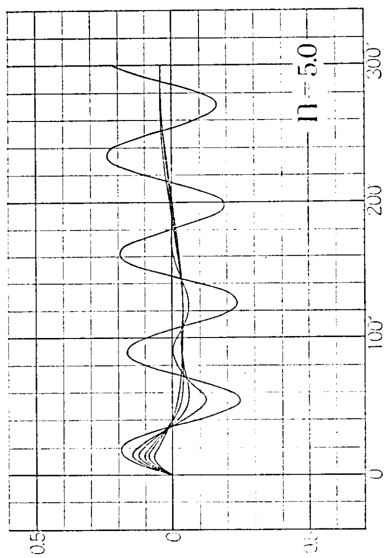

Fig. 9.

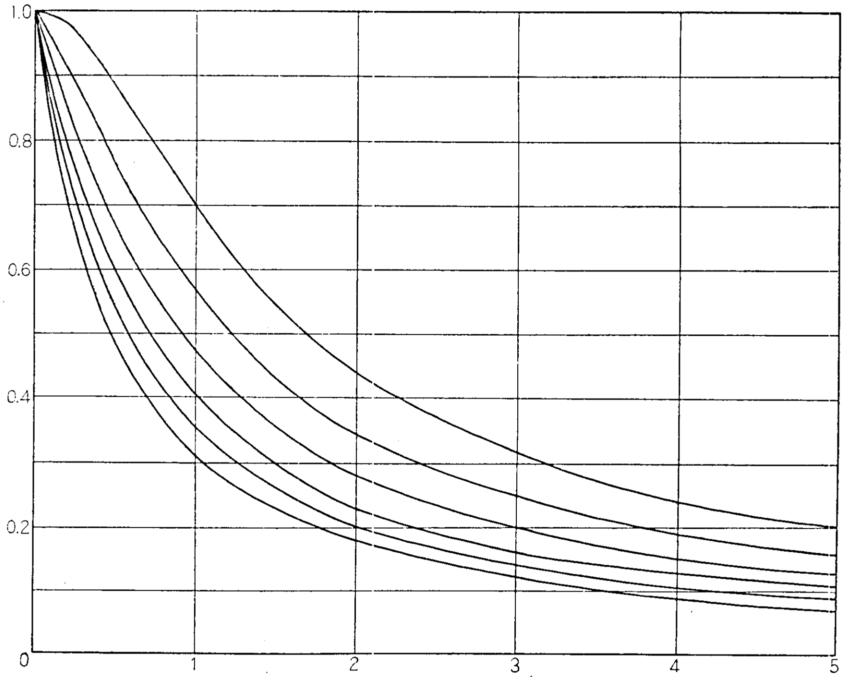

Fig. 10. Amplitude of the first maximum as the function n.

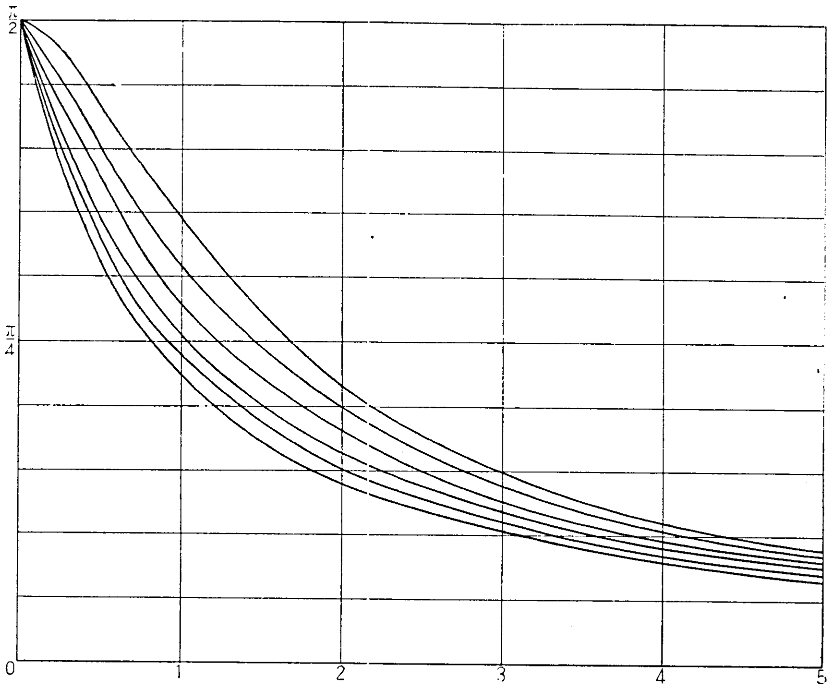

Fig. 11. Time of the first maximum.

