10. *On the Effects of Discontinuity Surfaces upon the Propagation of Elastic Wave. (I)*

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1. The theoretical investigation of the problem of reflection and refraction of elastic waves has come to the fore again, especially in Japan, in view of the marked development of seismometrical research relating to the construction of the earth's crust. T. Matuzawa,[1] H. Kawasumi,[2] T. Suzuki[2] and K. Sezawa[3] have all studied this problem of late. The boundary condition treated by these authorities is that in which two elastic solids adhere closely at the boundary surface, *i. e.*, in which the displacements and stresses of both solids are continuous at the common boundary. In the present paper, however, we study the effects of a boundary surface where the two elastic solids that come into contact slide upon each other without any friction on the propagating waves. The elasticity conditions at the boundary surface treated in this paper are not only perfectly correct from the theory of elasticity, but also furnish us with interesting properties of wave propagation which can not be obtained from the boundary condition in which the two solids adhere closely at the common boundary.

The present paper consists of two parts; in the first we study the effect of a boundary surface of two semi-infinite solids upon the propagation of plane waves, and in the second the effect of a surface layer of a stratified solid upon the propagation of these waves.

Part I. Effect of a Boundary Surface of Contact of Two Semi-infinite Solids upon the Propagation of Waves.

2. Let the boundary surface of the two elastic solids be $y=0$,

1) T. MATUZAWA, *Disin*, 4 (1932), (in Japanese).

2) H. KAWASUMI and T. SUZUKI, *Disin*, 4 (1932), (in Japanese).

3) K. SEZAWA and K. KANAI, "Reflection and Refraction of Seismic Waves in a Stratified Body," *Bull. Earthq. Res. Inst.*, 10 (1932).

and the axis x be on this contact surface. Again, let ρ and λ, μ be the density and Lame's elastic constants of the solid represented by I in Fig. 1, and ρ', λ', μ' be those of the solid II shown also in Fig. 1. We shall study the problems in the following two cases:

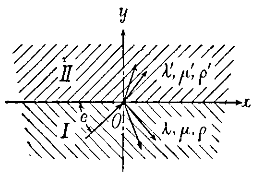

Fig. 1.

1. Incident Wave is Dilatational,
2. Incident Wave is Distortional.

a) Incident Wave is Dilatational.

3. When the incident wave is dilatational and expressed by

$$\Delta = \mathfrak{A} e^{i(pt-fx-ry)}, \quad \ldots\ldots (1)$$

the $x-$, $y-$ components of displacement u_1, v_1 and the normal, tangential and shear components of stress $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$ corresponding to this wave are respectively expressed by

$$\left.\begin{aligned} u_1 &= \frac{if}{h^2}\mathfrak{A} e^{i(pt-fx-ry)}, \\ v_1 &= \frac{ir}{h^2}\mathfrak{A} e^{i(pt-fx-ry)}, \end{aligned}\right\} \quad \ldots\ldots (2)$$

$$\left.\begin{aligned} \widehat{xx}_1 &= \mu\frac{(k^2-2h^2+2f^2)}{h^2}\mathfrak{A} e^{i(pt-fx-ry)}, \\ \widehat{yy}_1 &= \mu\frac{(k^2-2h^2+2r^2)}{h^2}\mathfrak{A} e^{i(pt-fx-ry)}, \\ \widehat{xy}_1 &= 2\mu\frac{fr}{h^2}\mathfrak{A} e^{i(pt-fx-ry)}, \end{aligned}\right\} \quad \ldots\ldots (3)$$

where

$$h^2 = \frac{\rho p^2}{\lambda+2\mu}, \quad k^2 = \frac{\rho p^2}{\mu}, \quad r^2+f^2=h^2, \quad \ldots\ldots (4)$$

and $p = \dfrac{2\pi}{T_1}$, T_1 being the period of incident wave.

Let the wave length and velocity of the incident dilatational wave be L_1 and V_1 respectively. Then we have the following relations.

$$L_1 = V_1 T_1 = \frac{2\pi}{\sqrt{r^2+f^2}}. \quad \ldots\ldots (5)$$

Now we know the reflected waves of two kinds, one of which is dilatational and the other distortional, such that

$$\Delta' = A' e^{i(pt-fx+ry)}, \quad \ldots\ldots (6)$$

$$2\varpi' = B' e^{i(pt - fx + sy)}, \qquad \qquad (7)$$

where A' and B' are arbitrary constants, and

$$s^2 = k^2 - f^2. \qquad \qquad (8)$$

The $x-$, $y-$ components of displacement u_1', v_1' and the stress components $\widehat{xx_1}'$, $\widehat{yy_1}'$, $\widehat{xy_1}'$ corresponding to Δ' expressed by (6) are written in the following forms:

$$\left.\begin{aligned} u_1' &= \frac{if}{h^2} A' e^{i(pt - fx + ry)}, \\ v_1' &= -\frac{ir}{h^2} A' e^{i(pt - fx + ry)}, \end{aligned}\right\} \qquad \qquad (9)$$

$$\left.\begin{aligned} \widehat{xx_1}' &= \mu \frac{(k^2 - 2h^2 + 2f^2)}{h^2} A' e^{i(pt - fx + ry)}, \\ \widehat{yy_1}' &= \mu \frac{(k^2 - 2h^2 + 2r^2)}{h^2} A' e^{i(pt - fx + ry)}, \\ \widehat{xy_1}' &= -\frac{2fr\mu}{h^2} A' e^{i(pt - fx + ry)}. \end{aligned}\right\} \qquad \qquad (10)$$

The displacement u_2', v_2' and the stress components $\widehat{xx_2}'$, $\widehat{yy_2}'$, $\widehat{xy_2}'$ corresponding to $2\varpi'$ expressed by (7) are respectively expressed in the forms:

$$\left.\begin{aligned} u_2' &= \frac{is}{k^2} B' e^{i(pt - fx + sy)}, \\ v_2' &= \frac{if}{k^2} B' e^{i(pt - fx + sy)}, \end{aligned}\right\} \qquad \qquad (11)$$

$$\left.\begin{aligned} \widehat{xx_2}' &= \frac{2sf\mu}{k^2} B' e^{i(pt - fx + sy)}, \\ \widehat{yy_2}' &= -\frac{2sf\mu}{k^2} B' e^{i(pt - fx + sy)}, \\ \widehat{xy_2}' &= \frac{(f^2 + s^2)\mu}{k^2} B' e^{i(pt - fx + sy)}. \end{aligned}\right\} \qquad \qquad (12)$$

The waves transmitted in the second medium also consist of two kinds such that

$$\Delta'' = A'' e^{i(pt - fx - r'y)}, \qquad \qquad (13)$$

$$2\varpi'' = B'' e^{i(pt - fx - s'y)}, \qquad \qquad (14)$$

where A'', B'' are arbitrary constants and

$$r'^2=h'^2-f^2,\quad s'^2=k'^2-f^2, \qquad \ldots\ldots\ldots\ldots\ldots(15)$$

$$h'^2=\frac{\rho' p^2}{\lambda'+2\mu'},\quad k'^2=\frac{\rho' p^2}{\mu'}. \qquad \ldots\ldots\ldots\ldots\ldots(16)$$

The $x-$, $y-$ components of displacement u_1'', v_1'' and the stress components $\widehat{xx_1}''$, $\widehat{yy_1}''$, $\widehat{xy_1}''$ corresponding to Δ'' expressed by (13) are also of the forms:

$$\left.\begin{aligned} u_1''&=\frac{if}{h'^2}A''e^{i(pt-fx-r'y)},\\ v_1''&=\frac{ir'}{h'^2}A''e^{i(pt-fx-r'y)},\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots(17)$$

$$\left.\begin{aligned} \widehat{xx_1}''&=\mu'\frac{(k'^2-2h'^2+2f^2)}{h'^2}A''e^{i(pt-fx-r'y)},\\ \widehat{yy_1}''&=\mu'\frac{(k'^2-2h'^2+2r'^2)}{h'^2}A''e^{i(pt-fx-r'y)},\\ \widehat{xy_1}''&=\frac{2fr'\mu'}{h'^2}A''e^{i(pt-fx-r'y)}.\end{aligned}\right\} \qquad \ldots\ldots\ldots(18)$$

The displacement u_2'', v_2'' and the stress $\widehat{xx_2}''$, $\widehat{yy_2}''$ $\widehat{xy_2}''$ corresponding to $2\varpi''$ expressed by (14) are written by

$$\left.\begin{aligned} u_2''&=-\frac{is'}{k'^2}B''e^{i(pt-fx-s'y)},\\ v_2''&=\frac{if}{k'^2}B''e^{i(pt-fx-s'y)},\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots(19)$$

$$\left.\begin{aligned} \widehat{xx_2}''&=-\frac{2s'f\mu'}{k'^2}B''e^{i(pt-fx-s'y)},\\ \widehat{yy_2}''&=\frac{2s'f\mu'}{k'^2}B''e^{i(pt-fx-s'y)},\\ \widehat{xy_2}''&=\frac{(f^2-s'^2)\mu'}{k'^2}B''e^{i(pt-fx-s'y)}.\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots(20)$$

Now the boundary conditions which we have discussed in Section 1 are expressed in the following forms:

When $y=0$,

$$\left.\begin{aligned} &\widehat{yy_1}+\widehat{yy_1}'+\widehat{yy_2}'=\widehat{yy_1}''+\widehat{yy_2}'',\\ &v_1+v_1'+v_2'=v_1''+v_2'',\\ &\widehat{xy_1}+\widehat{xy_1}'+\widehat{xy_2}'=0,\\ &\widehat{xy_1}''+\widehat{xy_2}''=0.\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots(21)$$

The boundary conditions expressed by (21) gives us the following values of A', B', A'', B'':

$$\left.\begin{aligned}
A' &= \mathfrak{A}\frac{\left[\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4f^2s'r'\}-r'(f^2+s'^2)\{(s^2-f^2)^2-4f^2sr\}\right]}{\left[\frac{\mu'}{\mu}r(f^2+s^2)\{(s'^2-f^2)^2+4f^2s'r'\}+r'(f^2+s'^2)\{(s^2-f^2)^2+4f^2sr\}\right]},\\
B' &= \mathfrak{A}\frac{4frr'k^2(s'^2+f^2)(s^2-f^2)}{h^2\left[\frac{\mu'}{\mu}r(f^2+s^2)\{(s'^2-f^2)^2+4f^2s'r'\}+r'(f^2+s'^2)\{(s^2-f^2)^2+4f^2sr\}\right]},\\
A'' &= \mathfrak{A}\frac{2rh'^2(s'^2-f^2)(s^2-f^2)(s^2+f^2)}{h^2\left[\frac{\mu'}{\mu}r(f^2+s^2)\{(s'^2-f^2)^2+4f^2s'r'\}+r'(f^2+s'^2)\{(s^2-f^2)^2+4f^2sr\}\right]},\\
B'' &= \mathfrak{A}\frac{4frr'k'^2(s^2-f^2)(s^2+f^2)}{h^2\left[\frac{\mu'}{\mu}r(f^2+s^2)\{(s'^2-f^2)^2+4f^2s'r'\}+r'(f^2+s'^2)\{(s^2-f^2)^2+4f^2sr\}\right]}.
\end{aligned}\right\}\quad(23)$$

