11. *On the Movement of Pendulum under Influence of the Motion of Shock Type.**

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The writer has studied in his previous paper[1] by comparison of the corresponding records of acceleration-seismogram and displacement-seismogram that the initial earthquake-motion is considered at the very commencement to begin with a displacement of a certain kind of shock, for example, given by the form e^{-t^2}. As already referred in the previous paper, the motion of pendulum corresponding to the earthquake-motions beginning with certain types of sinusoidal form[2] was investigated by Saem. Nakamura, T. Matuzawa, H. P. Berlage, and H. Kawasumi, in respective manners. Here, the writer will study the motion of pendulum when it is subjected to a kind of shock as above stated. Such an earthquake-motion is of course a special one; however, it may not be quite needless to know how the pendulum behaves for the said motion, which is not expressed in Fourier series.

Using customary notations, the equation of motion of a seismograph free from solid friction is given by

$$\frac{d^2a}{dt^2}+2\varepsilon\frac{da}{dt}+n^2a=-\frac{d^2x}{dt^2}. \quad \dots\dots\dots\dots\dots\dots (1)$$

The general solution of (1) is

$$a=e^{-\varepsilon t}[\Gamma_1\cos\gamma t+\Gamma_2\sin\gamma t]+\frac{e^{-\varepsilon t}}{\gamma}\Big[-\cos\gamma t\int^t e^{\varepsilon t'}\sin\gamma t'\left(-\frac{d^2x}{dt'^2}\right)dt'$$
$$+\sin\gamma t\int^t e^{\varepsilon t'}\cos\gamma t'\left(-\frac{d^2x}{dt'^2}\right)dt'\Big], \quad \dots\dots\dots (2)$$

where $n=2\pi/T_0$, $\gamma=\sqrt{n^2-\varepsilon^2}$, $\mu^2=1-h^2$, $h=\varepsilon/n$, and Γ_1 and Γ_2 are arbi-

* Communicated by T. Matuzawa.

1) T. SUZUKI, *Bull. Earthq. Res. Inst.*, 12 (1934), 15-18.

2) Lately C. TSUBOI has discussed a special case of this kind, the result of which has been read before the meeting of Earthquake Research Institute on Feb. 20, 1934.

trary constants to be determined by the initial conditions.

In the present case, if c' is taken as a parameter, the displacement of earthquake-motion is expressed by

$$x=e^{-c't^2}\ (c'>0). \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

As this motion starts gradually from a state of rest at $t=-\infty$, i.e. at $t=-\infty$, $x=0$ and $\dot{x}=0$, the same conditions as these must hold with respect to a. The solution which satisfies such conditions as above is given by

$$a=\frac{e^{-\varepsilon t}}{\gamma}\left[-\cos\gamma t\int_{-\infty}^{t}e^{\varepsilon t'}\sin\gamma t'\left(-\frac{d^2x}{dt'^2}\right)dt'+\sin\gamma t\int_{-\infty}^{t}e^{\varepsilon t'}\cos\gamma t'\left(-\frac{d^2x}{dt'^2}\right)dt'\right]. \qquad \ldots\ldots\ldots\ldots (4)$$

Putting $u=\dfrac{t}{T_0}$, in which T_0 is the proper period of seismograph without damping,

$$\begin{aligned} a=\frac{e^{-2\pi hu}}{2\pi\mu}\Big[&-\cos 2\pi\mu u\int_{-\infty}^{u}e^{2\pi hu'}\sin 2\pi\mu u'\left\{-2c(2cu'^2-1)\,e^{-cu'^2}\right\}du' \\ &+\sin 2\pi\mu u\int_{-\infty}^{u}e^{2\pi hu'}\cos 2\pi\mu u'\left\{-2c\,(2cu'^2-1)\,e^{-cu'^2}\right\}du'\Big] \\ =-e^{-cu^2}&+\frac{2\pi}{\mu}e^{\frac{\pi^2h^2}{c}}e^{-2\pi hu}\left\{\left[(h^2-\mu^2)\cos 2\pi\mu u+2\mu h\sin 2\pi\mu u\right]\right. \\ &\qquad\times\int_{-\infty}^{u-\frac{\pi h}{c}}e^{-cz^2}\sin 2\pi\mu\left(z+\frac{\pi h}{c}\right)dz \\ &+\left.\left[(\mu^2-h^2)\sin 2\pi\mu u+2\mu h\cos 2\pi\mu u\right]\int_{-\infty}^{u-\frac{\pi h}{c}}e^{-cz^2}\cos 2\pi\mu\left(z+\frac{\pi h}{c}\right)dz\right\}, \end{aligned} \qquad \ldots\ldots\ldots\ldots (5)$$