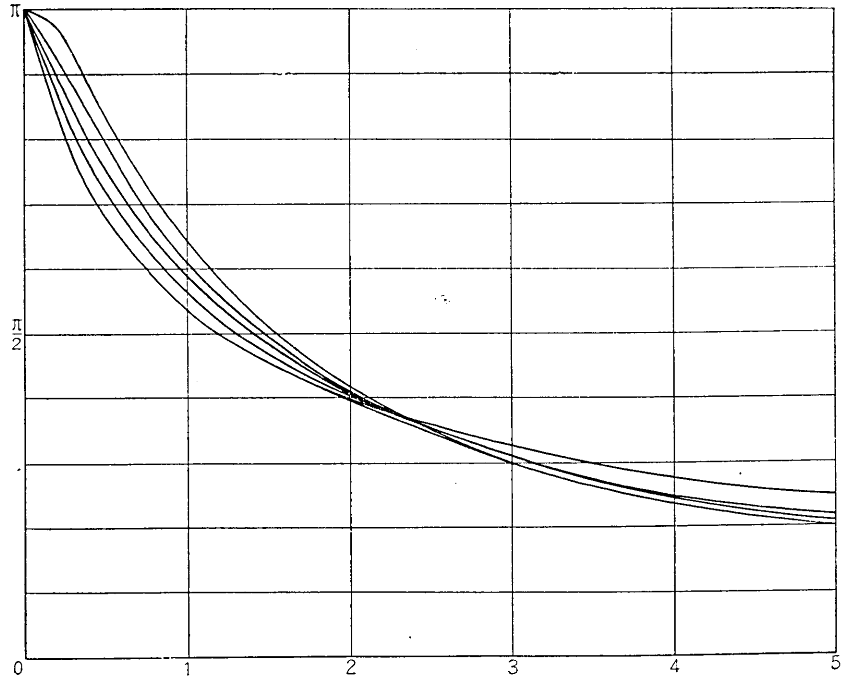

Fig. 12. Time of the first zero.

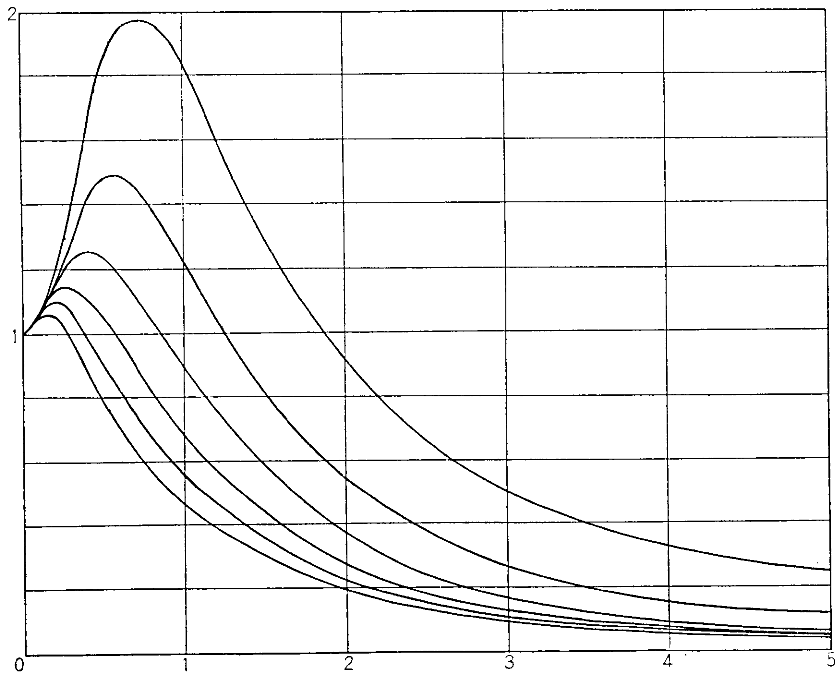

Fig. 13. Amplitude of the second maximum.