Substituting these values for A', B', A'' and B'' in the expressions (9), (11), (17) and (19), we obtain the reflected and transmitted waves which are generated by the primary incident wave expressed by (1) as in the following expressions:

Reflected dilatational wave;

$$\left.\begin{aligned}
u_1' &= \frac{if\mathfrak{A}}{h^2}\frac{\left[\frac{\mu'}{\mu}r(f^2+s^2)\{(s'^2-f^2)^2+4s'r'f^2\}-r'(s'^2+f^2)\{(s^2-f^2)^2-4rsf^2\}\right]}{\left[\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}+r'(s'^2+f^2)\{(s^2-f^2)^2+4rsf^2\}\right]}e^{i(pt-fx+ry)},\\
v_1' &= -\frac{i\mathfrak{A}r}{h^2}\frac{\left[\frac{\mu'}{\mu}r(f^2+s^2)\{(s'^2-f^2)^2+4s'r'f^2\}-r'(s'^2+f^2)\{(s^2-f^2)^2-4rsf^2\}\right]}{\left[\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}+r'(s'^2+f^2)\{(s^2-f^2)^2+4rsf^2\}\right]}e^{i(pt-fx+ry)}.
\end{aligned}\right\}\quad\ldots.(24)$$

Reflected distortional wave;

$$\left.\begin{aligned}
u_2' &= \frac{i\mathfrak{A}}{h^2}\frac{4sfrr'(s^2-f^2)(s'^2+f^2)}{\left[\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}+r'(s'^2+f^2)\{(s^2-f^2)^2+4rsf^2\}\right]}e^{i(pt-fx+sy)},\\
v_2' &= \frac{i\mathfrak{A}}{h^2}\frac{4f^2rr'(s^2-f^2)(s'^2+f^2)}{\left[\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}+r'(s'^2+f^2)\{(s^2-f^2)^2+4rsf^2\}\right]}e^{i(pt-fx+sy)},
\end{aligned}\right\}\quad\text{..(25)}$$

Transmitted dilatational wave ;

$$\left.\begin{aligned}
u_1'' &= \mathfrak{A}\frac{if}{h^2}\frac{2r(s^2+f^2)(s^2-f^2)(s'^2-f^2)}{\left[\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}+r'(s'^2+f^2)\{(s^2-f^2)^2+4rsf^2\}\right]}e^{i(pt-fx-r'y)},\\
v_1'' &= \mathfrak{A}\frac{ir'}{h^2}\frac{2r(s^2+f^2)(s^2-f^2)(s'^2-f^2)}{\left[\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}+r'(s'^2+f^2)\{(s^2-f^2)^2+4rsf^2\}\right]}e^{i(pt-fx-r'y)},
\end{aligned}\right\}\quad\text{..(26)}$$

Transmitted distortional wave ;

$$\left.\begin{aligned}
u_2'' &= -\mathfrak{A}\frac{is'}{h^2}\frac{4rr'f(s^2+f^2)(s^2-f^2)}{\left[\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}+r'(s'^2+f^2)\{(s^2-f^2)^2+4rsf^2\}\right]}e^{i(pt-fx-s'y)},\\
v_2'' &= \mathfrak{A}\frac{if}{h^2}\frac{4rr'f(s^2+f^2)(s^2-f^2)}{\left[\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}+r'(s'^2+f^2)\{(s^2-f^2)^2+4rsf^2\}\right]}e^{i(pt-fx-s'y)}.
\end{aligned}\right\}\quad\text{..(27)}$$

Let V_1 and V_2 be the velocities of dilatational and distortional waves in the medium I and V_1', V_2' be those of dilatational and distortional waves of the medium II respectively. And also let the emergency angle of incident dilatational wave expressed by (1) be e. Then the expressions (24), (25), (26) and (27) are written in the following forms:

$$\left.\begin{aligned} u_1' &= \frac{i\mathfrak{A}T_1V_1\cos e}{2\pi}\frac{\Psi}{\Phi}e^{i(pt-fx+ry)},\\ v_1' &= -\frac{i\mathfrak{A}T_1V_1\sin e}{2\pi}\frac{\Psi}{\Phi}e^{i(pt-fx+ry)}, \end{aligned}\right\} \qquad \text{(24)}'$$

$$\left.\begin{aligned} u_2' &= \frac{i\mathfrak{A}T_1V_1\cos e}{2\pi}\\ &\times\frac{4\sin e\left(\frac{V_1}{V_2'}\right)^2\left\{\left(\frac{V_1}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left\{\left(\frac{V_1}{V_2}\right)^2-2\cos^2 e\right\}\left\{\left(\frac{V_1}{V_2}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}}{\Phi}\\ &\times e^{i(pt-fx+sy)},\\ v_2' &= \frac{i\mathfrak{A}T_1V_1\cos^2 e}{2\pi}\\ &\times\frac{4\sin e\left(\frac{V_1}{V_2'}\right)^2\left\{\left(\frac{V_1}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left\{\left(\frac{V_1}{V_2}\right)^2-2\cos^2 e\right\}}{\Phi}e^{i(pt-fx+sy)}, \end{aligned}\right\} \text{(25)}'$$

$$\left.\begin{aligned} u_1'' &= \frac{i\mathfrak{A}T_1V_1\cos e}{2\pi}\\ &\times\frac{2\sin e\left(\frac{V_1}{V_2}\right)^2\left\{\left(\frac{V_1}{V_2}\right)^2-2\cos^2 e\right\}\left\{\left(\frac{V_1}{V_2'}\right)^2-2\cos^2 e\right\}}{\Phi}e^{i(pt-fx-r'y)},\\ v_1'' &= \frac{i\mathfrak{A}T_1V_1}{2\pi}\\ &\times\frac{2\sin e\left(\frac{V_1}{V_2}\right)^2\left\{\left(\frac{V_1}{V_2}\right)^2-2\cos^2 e\right\}\left\{\left(\frac{V_1}{V_2'}\right)^2-2\cos^2 e\right\}\left\{\left(\frac{V_1}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}}{\Phi}\\ &\times e^{i(pt-fx-r'y)}, \end{aligned}\right\} \text{(26)}'$$

$$\left.\begin{aligned}
u_2'' &= -\frac{i\mathfrak{A}T_1V_1\cos e}{2\pi} \\
&\quad\times\frac{4\sin e\left(\frac{V_1}{V_2}\right)^2\left\{\left(\frac{V_1}{V_2}\right)^2-2\cos^2 e\right\}\left\{\left(\frac{V_1}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left\{\left(\frac{V_1}{V_2'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}}{\Phi} \\
&\qquad\qquad\times e^{i(pt-fx-s'y)}, \\
v_2'' &= \frac{i\mathfrak{A}T_1V_1\cos^2 e}{2\pi} \\
&\quad\times\frac{4\sin e\left(\frac{V_1}{V_2}\right)^2\left\{\left(\frac{V_1}{V_2}\right)^2-2\cos^2 e\right\}\left\{\left(\frac{V_1}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}}{\Phi}e^{i(pt-fx-s'y)},
\end{aligned}\right\} \quad (27)'$$

where

$$\begin{aligned}
\Phi = \frac{\mu'}{\mu}\sin e\left(\frac{V_1}{V_2}\right)^2 &\Bigg[\left\{\left(\frac{V_1}{V_2'}\right)^2-2\cos^2 e\right\}^2 \\
&+4\cos^2 e\left\{\left(\frac{V_1}{V_2'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left\{\left(\frac{V_1}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\Bigg] \\
&+\left(\frac{V_1}{V_2'}\right)^2\left\{\left(\frac{V_1}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\Bigg[\left\{\left(\frac{V_1}{V_2}\right)^2-2\cos^2 e\right\}^2 \\
&+4\sin e\cos^2 e\left\{\left(\frac{V_1}{V_2}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\Bigg], \\
\Psi = \frac{\mu'}{\mu}\sin e\left(\frac{V_1}{V_2}\right)^2 &\Bigg[\left\{\left(\frac{V_1}{V_2'}\right)^2-2\cos^2 e\right\}^2 \\
&+4\cos^2 e\left\{\left(\frac{V_1}{V_2'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left\{\left(\frac{V_1}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\Bigg] \\
&-\left(\frac{V_1}{V_2'}\right)^2\left\{\left(\frac{V_1}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\Bigg[\left\{\left(\frac{V_1}{V_2}\right)^2-2\cos^2 e\right\}^2 \\
&-4\sin e\cos^2 e\left\{\left(\frac{V_1}{V_2}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\Bigg].
\end{aligned}$$

When $\rho=\rho'$, Poisson's ratio of the medium I is $\frac{1}{4}$, and that of the medium II is also $\frac{1}{4}$, we obtain the following expressions of waves generated at the boundary surface by the incident dilatational wave:

$$
\left.
\begin{aligned}
u_1' &= \frac{i\mathfrak{A}T_1V_1\cos e}{2\pi} \\
&\times \frac{\left[3\frac{\mu'}{\mu}\sin e\left\{\left(3\frac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(3\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\} - 3\frac{\mu}{\mu'}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{(3-2\cos^2 e)^2-4\sin e\cos^2 e(3-\cos^2 e)^{\frac{1}{2}}\right\}\right]}{\left[3\frac{\mu'}{\mu}\sin e\left\{\left(3\frac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(3\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\} + 3\frac{\mu}{\mu'}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{(3-2\cos^2 e)^2+4\sin e\cos^2 e(3-\cos^2 e)^{\frac{1}{2}}\right\}\right]} \\
&\times e^{i(\imath t-fx+ry)}, \\
v_1' &= -\frac{i\mathfrak{A}T_1V_1\sin e}{2\pi} \\
&\times \frac{\left[3\frac{\mu'}{\mu}\sin e\left\{\left(3\frac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(3\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\} - 3\frac{\mu}{\mu'}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{(3-2\cos^2 e)^2-4\sin e\cos^2 e(3-\cos^2 e)^{\frac{1}{2}}\right\}\right]}{\left[3\frac{\mu'}{\mu}\sin e\left\{\left(3\frac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(3\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\} + 3\frac{\mu}{\mu'}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{(3-2\cos^2 e)^2+4\sin e\cos^2 e(3-\cos^2 e)^{\frac{1}{2}}\right\}\right]} \\
&\times e^{i(pt-fx+ry)},
\end{aligned}
\right\} \quad (24)''
$$