where c stands for $c'T_0^2$.

As seen from the above expression, it is required to evaluate the integral of the form $\int_{-\infty}^{u}e^{-z^2}\begin{Bmatrix}\sin z\\ \cos z\end{Bmatrix}dz$, but this is in general not integrable except the case of definite integral taken between the limits $-\infty$ and 0. The above integral is written as follows:

$$\int_{-\infty}^{u}e^{-z^2}\begin{Bmatrix}\sin z\\ \cos z\end{Bmatrix}dz=\int_{-\infty}^{0}e^{-z^2}\begin{Bmatrix}\sin z\\ \cos z\end{Bmatrix}dz+\int_{0}^{u}e^{-z^2}\begin{Bmatrix}\sin z\\ \cos z\end{Bmatrix}dz,$$

so that this will be obtained if the second integral on the right is evaluated by the method of numerical integration.

It is known that

$$\int_{-\infty}^{0} e^{-k^2x^2}\cos 2cx\,dx = e^{-\frac{c^2}{k^2}}\frac{\sqrt{\pi}}{2k},$$

$$\int_{-\infty}^{0} e^{-k^2x^2}\sin 2cx\,dx = -\frac{1}{k^2}\int_{0}^{c} e^{\frac{z^2-c^2}{k^2}}\,dz,\text{[3)]}$$

so that by aid of these formulae the expression (5) becomes

$$\begin{aligned} a = -e^{-cu^2} + \frac{2\pi}{\mu} e^{\frac{\pi^2h^2}{c}} e^{-2\pi h u} \Bigg[&\left\{ \int_0^{u-\frac{\pi h}{c}} e^{-cz^2}\sin 2\pi\mu z\,dz \right. \\ &\left. -\frac{1}{c}\int_0^{\pi\mu} e^{\frac{y^2-\pi^2\mu^2}{c}}\,dy \right\} \sin\left[2\pi\mu\left(u-\frac{\pi h}{c}\right)+\Delta\right] \\ &+\left\{ \int_0^{u-\frac{\pi h}{c}} e^{-cz^2}\cos 2\pi\mu z\,dz + e^{-\frac{\pi^2\mu^2}{c}}\frac{\sqrt{\pi}}{2\sqrt{c}} \right\} \cos\left[2\pi\mu\left(u-\frac{\pi h}{c}\right)+\Delta\right] \Bigg], \end{aligned} \qquad \text{.....................(6)}$$

where $\sin\Delta = 1-2\mu^2$ and $\cos\Delta = 2\mu h$.

Especially in case of no damping $\varepsilon=0$, i.e. $h=0$, $\mu=1$,

$$\begin{aligned} a = -e^{-cu^2} &+ 2\pi\left\{\frac{1}{c}\int_0^{\pi} e^{\frac{y^2-\pi^2}{c}}\,dy - \int_0^{u} e^{-cz^2}\sin 2\pi z\,dz\right\}\cos 2\pi u \\ &+ 2\pi\left\{e^{-\frac{\pi^2}{c}}\frac{\sqrt{\pi}}{2\sqrt{c}} + \int_0^{u} e^{-cz^2}\cos 2\pi z\,dz\right\}\sin 2\pi u. \end{aligned} \qquad \text{..........(7)}$$