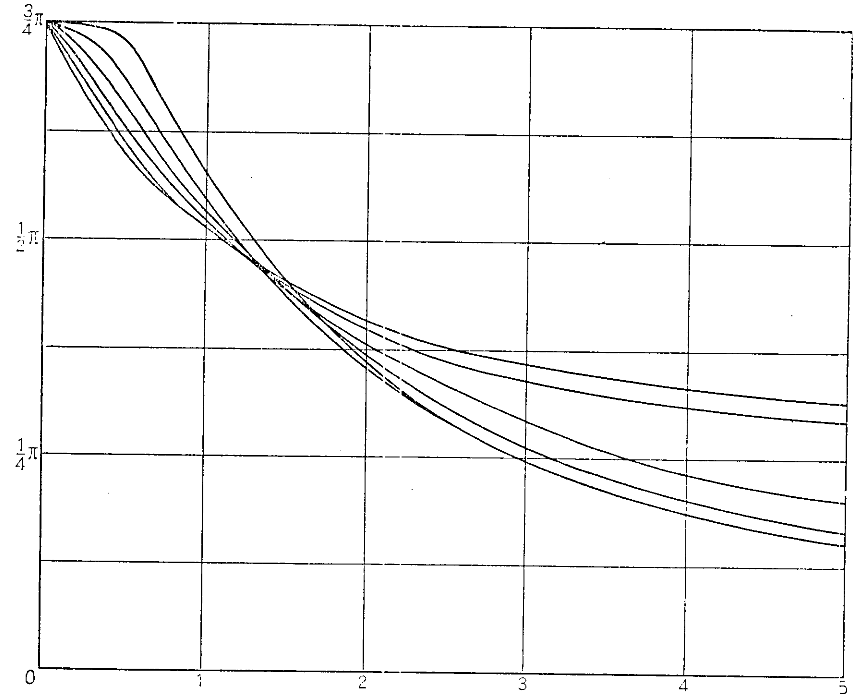

Fig. 14. Time of the second maximum.

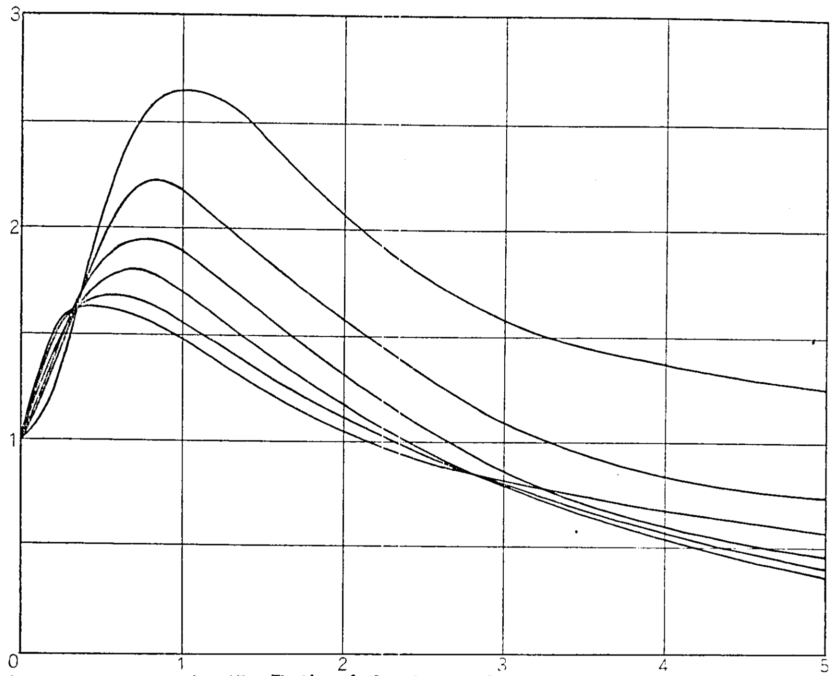

Fig. 15. Ratio of the first and second maxima.

Table I. Amplitude of the First Maximum.

n \ ε/n	0	0·2	0·4	0·6	0·8	1·0
0·2	0·98	0·91	0·85	0·79	0·75	0·71
0·4	0·92	0·81	0·72	0·65	0·59	0·54
0·6	0·85	0·71	0·62	0·55	0·49	0·44
0·8	0·77	0·63	0·54	0·46	0·41	0·36
1·0	0·70	0·57	0·47	0·40	0·35	0·31
2·0	0·44	0·34	0·28	0·23	0·20	0·18
3·0	0·32	0·25	0·20	0·16	0·14	0·12
4·0	0·24	0·19	0·15	0·13	0·11	0·09
5·0	0·20	0·16	0·13	0·11	0·09	0·07

Table II. Time of the First Maximum.

n \ ε/n	0	0·2	0·4	0·6	0·8	1·0
0·2	87°	83	80	77	75	73
0·4	82	76	72	67	64	61
0·6	75	68	64	58	55	52
0·8	68	62	56	52	48	45
1·0	63	55	50	46	43	41
2·0	39	36	33	30	27	25
3·0	27	25	24	23	21	19
4·0	20	19	17	16	15	14
5·0	16	16	14	13	13	12

Table III. Time of the First Zero.

n \ ε/n	0	0·2	0·4	0·6	0·8	1·0
0·2	174°	167	161	156	152	148
0·4	159	150	143	137	133	129
0·6	143	135	128	123	119	116
0·8	129	121	116	111	108	105
1·0	116	110	105	102	100	98
2.0	76	73	72	72	73	73
3.0	55	54	55	56	58	60
4·0	43	43	44	46	49	51
5·0	35	35	37	39	42	45

Table IV. Amplitude of the Second Maximum.