$$
\left.
\begin{aligned}
u_2' &= \frac{i\mathfrak{A}T_1V_1}{2\pi} \\
&\times \frac{12\sin e\cos e\frac{\mu}{\mu'}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}(3-2\cos^2 e)(3-\cos^2 e)^{\frac{1}{2}}\, e^{i(pt-fx+sy)}}{\left[3\frac{\mu'}{\mu}\sin e\left\{\left(3\frac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(3\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\} + 3\frac{\mu}{\mu'}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{(3-2\cos^2 e)^2+4\sin e\cos^2 e(3-\cos^2 e)^{\frac{1}{2}}\right\}\right]},
\end{aligned}
\right\} \quad (25)''
$$

$$
\left.\begin{aligned}
v_2' &= \frac{i\mathfrak{A}T_1V_1}{2\pi} \\
&\times \frac{12\sin e\cos^2 e\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}(3-2\cos^2 e)}{\left[3\frac{\mu'}{\mu}\sin e\left\{\left(3\frac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(3\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\} + 3\frac{\mu}{\mu'}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{(3-2\cos^2 e)^2+4\sin e\cos^2 e(3-\cos^2 e)^{\frac{1}{2}}\right\}\right]} e^{i(qt-fx+sy)}, \\
u_1'' &= \frac{i\mathfrak{A}T_1V_1}{2\pi} \\
&\times \frac{6\sin e\cos e(3-2\cos^2 e)\left(3\frac{\mu}{\mu'}-2\cos^2 e\right)}{\left[3\frac{\mu'}{\mu}\sin e\left\{\left(3\frac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(3\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\} + 3\frac{\mu}{\mu'}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{(3-2\cos^2 e)^2+4\sin e\cos^2 e(3-\cos^2 e)^{\frac{1}{2}}\right\}\right]} e^{i(pt-fx-r'y)}, \\
v_1'' &= \frac{i\mathfrak{A}T_1V_1}{2\pi} \\
&\times \frac{6\sin e(3-2\cos^2 e)\left(3\frac{\mu}{\mu'}-2\cos^2 e\right)\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}}{\left[3\frac{\mu'}{\mu}\sin e\left\{\left(3\frac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(3\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\} + 3\frac{\mu}{\mu'}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{(3-2\cos^2 e)^2+4\sin e\cos^2 e(3-\cos^2 e)^{\frac{1}{2}}\right\}\right]} e^{i(pt-fx-r'y)};
\end{aligned}\right\} \quad (26'')
$$

$$
\begin{aligned}
u_2'' &= -\frac{i\mathfrak{A}T_1V_1}{2\pi} \\
&\times \frac{12\sin e\cos e(3-2\cos^2 e)\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(3\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}}{\left[3\frac{\mu'}{\mu}\sin e\left\{\left(3\frac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(3\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\}\right.}
\end{aligned}
$$

$$\left.\begin{aligned}
&\frac{e^{i(pt-fx-s'y)}}{+3\frac{\mu}{\mu'}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{(3-2\cos^2 e)^2+4\sin e\cos^2 e(3-\cos^2 e)^{\frac{1}{2}}\right\}\Big]},\\
&v_2''=\frac{i\mathfrak{A}T_1V_1}{2\pi}\\
&\times\frac{12\sin e\cos^2 e(3-2\cos^2 e)\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}}{\left[3\frac{\mu'}{\mu}\sin e\left\{\left(3\frac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(3\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\}\right.}\\
&\frac{e^{i(pt-fx-s'y)}}{+3\frac{\mu}{\mu'}\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{(3-2\cos^2 e)^2+4\sin e\cos^2 e(3-\cos^2 e)^{\frac{1}{2}}\right\}\Big]}.
\end{aligned}\right\}\quad(27)''$$

When $\frac{\mu'}{\mu}=1$, the expressions (24)″, (25)″, (26)″, (27)″ are expressed by

$$\left.\begin{aligned}
u_1'&=\frac{i\mathfrak{A}T_1V_1}{2\pi}\cdot\frac{4\sin e\cos^3 e(3-\cos^2 e)^{\frac{1}{2}}}{\{(3-2\cos^2 e)^2+4\cos^2 e\sin e(3-\cos^2 e)^{\frac{1}{2}}\}}e^{i\left(pt-fx+\frac{2\pi\sin e}{T_1V_1}y\right)},\\
v_1'&=-\frac{i\mathfrak{A}T_1V_1}{2\pi}\cdot\frac{4\sin^2 e\cos^2 e(3-\cos^2 e)^{\frac{1}{2}}}{\{(3-2\cos^2 e)^2+4\cos^2 e\sin e(3-\cos^2 e)^{\frac{1}{2}}\}}e^{i\left(pt-fx+\frac{2\pi\sin e}{T_1V_1}y\right)},
\end{aligned}\right\}\quad(28)$$

$$\left.\begin{aligned}
u_2'&=\frac{i\mathfrak{A}T_1V_1}{2\pi}\cdot\frac{2\sin e\cos e(3-2\cos^2 e)(3-\cos^2 e)^{\frac{1}{2}}}{\{(3-2\cos^2 e)^2+4\cos^2 e\sin e(3-\cos^2 e)^{\frac{1}{2}}\}}e^{i\left(pt-fx+\frac{2\sqrt{3}\pi\sin e}{T_1V_1}y\right)},\\
v_2'&=\frac{i\mathfrak{A}T_1V_1}{2\pi}\cdot\frac{2\sin e\cos^2 e(3-2\cos^2 e)}{\{(3-2\cos^2 e)^2+4\cos^2 e\sin e(3-\cos^2 e)^{\frac{1}{2}}\}}e^{i\left(pt-fx+\frac{2\sqrt{3}\pi\sin e}{T_1V_1}y\right)},
\end{aligned}\right\}\quad(29)$$

$$\left.\begin{aligned}
u_1''&=\frac{i\mathfrak{A}T_1V_1}{2\pi}\cdot\frac{\cos e(3-2\cos^2 e)^2}{\{(3-2\cos^2 e)^2+4\cos^2 e\sin e(3-\cos^2 e)^{\frac{1}{2}}\}}e^{i\left(pt-fx-\frac{2\pi\sin e}{T_1V_1}y\right)},\\
v_1''&=\frac{i\mathfrak{A}T_1V_1}{2\pi}\cdot\frac{\sin e(3-2\cos^2 e)^2}{\{(3-2\cos^2 e)^2+4\cos^2 e\sin e(3-\cos^2 e)^{\frac{1}{2}}\}}e^{i\left(pt-fx-\frac{2\pi\sin e}{T_1V_1}y\right)},
\end{aligned}\right\}\ldots(30)$$

$$\left.\begin{aligned}
u_2''&=-\frac{i\mathfrak{A}T_1V_1}{2\pi}\cdot\frac{2\sin e\cos e(3-2\cos^2 e)(3-\cos^2 e)^{\frac{1}{2}}}{\{(3-2\cos^2 e)^2+4\cos^2 e\sin e(3-\cos^2 e)^{\frac{1}{2}}\}}e^{i\left(pt-fx-\frac{2\sqrt{3}\pi\sin e}{T_1V_1}y\right)},\\
v_2''&=\frac{i\mathfrak{A}T_1V_1}{2\pi}\cdot\frac{2\sin e\cos^2 e(3-2\cos^2 e)}{\{(3-2\cos^2 e)^2+4\cos^2 e\sin e(3-\cos^2 e)^{\frac{1}{2}}\}}e^{i\left(pt-fx-\frac{2\sqrt{3}\pi\sin e}{T_1V_1}y\right)}.
\end{aligned}\right\}(31)$$

The maximum amplitudes of these waves[4] expressed by (28), (29), (30) and (31) are given in Table I and II in which the amplitudes of the incident waves are also given, and Fig. 2 and 3 show us graphically these amplitudes.

Table I.

Max. Amp. \ e	0°	10°	20°	30°	45°	60°	70°	80°	90°
u_1	1	0·9848	0·9397	0·8659	0·70711	0·5000	0·3421	0·1736	0
u_1'	0	0·4590	0·5250	0·4320	0·2518	0·0939	0·0325	0·00394	0
u_2'	0	0·2479	0·3525	0·4328	0·5065	0·4665	0·3585	0·1958	0
u_1''	1	0·5313	0·4353	0·4335	0·4535	0·4065	0·3115	0·1696	0
u_2''	0	−0·2479	−0·3525	−0·4328	−0·5065	−0·4665	−0·3585	−0·1958	0

Table II.

Max. Amp. \ e	0°	10°	20°	30°	45°	60°	70°	80°	90°
v_1	0	0·1736	0·3420	0·5000	0·7071	0·8660	0·9397	0·9848	1
v_1'	0	−0·0805	−0·1828	−0·2492	−0·2518	−0·1615	−0·08325	−0·0224	0
v_2'	0	0·1714	0·2275	0·2500	0·2265	0·1408	0·0724	0·0197	0
v_1''	0	0·0937	0·1587	0·2505	0·4535	0·7032	0·8562	0·9620	1
v_2''	0	0·1714	0·2275	0·2500	0·2265	0·1408	0·0724	0·0197	0

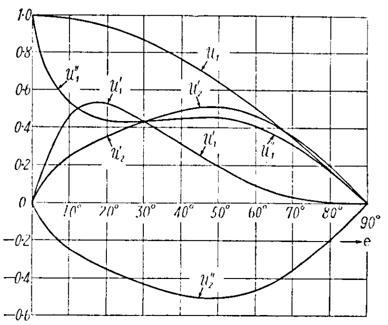

Fig. 2.

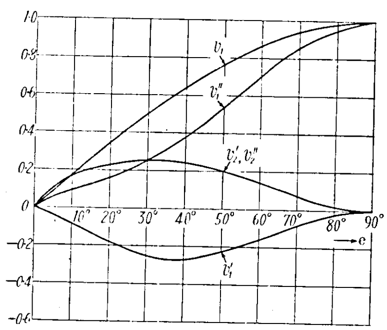

Fig. 3.

4) When the two solids adhere closely, these waves are never generated at the boundary surface.

These tables and figures show us many interesting facts, and some of them are summarised as follows:

1. When the incident wave is dilatational, reflected and transmitted waves are, in general, simultaneously generated at the contact surface, both of which consist of dilatational and distortional waves. When the two solids, however, adhere closely, these waves are never generated at the boundary.

2. When $e=0°$, the y-components of all generated waves vanish, and consequently there exist no reflected wave. And x-components of transmitted waves of dilatational type u_1'' of which the amplitude is equal to that of x-component of incident wave u_1 are generated at the contact surface of two solids.