The second and third terms represent the excited self-vibration of the pendulum. As u increases, the amplitude of the second term converges gradually to zero, while that of the third term tends to a finite value of $\frac{2\pi\sqrt{\pi}}{\sqrt{c}}e^{-\frac{\pi^2}{c}}$, which is graphically shown in Fig. 1 and becomes a maximum value when $c=2\pi^2$. The value shown in Fig. 1 represents the

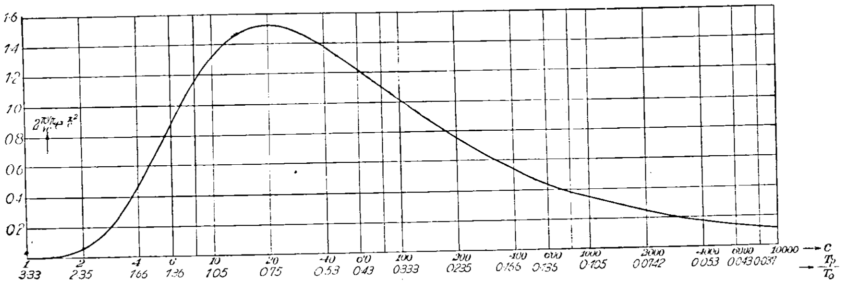

Fig. 1.

3) This formula is obtained as a subsidiary result in the process of evaluating $\int_{-\infty}^{0} e^{-x^2}\cos 2\beta x\,dx$ by a contour integration.

ratio of the amplitude of self-vibration to the maximum amplitude of given earth-movement. As seen from the following figures, $T_p/2$ is so defined as to be the time in which the amplitude of earth-motion diminishes to one half of its maximum amplitude and the ratio of thus defined T_p to T_0 is also given in Fig. 1 for the sake of comparison with c.

In case of critical damping $\varepsilon=n$, i.e. $h=1$ and $\mu=0$, the expression in (6) becomes an indeterminate form. Putting $\mu=\delta$, where δ is assumed as a small quantity and the magnitude of the order higher than the second may be neglected, then the following expression will be obtained from (6):

$$a=-\left(1+\frac{2\pi^2}{c}\right)e^{-cu^2}+2\sqrt{\frac{\pi}{c}}\,\pi e^{\frac{\pi^2}{c}}\,e^{-2\pi u}\left(1+\frac{\pi^2}{c}-\pi u\right)\left(1+\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{c}\left(u-\frac{\pi}{c}\right)}e^{-z^2}dz\right). \qquad \text{.....................(8)}$$

In this case the term as evaluated by numerical integration is reduced to the form of probability integral which is easily obtained from table.

In order to know a general aspect of matter, the numerical calculation has been carried out for several cases and these results are graphically shown in the following figures. In each figure the upper curve indicates the given earth-movement and the lower one the corresponding motion of a seismograph respectively.

As will be seen from the figures, when T_p becomes two or three times greater than T_0, the displacement of pendulum in the earlier stage is already considered to be nearly proportional to the acceleration of earth-motion. On the other hand, T_p/T_0 becomes one-tenth or much smaller, the displacement of pendulum in the beginning portion tends to nearly the same as the displacement of earth-motion.

Full line in the lower curve indicates the case of no damping.
Broken line in the lower curve indicates the case of critical damping.

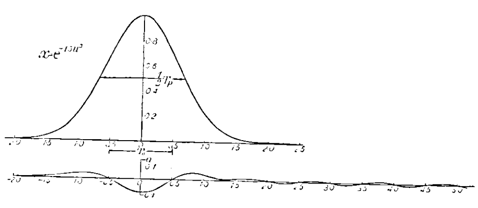

Fig. 2.

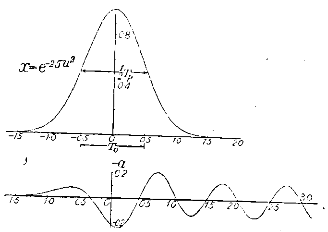

Fig. 3.

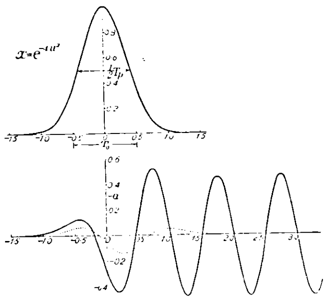

Fig. 4.