n \ ε/n	0	0·2	0·4	0·6	0·8	1·0
0·2	1·21	1·21	1·18	1·14	1·10	1.05
0.4	1·65	1·43	1·25	1·10	0·98	0·88
0·6	1·94	1·48	1·18	0·98	0·82	0·70
0·8	1·98	1·39	1·04	0·82	0·67	0·56
1·0	1·84	1·23	0·89	0·68	0·55	0.46
2·0	0·91	0·54	0·37	0·27	0·22	0·19
3·0	0·50	0·27	0·17	0·13	0·11	0·10
4·0	0·33	0·16	0·09	0·07	0·06	0·06
5·0	0·25	0·12	0·06	0·04	0·04	0·04

Table V. Time of the Second Maximum.

n \ ε/n	0	0·2	0·4	0·6	0·8	1·0
0·2	272°	265	260	255	250	245
0·4	267	254	245	236	230	225
0·6	253	236	226	217	212	207
0·8	230	216	207	202	196	196
1·0	208	197	192	186	185	184
2·0	131	128	132	137	144	147
3.0	90	90	95	107	123	130
4·0	67	67	73	87	112	120
5·0	55	55	59	74	107	115

Table VI. Ratio of the First and Second Maxima.

n \ ε/n	0	0·2	0·4	0·6	0.8	1·0
0·2	1·23	1·33	1·39	1·44	1·47	1.48
0·4	1·79	1·77	1·74	1·69	1·66	1·63
0·6	2·28	2·08	1·90	1·78	1·67	1·59
0·8	2·57	2·21	1·93	1·78	1·63	1·55
1·0	2·63	2·16	1·89	1·70	1·57	1·48
2·0	2·07	1·59	1·32	1·17	1·10	1·05
3·0	1·56	1·08	0·85	0·78	0·79	0·83
4·0	1·37	0·84	0·60	0·54	0·55	0·67
5·0	1·25	0·75	0·46	0·36	0·40	0·57

Twice the time of the first zero is the apparent period of the initial motion which is generally different from 2π. As the actual problem is seismometry, the ratio of the periods of the external vibration to that of the seismograph cannot be known and what we only know is the one of the apparent period of the recorded initial motion to that of the seismograph. In the present case, the ratio becomes as follows:

Table VII. Apparent Ratio of the Period of the Initial Earthquake Motion to the Period of the Seismograph.

n \ ε/n	0	0·2	0·4	0·6	0·8	1·0
0·2	0·19	0.19	0·18	0·17	0·17	0·16
0·4	0·35	0·33	0·32	0·30	0·29	0·29
0·6	0·48	0·45	0·43	0·41	0·40	0·39
0.8	0·57	0·54	0·51	0·49	0·48	0·47
1·0	0·64	0·61	0·58	0·57	0·55	0·55
2·0	0·84	0·81	0·80	0·80	0·81	0·81
3·0	0·91	0·90	0·91	0·93	0·96	1·00
4·0	0·95	0·95	0·97	1·01	1·08	1·12
5·0	0·96	0·97	1·01	1·07	1·17	1·24

If the amplitudes of the first and second maxima are plotted against these values, we have Figs. 16, 17 which are most important and useful in practical seismometry.

Several important conclusions which can be drawn from the above calculations are:

1. The first maximum on the seismogram is *always* smaller than unity.

2. The second maximum is largest for $n=0{\cdot}8$ and $\varepsilon=0$ but not for $n=1$ and $\varepsilon=0$.

3. However large the value of n may be, the first maximum on the seismogram indicates the sense of the earth's displacement but not that of its acceleration.