3. When $e=90°$, x-components of all generated waves vanish, and wherefore there exist no reflected wave. And x-component of transmitted wave of dilatational type v_1'' of which the amplitude is equal to that of y-component of incident wave v_1 is generated at the contact surface of two solids.

b) *Incident Wave is Distortional.*

4. Putting the incident wave of distortional type, referring to Fig. 1,

$$2\varpi=\mathfrak{B}e^{i(pt-fx-sy)}, \qquad \cdots\cdots\cdots\cdots (32)$$

where $\mathfrak{B}$ is the amplitude of this wave, we obtain the corresponding $x-$, $y-$ components of displacement u_2, v_2 and the normal, tangential and shear components of stress $\widehat{xx}_2$, $\widehat{yy}_2$, $\widehat{xy}_2$ as in the following forms:

$$\left.\begin{aligned} u_2&=-\frac{is}{k^2}\mathfrak{B}e^{i(pt-fx-sy)},\\ v_2&=\frac{if}{k^2}\mathfrak{B}e^{i(pt-fx-sy)}, \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots (33)$$

$$\left.\begin{aligned} \widehat{xx}_2&=-\frac{2sf\mu}{k^2}\mathfrak{B}e^{i(pt-fx-sy)},\\ \widehat{yy}_2&=\frac{2sf\mu}{k^2}\mathfrak{B}e^{i(pt-fx-sy)},\\ \widehat{xy}_2&=\frac{(f^2-s^2)\mu}{k^2}\mathfrak{B}e^{i(pt-fx-sy)}, \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots (34)$$

where $p=\frac{2\pi}{T_2}$, T_2 being the period of incident wave expressed by (32).

Let the wave length and velocity of the incident wave be L_2 and V_2 respectively. Then we have

$$L_2 = V_2 T_2 = \frac{2\pi}{\sqrt{s^2+f^2}}. \qquad \dots\dots\dots\dots (35)$$

By the general expressions of reflected waves expressed by (6) and (7) and those of transmitted waves expressed by (13) and (14), we obtain the boundary conditions which must be satisfied at the boundary surface $y=0$ as follows:

When $y=0$,

$$\left.\begin{aligned}
&\widehat{yy_1}' + \widehat{yy_2}' - \widehat{yy_1}'' - \widehat{yy_2}'' = -\widehat{yy_2},\\
&v_1' + v_2' - v_1'' - v_2'' = -v_2,\\
&\widehat{xy_1}' + \widehat{xy_2}' = -\widehat{xy_2},\\
&\widehat{xy_1}'' + \widehat{xy_2}'' = 0.
\end{aligned}\right\} \dots\dots\dots\dots (36)$$

These conditions give us the following four equations to determine A', B', A'', B'':

$$\left.\begin{aligned}
&\frac{2fr'}{h'^2}A'' + \frac{(f^2-s'^2)}{k'^2}B'' = 0,\\
&-\frac{2fr}{h^2}A' + \frac{(f^2-s^2)}{k^2}B' = -\frac{(f^2-s^2)}{k^2}\mathfrak{B},\\
&-\frac{r}{h^2}A' + \frac{f}{k^2}B' - \frac{r'}{h'^2}A'' - \frac{f}{k'^2}B'' = -\frac{f}{k^2}\mathfrak{B},\\
&\frac{\mu}{h^2}(s^2-f^2)A' - \frac{2\mu sf}{k^2}B' - \frac{\mu'}{h'^2}(s'^2-f^2)A'' - \frac{2\mu's'f}{k'^2}B'' = -\frac{4sf\mu}{k^2}\mathfrak{B}.
\end{aligned}\right\} (37)$$

Solving these equations, we obtain

$$\left.\begin{aligned}
A' &= \mathfrak{B}\frac{4sr'fh^2(f^2-s^2)(f^2+s'^2)}{k^2\left[r'(s'^2+f^2)\{(s^2-f^2)^2+4srf^2\} + \frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]},\\
B' &= \mathfrak{B}\frac{\left[r'(s'^2+f^2)\{4srf^2-(s^2-f^2)^2\} - \frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]}{k^2\left[r'(s'^2+f^2)\{(s^2-f^2)^2+4srf^2\} + \frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]},\\
A'' &= \mathfrak{B}\frac{4srfh'^2(s^2+f^2)(s'^2-f^2)}{k^2\left[r'(s'^2+f^2)\{(s^2-f^2)^2+4srf^2\} + \frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]},\\
B'' &= \mathfrak{B}\frac{8srr'k'^2f^2(s^2+f^2)}{k^2\left[r'(s'^2+f^2)\{(s^2-f^2)^2+4srf^2\} + \frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]}.
\end{aligned}\right\} (38)$$

Substituting these values for A', B', A'', B'' in (9), (11), (17), (19), we obtain the displacements of reflected waves and those of transmitted waves which are due to the incident wave expressed by (32) as in the following forms:

Reflected distortional wave;

$$u_1' = \frac{if\mathfrak{B}}{h^2} \frac{4sr'fh^2(f^2-s^2)(f^2+s'^2)}{k^2\left[r'(s'^2+f^2)\{(s^2-f^2)^2+4srf^2\}+\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]} \times e^{i(pt-fx+ry)},$$

$$v_1' = -\frac{ir\mathfrak{B}}{h^2} \frac{4sr'fh^2(f^2-s^2)(f^2+s'^2)}{k^2\left[r'(s'^2+f^2)\{(s^2-f^2)^2+4srf^2\}+\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]} \times e^{i(pt-fx+ry)}, \quad \ldots\ldots\ldots\ldots\ldots(39)$$

Reflected distortional wave;

$$u_2' = \frac{is\mathfrak{B}}{k^2} \frac{\left[r'(s'^2+f^2)\{4srf^2-(s^2-f^2)^2\}-\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]}{k^2\left[r'(s'^2+f^2)\{(s^2-f^2)^2+4srf^2\}+\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]} \times e^{i(pt-fx+sy)},$$

$$v_2' = \frac{if\mathfrak{B}}{k^2} \frac{\left[r'(s'^2+f^2)\{4srf^2-(s^2-f^2)^2\}-\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]}{k^2\left[r'(s'^2+f^2)\{(s^2-f^2)^2+4srf^2\}+\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]} \times e^{i(pt-fx+sy)}, \quad \ldots\ldots\ldots\ldots\ldots(40)$$

Transmitted dilatational wave;

$$u_1'' = \frac{if\mathfrak{B}}{h'^2} \frac{4srfh'^2(s^2+f^2)(s'^2-f^2)}{k^2\left[r'(s'^2+f^2)\{(s^2-f^2)^2+4srf^2\}+\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]} \times {}^{i(pt-fx-r'y)},$$

$$v_1'' = \frac{ir'\mathfrak{B}}{h'^2} \frac{4srfh'^2(s^2+f^2)(s'^2-f^2)}{k^2\left[r'(s'^2+f^2)\{(s^2-f^2)^2+4srf^2\}+\frac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]} \times {}^{i(pt-fx-r'y)}, \quad \ldots\ldots\ldots\ldots\ldots(41)$$

Transmitted distortional wave;

$$
\left.
\begin{aligned}
u_2'' &= -\frac{is'\mathfrak{B}}{k'^2}\frac{8srr'k'^2f^2(s^2+f^2)}{k^2\left[r'(s'^2+f^2)\{(s^2-f^2)^2+4srf^2\}+\dfrac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]} \\
&\qquad\qquad \times e^{i(pt-fx-s'\ )}, \\
v_2'' &= \frac{if\mathfrak{B}}{k'^2}\frac{8srr'k'^2f^2(s^2+f^2)}{k^2\left[r'(s'^2+f^2)\{(s^2-f^2)^2+4srf^2\}+\dfrac{\mu'}{\mu}r(s^2+f^2)\{(s'^2-f^2)^2+4s'r'f^2\}\right]} \\
&\qquad\qquad \times e^{i(pt-fx-s'y)}.
\end{aligned}
\right\}
$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

Let the emergency angle of incident distortional wave expressed by (32) be e. Then (39), (40), (41) and (42) are transformed into the following forms:

$$
\left.
\begin{aligned}
u_1' &= \frac{i\mathfrak{B}T_2V_2}{2\pi}\cdot\frac{4\cos^2 e\sin e\left(\dfrac{V_2}{V_2'}\right)^2(\cos^2 e-\sin^2 e)\left\{\left(\dfrac{V_2}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}}{\Phi}e^{i(pt-fx+ry)}, \\
v_1' &= -\frac{i\mathfrak{B}T_2V_2}{2\pi} \\
&\times\frac{4\cos e\sin e\left(\dfrac{V_2}{V_2'}\right)^2(\cos^2 e-\sin^2 e)\left\{\left(\dfrac{V_2}{V_1}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left\{\left(\dfrac{V_2}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}}{\Phi} \\
&\qquad\qquad \times e^{i(pt-fx+ry)},
\end{aligned}
\right\}(39)'
$$

$$
\left.
\begin{aligned}
u_2' &= \frac{i\mathfrak{B}T_2V_2}{2\pi}\cdot\frac{\Psi\sin e}{\Phi}e^{i(pt-fx+sy)}, \\
v_2' &= \frac{i\mathfrak{B}T_2V_2}{2\pi}\cdot\frac{\Psi\cos e}{\Phi}e^{i(pt-fx+sy)},
\end{aligned}
\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots (40)'
$$

$$
\left.
\begin{aligned}
u_1'' &= \frac{i\mathfrak{B}T_2V_2}{2\pi}\cdot\frac{4\sin e\cos^2 e\left\{\left(\dfrac{V_2}{V_2'}\right)^2-2\cos^2 e\right\}\left\{\left(\dfrac{V_2}{V_1}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}}{\Phi}e^{i(pt-fx-r'y)}, \\
v_1'' &= \frac{i\mathfrak{B}T_2V_2}{2\pi}\cdot\frac{4\sin e\cos e\left\{\left(\dfrac{V_2}{V_2'}\right)^2-2\cos^2 e\right\}\left\{\left(\dfrac{V_2}{V_1}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left\{\left(\dfrac{V_1}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}}{\Phi} \\
&\qquad\qquad \times e^{i(pt-fx-r'y)},
\end{aligned}
\right\}
$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41)'$$

$$\left.\begin{aligned} u_2'' &= -\frac{i\mathfrak{B}T_2V_2}{2\pi} \\ &\times \frac{8\sin e\cos^2 e\left\{\left(\frac{V_2}{V_1}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left\{\left(\frac{V_2}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left\{\left(\frac{V_2}{V_2'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}}{\Phi} \\ &\qquad\qquad \times e^{i(pt-fx-s'y)}, \\ v_2'' &= \frac{i\mathfrak{B}T_2V_2}{2\pi}\,\frac{8\sin e\cos^3 e\left\{\left(\frac{V_2}{V_1}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left\{\left(\frac{V_2}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}}{\Phi}\,e^{i(pt-fx-s'y)}, \end{aligned}\right\} \quad (42)'$$

where

$$\begin{aligned} \Phi = &\left\{\left(\frac{V_2}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left(\frac{V_2}{V_2'}\right)^2\left[4\cos^2 e\sin e\left\{\left(\frac{V_2}{V_1}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}+(\sin^2 e-\cos^2 e)^2\right] \\ &+\frac{\mu'}{\mu}\left\{\left(\frac{V_2}{V_1}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left[\left\{\left(\frac{V_2}{V_2'}\right)^2-2\cos^2 e\right\}^2\right. \\ &\left.+4\cos^2 e\left\{\left(\frac{V_2}{V_2'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left\{\left(\frac{V_2}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\right], \end{aligned}$$

$$\begin{aligned} \Psi = &\left\{\left(\frac{V_2}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left(\frac{V_2}{V_2'}\right)^2\left[4\cos^2 e\sin e\left\{\left(\frac{V_2}{V_1}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}-(\sin^2 e-\cos^2 e)^2\right] \\ &-\frac{\mu'}{\mu}\left\{\left(\frac{V_2}{V_1}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left[\left\{\left(\frac{V_2}{V_2'}\right)^2-2\cos^2 e\right\}^2\right. \\ &\left.+4\cos^2 e\left\{\left(\frac{V_2}{V_2'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\left\{\left(\frac{V_2}{V_1'}\right)^2-\cos^2 e\right\}^{\frac{1}{2}}\right]. \end{aligned}$$