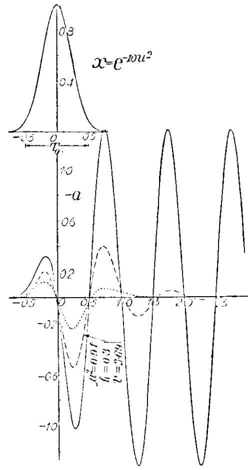

Fig. 5.

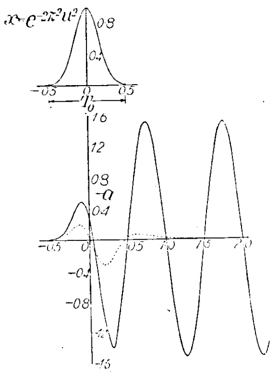

Fig. 6.

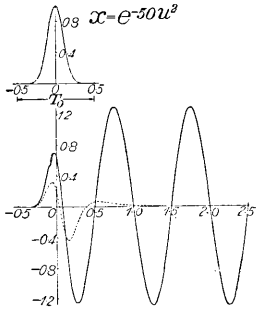

Fig. 7.

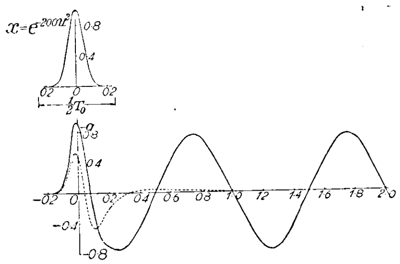

Fig. 8.

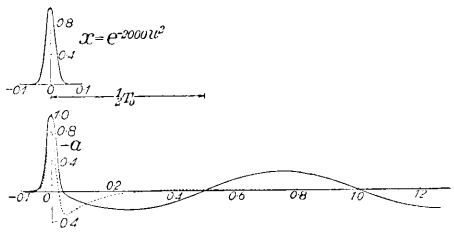

Fig. 9.

The earth-motion here studied is expressed by Fourier's double integral, that is,

$$e^{-cu^2}=\frac{1}{\pi}\int_0^\infty dp\int_{-\infty}^\infty e^{-c\lambda^2}\cos p(u-\lambda)\,d\lambda$$
$$=\int_0^\infty \frac{e^{-\frac{p^2}{4c}}}{\sqrt{\pi c}}\cos pu\,dp, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

which implies that the motion is given by the superposition of simple harmonic motions of frequencies extending from 0 to ∞, such that they should have all the same phase at $u=0$. Among them, the one component corresponding to $p=2\pi$ is a simple harmonic motion with just the same period as that of seismograph (T_0). For a given value of parameter c, we will consider how the amplitude at $u=0\left(\int_0^\infty \frac{e^{-\frac{p^2}{4c}}}{\sqrt{\pi c}}dp\right)$ is expressed by the terms of frequencies between 0 and 2π as well as by those of frequencies greater than 2π.

$$\int_0^\infty \frac{e^{-\frac{p^2}{4c}}}{\sqrt{\pi c}}dp=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\beta^2}d\beta, \qquad p=2\sqrt{c}\beta.$$

The above integral is separated into two parts, i.e. $\int_0^{2\pi}+\int_{2\pi}^\infty$; then the former gives the amplitude contributed by the terms of the period greater than T_0 and the latter by those of the period smaller than T_0. These are shown in Table I in the third column and the fourth column respectively.