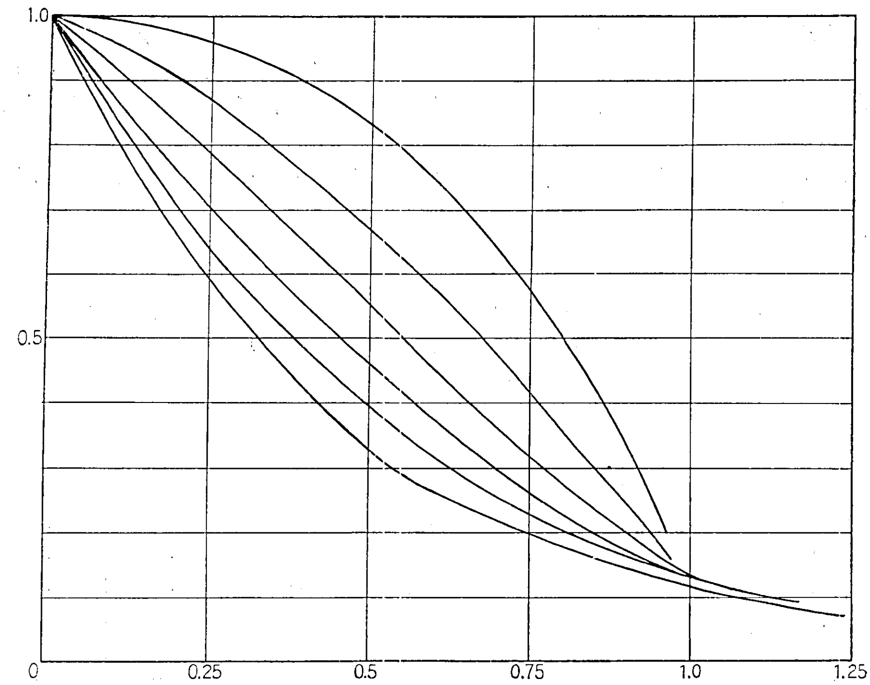

Fig. 16. Amplitude of the first maximum.

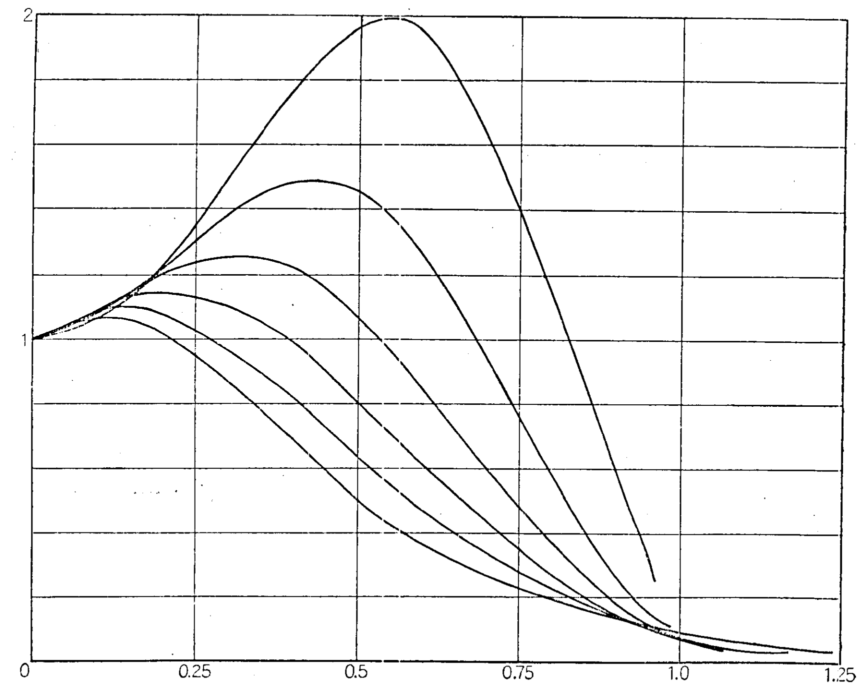

Fig. 17. Amplitude of the second maximum.

We will now poceed to the case in which the commencement of the external vibration is gradual. One expression for such a motion is

$$\begin{cases} t<0 & f(t)=0, \\ t\geqq 0 & f(t)=2\sin t-\sin 2t. \end{cases}$$

The velocity and acceleration are both continuous at $t=0$ for we have

$$\frac{d}{dx}f(t)=2\cos t-2\cos 2t=0,$$

$$\frac{d^2}{dx^2}f(t)=2\sin t+4\sin 2t=0.$$

At $t=120°$, the value of $f(t)$ is maximum which is 2·598. The solution of the equation of motion of a pendulum subjected to this vibration is given by

$$x=\left\{\frac{2}{\sqrt{(n^2-1)^2+4\varepsilon^2}}\sin\tau-\frac{4}{\sqrt{(n^2-4)^2+16\varepsilon^2}}\sin 2\tau'\right\}e^{-\varepsilon t}\cos\sqrt{n^2\varepsilon^2}\,t$$

$$+\frac{1}{\sqrt{n^2-\varepsilon^2}}\left\{\frac{2\varepsilon}{\sqrt{(2^2-1)^2+4\varepsilon^2}}\sin\tau-\frac{4\varepsilon}{\sqrt{(n^2-4)^2+16\varepsilon^2}}\sin 2\tau'\right.$$

$$\left.-\frac{2}{\sqrt{(n^2-1)^2+4\varepsilon^2}}\cos\tau+\frac{8}{\sqrt{(n^2-4)^2+16\varepsilon^2}}\cos 2\tau'\right\}+e^{-\varepsilon t}\sin\sqrt{n^2-\varepsilon^2}\,t$$

$$+\frac{2}{\sqrt{(n^2-1)^2+4\varepsilon^2}}\sin(t-\tau)-\frac{4}{\sqrt{(n^2-4)^2+16\varepsilon^2}}\sin 2(t-\tau'),$$