When $\rho=\rho'$ and the respective Poisson's ratios of two media are $\frac{1}{4}$,

$$u_1' = \frac{i\mathfrak{B}T_2V_2}{2\pi}\,\frac{4\frac{\mu}{\mu'}\cos^2 e\sin e(\cos^2 e-\sin^2 e)\left(\frac{1}{3}\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}}{\left[\frac{\mu}{\mu'}\left(\frac{1}{3}\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{4\cos^2 e\sin e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}+(\sin^2 e-\cos^2 e)^2\right\}\right.}$$

$$\frac{e^{i(pt-fx+ry)}}{\left.+\frac{\mu'}{\mu}\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\left\{\left(\frac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\frac{1}{3}\frac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\}\right]},$$

$$v_1' = -\frac{i\mathfrak{B}T_2V_2}{2\pi}\,\frac{4\dfrac{\mu}{\mu'}\cos e\sin e(\cos^2 e-\sin^2 e)\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac12}\left(\dfrac13\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}}{\left[\dfrac{\mu}{\mu'}\left(\dfrac13\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\left\{4\cos^2 e\sin e\left(\dfrac13-\cos^2 e\right)^{\frac12}+(\sin^2 e-\cos^2 e)^2\right\}\right.}$$
$$\frac{e^{i(pt-fx+ry)}}{\left.+\dfrac{\mu'}{\mu}\left(\dfrac13-\cos^2 e\right)^{\frac12}\left\{\left(\dfrac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\left(\dfrac13\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\right\}\right]},$$

$$\dots\dots\dots\dots\dots\dots\dots(39)''$$

$$u_2' = \frac{i\mathfrak{B}T_2V_2}{2\pi}$$
$$\times\frac{\sin e\left[\dfrac{\mu}{\mu'}\left(\dfrac13\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\left\{4\cos^2 e\sin e\left(\dfrac13-\cos^2 e\right)^{\frac12}-(\sin^2 e-\cos^2 e)^2\right\}\right.}{\left[\dfrac{\mu}{\mu'}\left(\dfrac13\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\left\{4\cos^2 e\sin e\left(\dfrac13-\cos^2 e\right)^{\frac12}+(\sin^2 e-\cos^2 e)^2\right\}\right.}$$
$$\frac{\left.-\dfrac{\mu'}{\mu}\left(\dfrac13-\cos^2 e\right)^{\frac12}\left\{\left(\dfrac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\left(\dfrac13\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\right\}\right]}{\left.+\dfrac{\mu'}{\mu}\left(\dfrac13-\cos^2 e\right)^{\frac12}\left\{\left(\dfrac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\left(\dfrac13\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\right\}\right]}$$
$$\times e^{i(pt-fx+sy)},$$

$$v_2' = \frac{i\mathfrak{B}T_2V_2}{2\pi}$$
$$\times\frac{\cos e\left[\dfrac{\mu'}{\mu}\left(\dfrac13\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\left\{4\cos^2 e\sin e\left(\dfrac13-\cos^2 e\right)^{\frac12}-(\sin^2 e-\cos^2 e)^2\right\}\right.}{\left[\dfrac{\mu}{\mu'}\left(\dfrac13\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\left\{4\cos^2 e\sin e\left(\dfrac13-\cos^2 e\right)^{\frac12}+(\sin^2 e-\cos^2 e)^2\right\}\right.}$$
$$\frac{\left.-\dfrac{\mu'}{\mu}\left(\dfrac13-\cos^2 e\right)^{\frac12}\left\{\left(\dfrac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\left(\dfrac13\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\right\}\right]}{\left.+\dfrac{\mu'}{\mu}\left(\dfrac13-\cos^2 e\right)^{\frac12}\left\{\left(\dfrac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\left(\dfrac13\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac12}\right\}\right]}$$
$$\times e^{i(pt-fx+sy)},$$

$$\dots\dots\dots\dots\dots\dots\dots(40)''$$

$$\left.\begin{aligned}
u_1'' &= \frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{4\sin e\cos^2 e\left(\dfrac{\mu}{\mu'}-2\cos^2 e\right)\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}}{\left[\dfrac{\mu}{\mu'}\left(\dfrac{1}{3}\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{4\cos^2 e\sin e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}+(\sin^2 e-\cos^2 e)^2\right\}\right.} \\
&\qquad \frac{e^{i(pt-fx-r'y)}}{\left.+\dfrac{\mu'}{\mu}\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\left\{\left(\dfrac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\dfrac{1}{3}\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\}\right]}, \\
v_1'' &= \frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{4\sin e\cos e\left(\dfrac{\mu}{\mu'}-2\cos^2 e\right)\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\left(\dfrac{1}{3}\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}}{\left[\dfrac{\mu}{\mu'}\left(\dfrac{1}{3}\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{4\cos^2 e\sin e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}+(\sin^2 e-\cos^2 e)^2\right\}\right.} \\
&\qquad \frac{e^{i(pt-fx-r'y)}}{\left.+\dfrac{\mu'}{\mu}\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\left\{\left(\dfrac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\dfrac{1}{3}\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\}\right]},
\end{aligned}\right\} \qquad \ldots\ldots(41)''$$

$$\left.\begin{aligned}
u_2'' &= -\frac{i\mathfrak{B}T_2V_2}{2\pi} \\
&\times\frac{8\sin e\cos^2 e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\left(\dfrac{1}{3}\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}}{\left[\dfrac{\mu}{\mu'}\left(\dfrac{1}{3}\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{4\cos^2 e\sin e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}+(\sin^2 e-\cos^2 e)^2\right\}\right.} \\
&\qquad \frac{e^{i(pt-fx-s'y)}}{\left.+\dfrac{\mu'}{\mu}\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\left\{\left(\dfrac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\dfrac{1}{3}\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\}\right]}, \\
v_2'' &= \frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{8\sin e\cos^3 e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\left(\dfrac{1}{3}\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}}{\left[\dfrac{\mu}{\mu'}\left(\dfrac{1}{3}\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left\{4\cos^2 e\sin e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}+(\sin^2 e-\cos^2 e)^2\right\}\right.} \\
&\qquad \frac{e^{i(pt-fx-s'y)}}{\left.+\dfrac{\mu'}{\mu}\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\left\{\left(\dfrac{\mu}{\mu'}-2\cos^2 e\right)^2+4\cos^2 e\left(\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\left(\dfrac{1}{3}\dfrac{\mu}{\mu'}-\cos^2 e\right)^{\frac{1}{2}}\right\}\right]}.
\end{aligned}\right\} \qquad \ldots\ldots(42)''$$

When $\frac{\mu'}{\mu}=1$, expressions (39)″, (40)″, (41)″, (42)″ are rewritten by

$$\left.\begin{aligned} u_1'&=-\frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{2\cos^2 e\sin e(\sin^2 e-\cos^2 e)}{\left\{(\sin^2 e-\cos^2 e)^2+4\cos^2 e\sin e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}e^{i\left(pt-fx+\frac{2\pi\sin e}{\sqrt{3}T_2V_2}y\right)},\\ v_1'&=\frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{2\cos e\sin e(\sin^2 e-\cos^2 e)\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^2+4\cos^2 e\sin e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}e^{i\left(pt-fx+\frac{2\pi\sin e}{\sqrt{3}T_2V_2}y\right)}, \end{aligned}\right\} \quad \ldots\ldots\ldots(43)$$

$$\left.\begin{aligned} u_2'&=-\frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{\sin e(\sin^2 e-\cos^2 e)^2}{\left\{(\sin^2 e-\cos^2 e)^2+4\cos^2 e\sin e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}e^{i\left(pt-fx+\frac{2\pi\sin e}{T_2V_2}y\right)},\\ v_2'&=-\frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{\cos e(\sin^2 e-\cos^2 e)^2}{\left\{(\sin^2 e-\cos^2 e)^2+4\cos^2 e\sin e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}e^{i\left(pt-fx+\frac{2\pi\sin e}{T_2V_2}y\right)}, \end{aligned}\right\} \quad \ldots\ldots\ldots(44)$$

$$\left.\begin{aligned} u_1''&=\frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{2\sin e\cos^2 e(\sin^2 e-\cos^2 e)}{\left\{(\sin^2 e-\cos^2 e)^2+4\cos^2 e\sin e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}e^{i\left(pt-fx-\frac{2\pi\sin e}{\sqrt{3}T_2V_2}y\right)},\\ v_1''&=\frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{2\sin e\cos e(\sin^2 e-\cos^2 e)\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^2+4\cos^2 e\sin e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}e^{i\left(pt-fx-\frac{2\pi\sin e}{\sqrt{3}T_2V_2}y\right)}, \end{aligned}\right\} \quad \ldots\ldots\ldots(45)$$

$$\left.\begin{aligned} u_2''&=-\frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{4\sin^2 e\cos^2 e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^2+4\cos^2 e\sin e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}e^{i\left(pt-fx-\frac{2\pi\sin e}{T_2V_2}y\right)},\\ v_2''&=\frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{4\sin e\cos^3 e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^2+4\cos^2 e\sin e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}e^{i\left(pt-fx-\frac{2\pi\sin e}{T_2V_2}y\right)}. \end{aligned}\right\} \quad \ldots\ldots\ldots(46)$$