Table I.

c	$(p=2\pi)\ \beta=\frac{\pi}{\sqrt{c}}$	$\frac{2}{\sqrt{\pi}}\int_0^{\frac{\pi}{\sqrt{c}}}e^{-\beta^2}d\beta$	$1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{\pi}{\sqrt{c}}}e^{-\beta^2}d\beta$	$\frac{T_p}{T_0}$
1	3·1416	$\doteqdot 1$	$\doteqdot 0$	3·330
1.5	2·5651	0·9997	0·0003	2·719
2·5	1·9869	0·9949	0·0051	2·106
4·0	1·5708	0·9736	0·0264	1·665
10·0	0·9935	0·8385	0·1615	1·053
11·09	0·9434	0·8163	0·1837	1·000
$2\pi^2$	0·7071	0·6826	0·3174	0·7496
50	0·4443	0·4662	0·5338	0·4710
200	0·2221	0·2443	0·7557	0·2355
2000	0·0703	0·0788	0·9212	0·0742
10000	0·0314	0·0338	0·9662	0·0333

As seen from Table I, when T_p becomes about two times greater than T_0, then the amplitude of e^{-cu^2} at $u=0$ is expressed enough by taking only the components, whose periods are greater than T_0. If T_p becomes one-tenth or much smaller compared with T_0, then the most part of the amplitude at $u=0$ is given by the components, whose periods are smaller than T_0. The value of e^{-cu^2} for any value of u is of course not sufficiently expressed only by taking those terms, which well represent e^{-cu^2} at least in the neighbourhood of $u=0$. But generally speaking, it may be plausibly assumed, though not rigorous, that the principal portion of e^{-cu^2} is also approximately represented by those terms which mainly represent the value at $u=0$[4]. If such an assumption is taken for granted, it will be said qualitatively that for large value of T_p/T_0 (say two or more), the seismograph gives the acceleration of earth-motion, that is, an acceleration-seismograph, while for small value of T_p/T_0 (one-tenth or much smaller) the seismograph gives the displacement of earth-motion, that is, a displacement-seismograph, by the reason that the displacement of pendulum affected by a steady simple harmonic earth-motion is nearly proportional to acceleration of or displacement of earth-motion according as the period of seismograph is suitably smaller or greater than that of earth-motion.

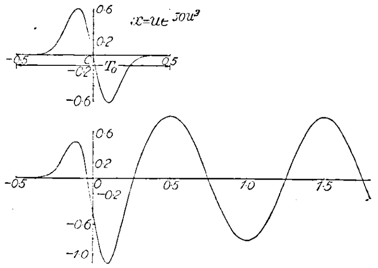

Fig. 10.

It is again remarked that comparatively larger difference between T_p and T_0 is necessary for describing displacement of earth-motion of the type e^{-t^2} than for its acceleration. It is probably due to the considerable existence of components with period larger than T_0 in representing e^{-t^2} by superposition of harmonic terms.

If the displacement of earth-motion is given in such a form as te^{-bt^2}, then the corresponding motion of a seismograph is obtained by a quite similar way as before and in case of no damping expressed as follows:

$$a=-ue^{-bu^2}+\frac{2\pi^2}{b}\left\{\left[e^{-\frac{\pi^2}{b}}\frac{\sqrt{\pi}}{2\sqrt{b}}+\int_0^u e^{-bu'^2}\cos 2\pi u'du'\right](-\cos 2\pi u)\right.$$
$$\left.+\left[\frac{1}{b}\int_0^\pi e^{\frac{y^2-\pi^2}{b}}dy-\int_0^u e^{-bu'^2}\sin 2\pi u'du'\right]\sin 2\pi u\right\}. \quad ..(10)$$

4) For example, when $c=1{\cdot}5$, the value of e^{-cu^2} at $u=1$ is 0·2231, while the value calculated by (9) from $p=0$ to $p=2\pi$ is 0·2226.

One example of this case is shown in Fig. 10.

In conclusion, the writer wishes to express his sincere thanks to Professor M. Ishimoto and Professor T. Matuzawa for their kind guidance and valuable advices.

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11. 衝撃性變位が作用した場合の振子の運動

地震學教室 鈴 木 武 夫

振子に變位が e^{-t^2} にて表される様な衝撃性變位が作用した場合の振子の運動を二三の場合につき調べた．擾亂部分の長さと振子の自己週期との比によりて振子に起される自己振動の振幅も種々に變化し又その比が 2 以上にもなれば擾亂の著しい部分に於ては振子の運動は加速度に近きものを示し反對に 0·1 以下ともなれば變位に極めて近きものを與へる．この事は定性的には簡單な考察によりて説明される様である．