where

$$\tan\tau=\frac{2\varepsilon}{n^2-1},$$

$$\tan 2\tau'=\frac{4\varepsilon}{n^2-4}.$$

As in the preceding section, the numerical values of x for all possible combinations of

$$\begin{cases} n=0{\cdot}2.\ 0{\cdot}4,\ 0{\cdot}6,\ 0{\cdot}8,\ 1{\cdot}0,\ 2{\cdot}0,\ 3{\cdot}0,\ 4{\cdot}0,\ 5{\cdot}0\,, \\ \dfrac{\varepsilon}{n}=0,\ 0{\cdot}2,\ 0{\cdot}4,\ 0{\cdot}6,\ 0{\cdot}8,\ 1{\cdot}0, \end{cases}$$

for every $10°$ interval of t from $t=0$ to $t=300°$ were calculated by making use of the results for the former case. The magnifications and the times with and at which the first and second maxima take place and the time of the first zero are given in Tables VIII–XII and shown graphically in Figs. 18–20.

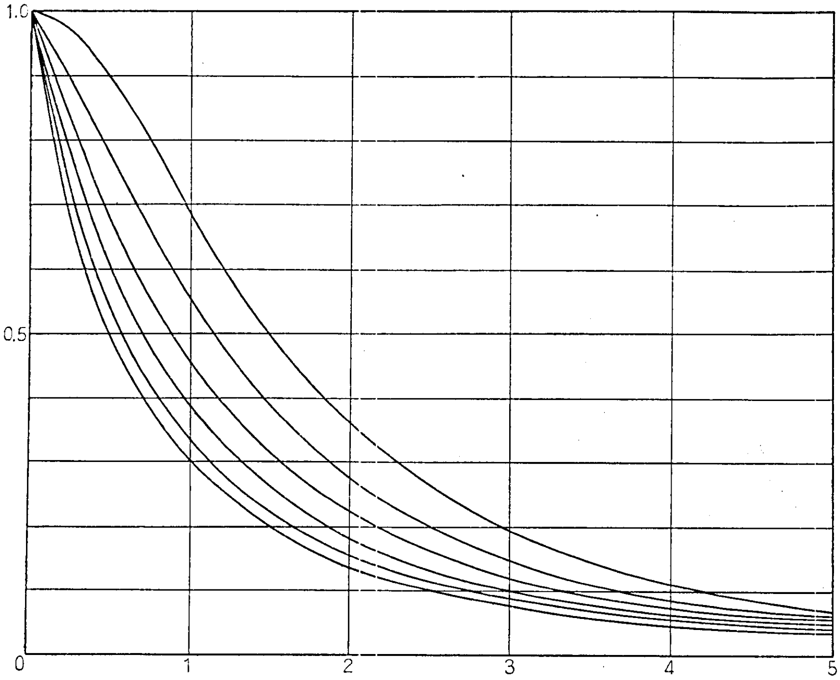

Fig. 18. Magnification for the first maximum as the function of n.

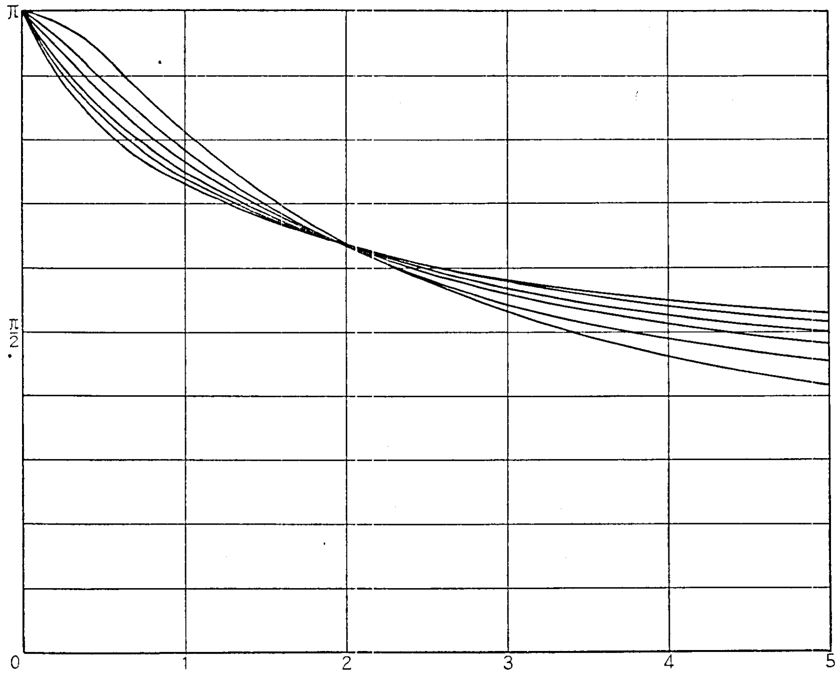

Fig. 19. Time of the first zero.