When the emergency angle is smaller than 54°44′, the expressions (43), (44), (45), (46) should be replaced by the following formulae:

$$\left.\begin{aligned} u_1' &= -\frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{2\cos^2 e\sin e(\sin^2 e-\cos^2 e)}{\left\{(\sin^2 e-\cos^2 e)^4+16\cos^4 e\sin^2 e\left(\cos^2 e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}} \times e^{i\left(pt-fx+\frac{2\pi\sin e}{\sqrt{3}\,T_2V_2}y-\varphi\right)},\\ v_1' &= \frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{2\cos e\sin e(\sin^2 e-\cos^2 e)\left(\cos^2 e-\frac{1}{3}\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^4+16\cos^4 e\sin^2 e\left(\cos^2 e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}} \times e^{i\left(pt-fx+\frac{2\pi\sin e}{\sqrt{3}\,T_2V_2}y-\varphi+\frac{\pi}{2}\right)}, \end{aligned}\right\}\quad\text{..(47)}$$

$$\left.\begin{aligned} u_2' &= -\frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{\sin e(\sin^2 e-\cos^2 e)^2}{\left\{(\sin^2 e-\cos^2 e)^4+16\cos^4 e\sin^2 e\left(\cos^2 e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}} \times e^{i\left(pt-fx+\frac{2\pi\sin e}{T_2V_2}y-\varphi\right)},\\ v_2' &= -\frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{\cos e(\sin^2 e-\cos^2 e)^2}{\left\{(\sin^2 e-\cos^2 e)^4+16\cos^4 e\sin^2 e\left(\cos^2 e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}} \times e^{i\left(pt-fx+\frac{2\pi\sin e}{T_2V_2}y-\varphi\right)}, \end{aligned}\right\}\quad\text{..(48)}$$

$$\left.\begin{aligned} u_1'' &= \frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{2\sin e\cos^2 e(\sin^2 e-\cos^2 e)}{\left\{(\sin^2 e-\cos^2 e)^4+16\cos^4 e\sin^2 e\left(\cos^2 e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}} \times e^{i\left(pt-fx-\frac{2\pi\sin e}{\sqrt{3}\,T_2V_2}y-\varphi\right)},\\ v_1'' &= \frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{2\sin e\cos e(\sin^2 e-\cos^2 e)\left(\cos^2 e-\frac{1}{3}\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^4+16\cos^4 e\sin^2 e\left(\cos^2 e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}} \times e^{i\left(pt-fx-\frac{2\pi\sin e}{\sqrt{3}\,T_2V_2}y-\varphi+\frac{\pi}{2}\right)}, \end{aligned}\right\}\quad\text{..(49)}$$

$$\left.\begin{aligned} u_2''&=-\frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{4\sin^2 e\cos^2 e\left(\cos^2 e-\frac{1}{3}\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^4+16\cos^4 e\sin^2 e\left(\cos^2 e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}} \\ &\qquad\times e^{i\left(pt-fx-\frac{2\pi\sin e}{T_2V_2}y-\varphi+\frac{\pi}{2}\right)}, \\ v_2''&=\frac{i\mathfrak{B}T_2V_2}{2\pi}\frac{4\sin e\cos^3 e\left(\cos^2 e-\frac{1}{3}\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^4+16\cos^4 e\sin^2 e\left(\cos^2 e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}} \\ &\qquad\times e^{i\left(pt-fx-\frac{2\pi\sin e}{T_2V_2}y-\varphi+\frac{\pi}{2}\right)}, \end{aligned}\right\}\quad\ldots(50)$$

where $$\varphi=\tan^{-1}\left\{\frac{4\sin e\cos^2 e\left(\cos^2 e-\frac{1}{3}\right)^{\frac{1}{2}}}{(\sin^2 e-\cos^2 e)^2}\right\}.$$

By the numerical calculations we obtain the following Table III, IV, in which the maximum amplitudes of the generated waves expressed by (43), (44), (45), (46), (47), (48), (49), (50), together with the amplitude of incident wave expressed by (33) are tabulated. (47), (48), (49), (50) shew us that when the emergency angle e is smaller than 54°44′, the

Table III.

Max. Amp. \ e	0°	10°	20°	30°	40°	45°	50°	52°30′	53°30′	54°
u_2	0	−0·1736	−0·3420	−0·5000	−0·6428	−0·7071	−0·7661	−0·7935	−0·8038	−0·8090
u_1'	0	0·3061	0·4315	0·375	0·1722	0	−0·3057	−0·645	−0·903	−1·106
u_2'	0	−0·1481	−0·1875	−0·125	−0·0255	0	−0·0643	−0·226	−0·374	−0·4435
u_1''	0	−0·3061	−0·4315	−0·375	−0·1722	0	0·3057	0·645	0·903	1·106
u_2''	0	−0·0902	−0·2865	−0·485	−0·643	−0·7071	−0·763	−0·76	−0·711	−0·637

Max. Amp. \ e	54°44′	55°	56°30′	57°30′	60°	65°	70°	80°	90°
u_2	−0·8165	−0·820	−0·8339	−0·8435	−0·866	−0·9064	−0·9397	−0·9848	−1
u_1'	−1·634	−0·997	−0·6115	−0·534	−0·433	−0·3115	−0·213	−0·0588	0
u_2'	−0·8165	−0·520	−0·392	−0·392	−0·433	−0·5605	−0·699	−0·919	−1
u_1''	1·634	0·997	0·6115	0·534	0·433	0·3115	0·213	0·0588	0
u_2''	0	−0·2993	−0·442	−0·4535	−0·433	−0·3455	−0·242	−0·0679	0

Table IV.

Max. Amp. \ e	0°	10°	20°	30°	40°	45°	50°	52°30′	53°30′	54°
v_2	1	0·9848	0·9397	0·8659	0·7661	0·7071	0·6428	0·6086	0·5949	0·5878
v_1'	0	−0·2480	−0·3405	−0·279	−0·1133	0	0·1345	0·204	0·217	0·2075
v_2'	−1	−0·8430	−0·515	−0·2165	−0·0304	0	−0·054	−0·173	−0·276	−0·3225
v_1''	0	−0·2480	−0·3405	−0·279	−0·1133	0	0·1345	0·204	0·217	0·2075
v_2''	0	0·5119	0·7875	0·840	0·766	0·7071	0·641	0·582	0·527	0·463

Max. Amp. \ e	54°44′	55°	56°30′	57°30′	60°	65°	70°	80°	90°
v_2	0·5773	0·574	0·552	0·5372	0·500	0·4224	0·3421	0·1736	0
v_1'	0	0·110	0·1875	0·2105	0·25	0·2905	0·2893	0·1867	0
v_2'	−0·5773	−0·364	−0·2595	−0·25	−0·25	−0·2615	−0·254	−0·1612	0
v_1''	0	0·110	0·1875	0·2105	0·25	0·2905	0·2893	0·1867	0
v_2''	0	0·210	0·2925	0·2885	0·25	0·1613	0·0883	0·0119	0

Table V.

e	0°	10°	20°	30°	40°	45°	50°	52°30′	54°44′	60°	70°	80°	90°
φ	0	0·546	0·992	1·318	1·531	1·570	1·487	1·282	0	0	0	0	0

waves generated by the incident distortional wave accumulate on the surface $y=0$, and have the phase differences which vary with the emergency angle e. The phase difference φ is also calculated and tabulated in Table V, and graphically shewn in Fig. 6.

From Fig. 4, 5 and 6 in which the values in Table III, IV, V are graphically shewn, we can see many remarkable facts, and some of them are summarised as follows:

1. When the incident wave is distortional, reflected and transmitted waves are, in general, simultaneously generated at the contact surface, both of which consist of dilatational and distortional waves. When the two solids, however, adhere closely, these waves are never generated at this boundary.

2. When $e=0°$, x-components of all generated waves vanish, and moreover there exist no transmitted wave. The amplitude of y-component of reflected wave (v_2'), in this case, is equal to that of incident wave v_2.

3. When $0°<e<54°44'$, all generated waves become to have the properties of surface waves of which the energies are accumulated on

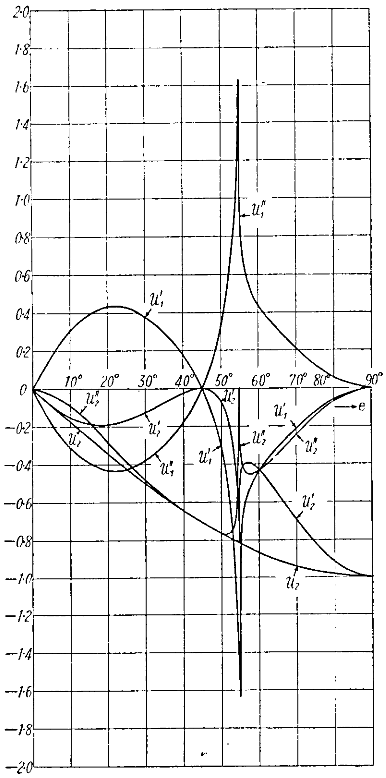

Fig. 4.

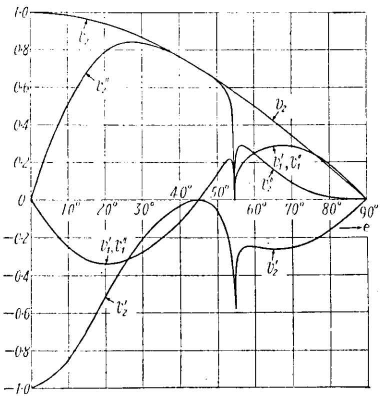

Fig. 5.

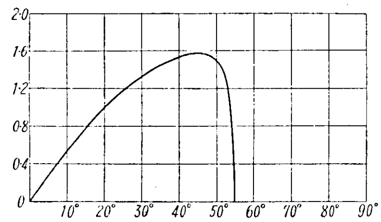

Fig. 6.

the contact surface mainly. In this case, the phase differences of which the magnitudes depend upon the magnitude of the emergency angle of the incident wave exist between these surface waves and the incident wave. (See (47), (48), (49), (50) and Fig. 6.)

4. When $e=45°$, the transmitted distortional wave (u_2'', v_2'') are only generated, and the other waves such as transmitted dilatational wave, and reflected dilatational and distortional ones vanish.

5. When $e\fallingdotseq 54°44'$, the amplitude of the x-component of reflected dilatational wave (u_1') and that of transmitted dilatational wave (u_1'') become very large, and that of the reflected distortional one (u_2') is also moderately large. In this case, however, there is no x-component of transmitted distortional wave (u_2''). And the amplitude of y-component of reflected dilatational wave (v_1'), and those of the transmitted dilatational (v_1'') and distortional waves (v_2'') are zero, and in the contrary the amplitude of y-component of reflected distortional wave

(v_2') is moderately large.

6. When $e=90°$, the y-components of the generated waves vanish and there exist no transmitted waves.

Part II. Initial Movement of the Free Surface of a Stratified Solid.

5. We intend to investigate the problems that when the transmitted waves generated by the waves, which are primarily incident obliquely to the bottom surface $x=0$ of a stratified layer residing on the surface of a semi-infinite elastic solid, are incident to the free surface $y=H$, what kinds of waves are generated at this free surface, and what movements should the free surface take at initial stage. It is not necessary in the present study, therefore, to consider the effect of multiple reflections of waves at $y=H$ and $y=0$. Of course, at the boundary surface the two elastic solids slide upon each other without any friction.

Before entering this investigation, we shall study the following two problems, as the preliminary calculation, relating to the reflections of the waves which are generated by a transmitted wave when it is incident to the free surface $y=H$.

a) Transmitted wave is dilatational.

b) Transmitted wave is distortional.

a) Transmitted Wave is Dilatational.

6. Let the transmitted wave, which is generated by a primary wave of any type when it is incident to the boundary surface, be a dilatational one which is expressed by

$$\Delta_U = A_U e^{i(pt-fx-r'y)}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots (51)$$

Then the corresponding displacement u_{1U}, v_{1U} and the stress $\widehat{xx}_{1U}$, $\widehat{yy}_{1U}$, $\widehat{xy}_{1U}$ are expressed by

$$\left.\begin{aligned} u_{1U} &= \frac{if(\lambda'+2\mu')}{\rho' p^2} A_U e^{i(pt-fx-r'y)}, \\ v_{1U} &= \frac{ir'(\lambda'+2\mu')}{\rho' p^2} A_U e^{i(pt-fx-r'y)}, \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots (52)$$

$$\left.\begin{aligned} \widehat{xx}_{1U} &= \left\{\lambda' + \frac{2f^2\mu'(\lambda'+2\mu')}{\rho' p^2}\right\} A_U e^{i(pt-fx-r'y)}, \\ \widehat{yy}_{1U} &= \left\{\lambda' + \frac{2r'^2\mu'(\lambda'+2\mu')}{\rho' p^2}\right\} A_U e^{i(pt-fx-r'y)}, \end{aligned}\right\} \qquad \ldots\ldots (53)$$

$$\widehat{xy}_{1U}=\frac{2fr'\mu'(\lambda'+2\mu')}{\rho'p^2}A_U e^{i(pt-fx-r'y)}, \Big\}$$

where A_U is a constant determined by the boundary conditions at $y=0$, and is a certain function of the emergency angle, the velocity and the period of a primary wave incident to the plane $y=0$.

Now we know the reflected waves of two types such that

$$\Delta_U'=A_U'e^{i(pt-fx+r'y)}, \qquad \text{.................}(54)$$

$$2\varpi_U'=B_U'e^{i(pt-fx+s'y)}, \qquad \text{.................}(55)$$

where A_U', B_U' are arbitrary constants and

$$r'^2=h'^2-f^2,\quad s'^2=k'^2-f^2,\quad h'^2=\frac{\rho'p^2}{(\lambda'+2\mu')},\quad k'^2=\frac{\rho'p^2}{\mu'}.$$

The displacement u_{1U}', v_{1U}' and the stress $\widehat{xx}_{1U}'$, $\widehat{yy}_{1U}'$, $\widehat{xy}_{1U}'$ corresponding to Δ_U' are written as follows:

$$\left.\begin{aligned} u_{1U}'&=\frac{if(\lambda'+2\mu')}{\rho'p^2}A_U'e^{i(pt-fx+r'y)},\\ v_{1U}'&=-\frac{ir'(\lambda'+2\mu')}{\rho'p^2}A_U'e^{i(pt-fx+r'y)}, \end{aligned}\right\} \qquad \text{............}(56)$$

$$\left.\begin{aligned} \widehat{xx}_{1U}'&=\left\{\lambda'+\frac{2f^2\mu'(\lambda'+2\mu')}{\rho'p^2}\right\}A_U'e^{i(pt-fx+r'y)},\\ \widehat{yy}_{1U}'&=\left\{\lambda'+\frac{2r'^2\mu'(\lambda'+2\mu')}{\rho'p^2}\right\}A_U'e^{i(pt-fx+r'y)},\\ \widehat{xy}_{1U}'&=-\frac{2fr'\mu'(\lambda'+2\mu')}{\rho'p^2}A_U'e^{i(pt-fx+r'y)}. \end{aligned}\right\} \qquad \text{......}(57)$$

The displacement u_{2U}', v_{2U}' and the stress $\widehat{xx}_{2U}'$, $\widehat{yy}_{2U}'$, $\widehat{xy}_{2U}'$ corresponding to $2\varpi'_U$ expressed by (55) are expressed by

$$\left.\begin{aligned} u_{2U}'&=\frac{is'\mu'}{\rho'p^2}B_U'e^{i(pt-fx+s'y)},\\ v_{2U}'&=\frac{if\mu'}{\rho'p^2}B_U'e^{i(pt-fx+s'y)}, \end{aligned}\right\} \qquad \text{..................}(58)$$

$$\left.\begin{aligned} \widehat{xx}_{2U}'&=\frac{2s'f\mu'^2}{\rho'p^2}B_U'e^{i(pt-fx+s'y)},\\ \widehat{yy}_{2U}'&=-\frac{2s'f\mu'^2}{\rho'p^2}B_U'e^{i(pt-fx+s'y)},\\ \widehat{xy}_{2U}'&=\frac{(f^2-s'^2)\mu'^2}{\rho'p^2}B_U'e^{i(pt-fx+s'y)}. \end{aligned}\right\} \qquad \text{............}(59)$$

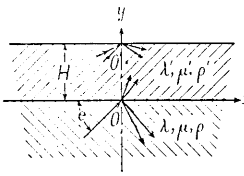

Fig. 7.

Using (54) and (55), we shall adjust A_U' and B_U' to make shear stress on the free surface $y=H$ due to the incident wave expressed by (51) be nil and to make a certain normal stress and tangential stress existent on this surface. The boundary conditions thus confined at $y=H$ are written as follows:

When $y=H$,

$$\left.\begin{aligned}\widehat{xy}_{1U}+\widehat{xy}_{1U}'+\widehat{xy}_{2U}'&=0,\\ \widehat{yy}_{1U}'+\widehat{yy}_{2U}'&=\widehat{yy}_{1U}.\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\quad(60)$$

These conditions give us two equations to determine the values of A_U' and B_U' such that

$$\left.\begin{aligned}&\frac{2fr'}{h'^2}A_U'e^{ir'H}-\frac{(f^2-s'^2)}{k'^2}B_U'e^{is'H}=\frac{2fr'}{h'^2}A_Ue^{-ir'H},\\ &\left(\frac{\lambda'}{\mu'}+\frac{2r'^2}{h'^2}\right)A_U'e^{ir'H}-\frac{2s'f}{k'^2}B_U'e^{is'H}=\left(\frac{\lambda'}{\mu'}+\frac{2r'^2}{h'^2}\right)A_Ue^{-ir'H}.\end{aligned}\right\}\cdots(61)$$

Solving these equations, we obtain

$$\left.\begin{aligned}A_U'&=A_Ue^{-2ir'H},\\ B_U'&=0.\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots(62)$$

Substituting (62) in (56), (58), we get

$$\left.\begin{aligned}u_{1U}'&=\frac{if}{h'^2}A_Ue^{i(pt-fx+r'y-2r'H)},\\ v_{1U}'&=-\frac{ir'}{h'^2}A_Ue^{i(pt-fx+r'y-2r'H)},\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots(63)$$

$$\left.\begin{aligned}u_{2U}'&=0,\\ v_{2U}'&=0.\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(64)$$

To make shear stress at $y=H$ due to the incident wave (51) be nil, it is sufficient to use the reflected waves of dilatational type expressed by (63) only, and there is no necessity to take distortional wave (55) of reflected type. And the normal stress in the medium II due to (51) and (54) is expressed by

$$\widehat{yy}_U=\frac{\mu'}{h'^2}(k'^2-2h'^2+2r'^2)A_U\{e^{-ir'y}+e^{i(r'y-2r'H)}\}e^{i(pt-fx)}.\quad\cdots(65)$$

Next, to make the normal stress (65) at the plane $y=H$ be nil in order to make the surface $y=H$ be completely free from traction, we use the following two waves which predominate on the surface $y=H$ such that

$$\Delta_U''=A_U''e^{r''(y-H)}e^{i(pt-fx)}, \qquad \text{(66)}$$

$$2\varpi_U''=B_U''e^{s''(y-H)}e^{i(pt-fx)}, \qquad \text{(67)}$$

where A_U'' and B_U'' are arbitrary constants and

$$r''^2=f^2-h'^2, \quad s''^2=f^2-k'^2. \qquad \text{(68)}$$

The displacement u_{1U}'', v_{1U}'' and the stress $\widehat{xx}_{1U}''$, $\widehat{yy}_{1U}''$, $\widehat{xy}_{1U}''$ corresponding to (66), and the displacement u_{2U}'', v_{2U}'' and the stress $\widehat{xx}_{2U}''$, $\widehat{yy}_{2U}''$, $\widehat{xy}_{2U}''$ corresponding to (67) are easily obtained as follows:

$$\left.\begin{aligned} u_{1U}''&=\frac{if}{h'^2}A_U''e^{r''(y-H)+i(pt-fx)},\\ v_{1U}''&=-\frac{r''}{h'^2}A_U''e^{r''(y-H)+i(pt-fx)}, \end{aligned}\right\} \qquad \text{(69)}$$

$$\left.\begin{aligned} \widehat{xx}_{1U}''&=\left(\lambda'+2\mu'\frac{f^2}{h'^2}\right)A_U''e^{r''(y-H)+i(pt-fx)},\\ \widehat{yy}_{1U}''&=\left(\lambda'-2\mu'\frac{r''^2}{h'^2}\right)A_U''e^{r''(y-H)+i(pt-fx)},\\ \widehat{xy}_{1U}''&=\frac{2i\mu'fr''}{h'^2}A_U''e^{r''(y-H)+i(pt-fx)}, \end{aligned}\right\} \qquad \text{(70)}$$

$$\left.\begin{aligned} u_{2U}''&=\frac{s''}{k'^2}B_U''e^{s''(y-H)+i(pt-fx)},\\ v_{2U}''&=\frac{if}{k'^2}B_U''e^{s''(y-H)+i(pt-fx)}, \end{aligned}\right\} \qquad \text{(71)}$$

$$\left.\begin{aligned} \widehat{xx}_{2U}''&=-\frac{2i\mu'fs''}{k'^2}B_U''e^{s''(y-H)+i(pt-fx)},\\ \widehat{yy}_{2U}''&=\frac{2i\mu'fs''}{k'^2}B_U''e^{s''(y-H)+i(pt-fx)},\\ \widehat{xy}_{2U}''&=\frac{\mu'}{k'^2}(f^2+s''^2)B_U''e^{s''(y-H)+i(pt-fx)}. \end{aligned}\right\} \qquad \text{(72)}$$