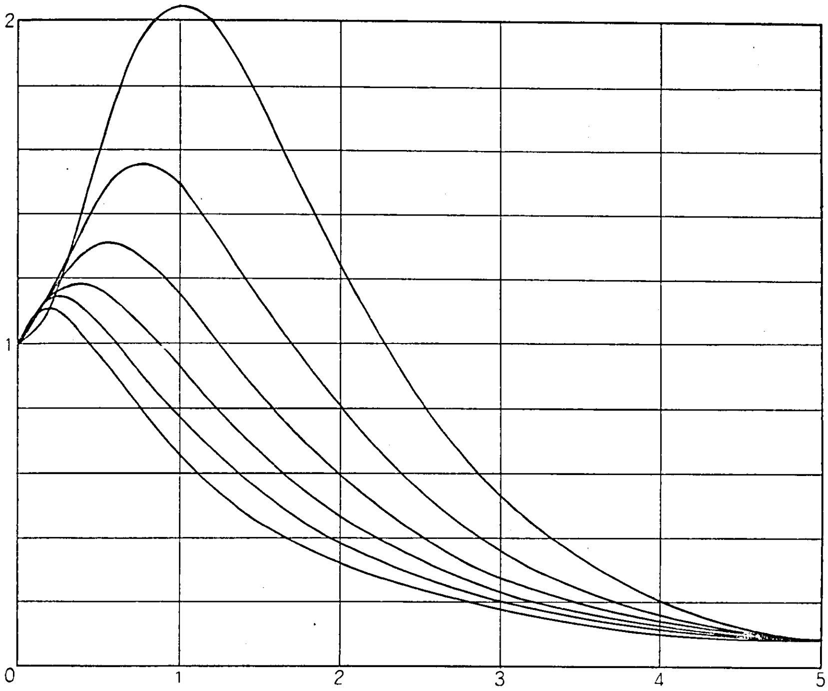

Fig. 20. Magnification for the second maximum.

Table VIII. Magnification for the First Maximum.

n \ ε/n	0	0·2	0·4	0·6	0·8	1·0
0·2	0·98	0·91	0·85	0·81	0·77	0·73
0·4	0·93	0·82	0·74	0·67	0·60	0·55
0·6	0·86	0·73	0·62	0·55	0·49	0·44
0·8	0·79	0·63	0·54	0·46	0·40	0·36
1·0	0·69	0·55	0·45	0·39	0·33	0·29
2·0	0·36	0·28	0·22	0·18	0·16	0·14
3·0	0·19	0·15	0·12	0·10	0·09	0·08
4·0	0·11	0·09	0·07	0·06	0·06	0·05
5·0	0·07	0·06	0·05	0·04	0·04	0·03

Table IX. Time of the First Maximum.

n \ ε/n	0	0·2	0·4	0·6	0·8	1·0
0·2	119°	116	115	113	111	110
0·4	116	113	110	107	104	103
0·6	112	108	105	102	100	97
0·8	108	103	100	97	95	92
1·0	103	99	95	93	91	88
2·0	82	79	79	78	76	76
3·0	67	67	67	67	68	68
4·0	57	58	59	61	62	64
5·0	49	51	54	56	58	60

Table X. Time of the First Zero.

n \ ε/n	0	0·2	0·4	0·6	0·8	1·0
0·2	178°	174	170	167	165	163
0·4	171	165	161	157	154	151
0·6	163	157	152	149	146	143
0·8	155	149	144	141	139	136
1·0	147	141	138	135	133	131
2·0	116	114	114	114	114	115
3·0	96	98	100	102	104	105
4·0	83	88	92	95	98	100
5·0	75	82	87	91	93	95

Table XI. Magnification for the Second Maximum.

n \ ε/n	0	0·2	0·4	0·6	0·8	1·0
0·2	1·12	1·15	1·15	1·15	1·15	1·11
0·4	1·42	1·35	1·28	1·18	1·10	1·02
0·6	1·76	1·51	1·30	1·13	1·00	0·90
0.8	1·93	1·55	1·25	1·04	0·88	0·77
1·0	2·05	1·50	1·15	0·93	0·77	0·65
2·0	1·23	0·81	0·59	0·46	0·37	0·32
3·0	0·52	0·36	0.28	0·23	0·20	0·18
4·0	0·20	0·16	0·15	0·13	0·12	0.11
5·0	0·09	0·09	0·09	0·08	0·08	0·08

Table XII. Ratio of the First and Second Maxima.

n \ ε/n	0	0·2	0.4	0·6	0·8	1·0
0·2	1·14	1·26	1·35	1·42	1·48	1·53
0·4	1·54	1·65	1·73	1·78	1·84	1·85
0·6	2·05	2·07	2·10	2·06	2·06	2·04
0·8	2·52	2·46	2·34	2·28	2·22	2·12
1·0	2·98	2·70	2·53	2·42	2·30	2·22
2·0	3·39	2·92	2·67	2·52	2·40	2·33
3·0	2·66	2·41	2·34	2·33	2·32	2·29
4·0	1·80	1.88	2·05	2·17	2·20	2·24
5·0	1·32	1·68	1·98	2·08	2·16	2·19

The ratio of the first and second maxima tends to an asymptotic value as n increases to infinity. The value corresponds to the ratio of the first and second maxima in the acceleration of the external vibration which is 2·06. Twice the time of the first zero is the apparent period of the initial motion and its ratio to the period of the seismograph is given in Table XIII.

Table XIII. Ratio of the Apparent Period of the Initial Earthquake Motion to the Period of the Seismograph.

n \ ε/n	0	0·2	0·4	0·6	0·8	1·0
0·2	0·20	0·19	0·19	0·19	0·18	0·18
0·4	0·38	0·37	0·36	0·35	0·34	0·34
0·6	0·54	0·52	0·51	0·50	0·49	0·48
0·8	0·69	0·66	0·64	0·63	0·62	0·61
1·0	0·81	0·79	0·76	0·75	0·74	0·73
2·0	1·29	1·27	1·27	1·27	1·27	1.27
3·0	1·60	1·63	1·68	1·71	1·73	1·76
4·0	1·86	1·96	2·04	2·11	2·17	2.21
5·0	2·08	2·28	2·42	2·51	2·60	2·65

If the magnification for the first maximum is plotted against these values, we have Fig. 21. In this case also, the magnification for the first maximum is always smaller than unity.

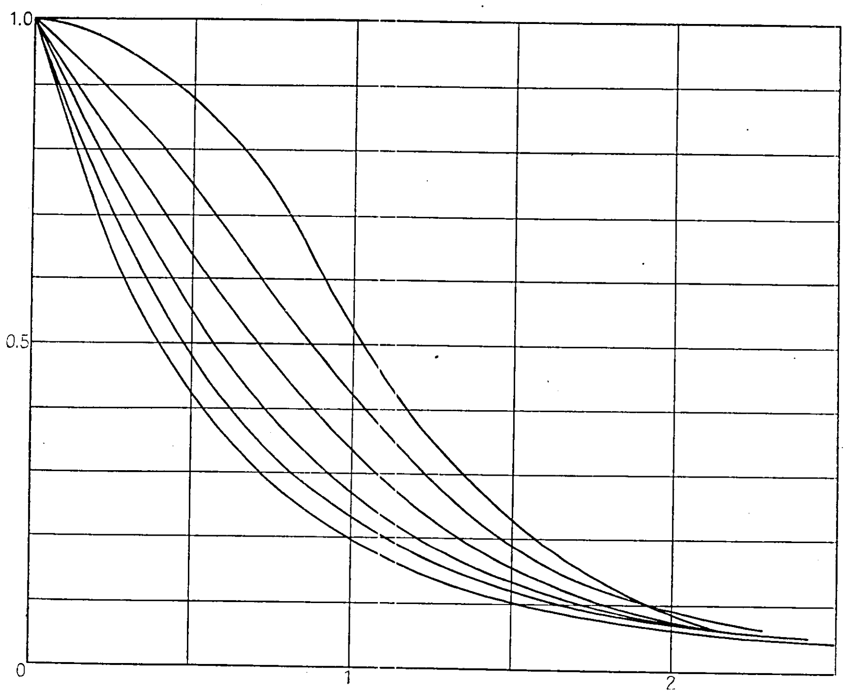

Fig. 21. Magnification for the first maximum.

28. 急激又は漸次に始まる振動によつて起される振子の過渡運動

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始め靜止して居る振子が外から振動を受けた場合に如何なる運動をするかと云ふ問題は，地震の初動を論ずる場合に重要であるに拘らず精しく計算した例が少い樣である．本文に於ては振子が

$$(1)\quad \begin{cases} t<0 & f(t)=0 \\ t\geqq 0 & f(t)=\sin t \end{cases}$$

$$(2)\quad \begin{cases} t<0 & f(t)=0 \\ t\geqq 0 & f(t)=2\sin t-\sin 2t \end{cases}$$

なる振動を受けた2つの場合について，振子の振動の模樣を，振子の週期と減衰度の色々な組合せ

に就いて計算した. (1) は外界の振動が急激に始まる場合. (2) は漸次に始まる場合である.

計算は少し面倒なだけで少しも困難な所はない. 本文中の第1動や第2動に對する倍率曲線が驗震家に用ひられれば幸である.