Making the shear stress on the surface $y=H$ due to (66) and (67) be nil, we obtained the following relations between A_U'' and B_U'' such that

$$B_U''=-\frac{2ifr''k'^2}{h'^2(f^2+s''^2)}A_U''. \qquad \text{(73)}$$

Then (69) and (71) give us

$$\left.\begin{aligned} u_{1U}'' &= \frac{if}{h'^2} A_U'' e^{r''(y-H)+i(pt-fx)}, \\ v_{1U}'' &= -\frac{r''}{h'^2} A_U'' e^{r''(y-H)+i(pt-fx)}, \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (69)'$$

$$\left.\begin{aligned} u_{2U}'' &= -\frac{2ifr''s''}{h'^2(f^2+s''^2)} A_U'' e^{s''(y-H)+i(pt-fx)}, \\ v_{2U}'' &= \frac{2f^2r''}{h'^2(f^2+s''^2)} A_U'' e^{s''(y-H)+i(pt-fx)}. \end{aligned}\right\} \qquad \ldots\ldots\ldots (71)'$$

The normal stress $\widehat{yy}_U''$ due to (69) and (71) are expressed by

$$\widehat{yy}_U'' = \frac{\mu'}{h'^2(f^2+s''^2)} \{ (f^2+s''^2)(k'^2-2h'^2-2r''^2) e^{r''(y-H)} + 4f^2r''s''e^{s''(y-H)} \} A_U'' e^{i(pt-fx)}. \qquad \ldots (74)$$

Now, using these expressions (69)′, (71)′ and (74), we obtain the following conditions at the surface $y=H$ in order to satisfy completely the free surface conditions at $y=H$ when the waves expressed by (51) are incident to the surface $y=H$:

When $y=H$,

$$\widehat{yy}_{U_{y=H}} \text{expressed by (74)} + \widehat{yy}_{U_{y=H}} \text{expressed by (65)} = 0, \qquad \ldots\ldots (75)$$

which gives us the following values of A_U'' after some reductions:

$$A_U'' = -\frac{2\left(\frac{k'^2}{f^2}-2\right)^2}{\left\{\left(\frac{k'^2}{f^2}-2\right)^2 - 4\frac{r''s''}{f^2}\right\}} A_U e^{-ir'H}. \qquad \ldots\ldots\ldots\ldots\ldots (76)$$

Substituting from (76) in (69)′ and (71)′, we get

$$\left.\begin{aligned} u_{1U}'' &= -\frac{2if\left(\frac{k'^2}{f^2}-2\right)^2}{h'^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2 - 4\frac{r''s''}{f^2}\right\}} A_U e^{r''(y-H)+i(pt-fx-r'H)}, \\ v_{1U}'' &= \frac{2r''\left(\frac{k'^2}{f^2}-2\right)^2}{h'^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2 - 4\frac{r''s''}{f^2}\right\}} A_U e^{r''(y-H)+i(pt-fx-r'H)}, \end{aligned}\right\} \qquad \ldots (77)$$

$$\left.\begin{aligned} u_{2U}'' &= -\frac{4ir''s''\left(\frac{k'^2}{f^2}-2\right)}{fh'^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2-4\frac{r''s''}{f^2}\right\}}A_U e^{s''(y-H)+i(pt-fx-r'H)}, \\ v_{2U}'' &= \frac{4r''\left(\frac{k'^2}{f^2}-2\right)}{h'^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2-4\frac{r''s''}{f^2}\right\}}A_U e^{s''(y-H)+i(pt-fx-r'H)}. \end{aligned}\right\} \quad \ldots\ldots(78)$$

For the convenience sake, we summarise the results obtained in this section as follows:

When the transmitted wave of dilatational type expressed by Δ_U'51) is primarily incident to the free surface $y=H$, the following waves are reflected at the surface $y=H$:

$$\left.\begin{aligned} u_{1U}' &= \frac{if}{h'^2}A_U e^{i(pt-fx+r'y-2r'H)}, \\ v_{1U}' &= -\frac{ir'}{h'^2}A_U e^{i(pt-fx+r'y-2r'H)}, \end{aligned}\right\}\text{dilatational type.} \quad \ldots\ldots(63)$$

$$\left.\begin{aligned} u_{1U}'' &= -\frac{2if\left(\frac{k'^2}{f^2}-2\right)^2}{h'^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2-4\frac{r''s''}{f^2}\right\}}A_U e^{r''(y-H)+i(pt-fx-r'H)}, \\ v_{1U}'' &= \frac{2r''\left(\frac{k'^2}{f^2}-2\right)^2}{h'^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2-4\frac{r''s''}{f^2}\right\}}A_U e^{r''(y-H)+i(pt-fx-r'H)}, \end{aligned}\right\}\text{dilatational type.} \quad (77)$$

$$\left.\begin{aligned} u_{2U}'' &= -\frac{4ir''s''\left(\frac{k'^2}{f^2}-2\right)}{fh'^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}}A_U e^{s''(y-H)+i(pt-fx-r'H)}, \\ v_{2U}'' &= \frac{4r''\left(\frac{k'^2}{f^2}-2\right)}{h'^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}}A_U e^{s''(y-H)+i(pt-fx-r'H)}. \end{aligned}\right\}\text{distortional type.} \quad (78)$$

b) *Transmitted Wave is Distortional.*

7. Let the transmitted wave, which is generated by a primary wave of any type when it is incident to the boundary surface $y=0$, be a distortional one $2\varpi_U$ which is expressed by

$$2\varpi_U = B_U e^{i(pt-fx-s'y)}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots (79)$$

Then the displacement u_{2U}, v_{2U} and the stress $\widehat{xx}_{2U}$, $\widehat{yy}_{2U}$, $\widehat{xy}_{2U}$ corresponding to (79) are written by

$$\left.\begin{aligned} u_{2U} &= -\frac{is'}{k'^2} B_U e^{i(pt-fx-s'y)}, \\ v_{2U} &= \frac{if}{k'^2} B_U e^{i(pt-fx-s'y)}, \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (80)$$

$$\left.\begin{aligned} \widehat{xx}_{2U} &= -\frac{2s'f\mu'}{k'^2} B_U e^{i(pt-fx-s'y)}, \\ \widehat{yy}_{2U} &= \frac{2s'f\mu'}{k'^2} B_U e^{i(pt-fx-s'y)}, \\ \widehat{xy}_{2U} &= \frac{(f^2-s'^2)\mu'}{k'^2} B_U e^{i(pt-fx-s'y)}, \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (81)$$

where B_U is a constant to be determined by the boundary conditions of elasticity at $y=0$ and is a certain function of the emergency angle, the velocity and the period of the primary wave incident to the plane $y=0$.

Using the two waves (54), (55) of reflected type, we shall adjust A_U' and B_U' to make shear stress on the free surface $y=H$ due to the incident wave expressed by (79) be nil and to make a certain normal stress and the tangential stress only existent on this surface.

The boundary conditions at the surface $y=H$ thus confined are written as follows:

When $y=H$,

$$\left.\begin{aligned} \widehat{xy}_{2U} + \widehat{xy}_{1U}' + \widehat{xy}_{2U}' &= 0, \\ \widehat{yy}_{1U}' + \widehat{yy}_{2U}' &= \widehat{yy}_{2U}, \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (82)$$

where $\widehat{yy}_{1U}'$, $\widehat{xy}_{1U}'$ and $\widehat{yy}_{2U}'$, $\widehat{xy}_{2U}'$ are already shewn by (57) and (59).

These conditions give us the two equations to determine A_U' and B_U' such that

$$\left.\begin{aligned} \frac{2fr'}{h'^2} A_U' e^{ir'H} - \frac{(f^2-s'^2)}{k'^2} B_U' e^{is'H} &= \frac{(f^2-s'^2)}{k'^2} B_U e^{-is'H}, \\ \frac{1}{h'^2}(s'^2-f^2) A_U' e^{ir'H} - \frac{2s'f}{k'^2} B_U' e^{is'H} &= \frac{2s'f}{k'^2} B_U e^{-is'H}. \end{aligned}\right\} \qquad \ldots (83)$$

Solving (83), we obtain

$$\left.\begin{aligned} A'_U&=0,\\ B_U'&=-B_U e^{-2is'H}.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(84)$$

Substituting from (84) in (56) and (58), we get

$$\left.\begin{aligned} u_{1U}'&=0,\\ v_{1U}'&=0,\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(85)$$

$$\left.\begin{aligned} u_{2U}'&=-\frac{is'}{k'^2}B_U e^{i(pt-fx+s'y-2s'H)},\\ v_{2U}'&=-\frac{if}{k'^2}B_U e^{i(pt-fx+s'y-2s'H)}.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(86)$$

To make shear stress at $y=H$ due to the incident wave (79) be nil, it is sufficient to use the reflected distortional wave (86) only, and there is no necessity to take the dilatational wave (85). The normal stress in the medium II due to (80) and (86) is expressed by

$$\widehat{yy}_U=\frac{2\mu's'f}{k'^2}B_U\{e^{-is'y}+e^{i(s'y-2s'H)}\}e^{i(pt-fx)}.\quad\ldots\ldots\ldots(87)$$

Now we have obtained in the preceding section the two waves expressed by (69) and (71) which apparently predominate on the surface $y=H$ and make shear stress at $y=H$ be nil and the normal stress be equal to the one expressed by (74). Using these two waves, we obtain the following condition at the surface $y=H$ in order to satisfy completely the free surface conditions at $y=H$ when the wave expressed by (79) are incident to the surface $y=H$:

When $y=H$,

$$\widehat{yy}_U''{}_{y=H}\text{expressed by (74)}+\widehat{yy}_{U_{y=H}}\text{expressed by (87)}=0.\quad\ldots.(88)$$

By this condition we obtain the following values of A_U'' after some reductions:

$$A_U''=-\frac{4s'h'^2\left(\dfrac{k'^2}{f^2}-2\right)}{fk'^2\left\{\left(\dfrac{k'^2}{f^2}-2\right)^2-\dfrac{4s''r''}{f^2}\right\}}B_U e^{-is'H}.\quad\ldots\ldots(89)$$

Substituting from (89) in (69)′ and (71)′, we get

$$\left.\begin{aligned} u_{1U}''=-\frac{4is'\left(\dfrac{k'^2}{f^2}-2\right)}{k'^2\left\{\left(\dfrac{k'^2}{f^2}-2\right)^2-\dfrac{4r''s''}{f^2}\right\}}B_U e^{r''(y-H)+i(pt-fx-s'H)},\end{aligned}\right\}\ldots.(90)$$

$$\left.\begin{aligned}
v_{1U}''&=\frac{4r''s'\left(\dfrac{k'^2}{f^2}-2\right)}{fk'^2\left\{\left(\dfrac{k'^2}{f^2}-2\right)^2-\dfrac{4r''s''}{f^2}\right\}}B_U e^{r''(y-H)+i(pt-fx-s'H)},\\
u_{2U}''&=-\frac{8ir''s's''}{f^2k'^2\left\{\left(\dfrac{k'^2}{f^2}-2\right)^2-\dfrac{4r''s''}{f^2}\right\}}B_U e^{s''(y-H)+i(pt-fx-s'H)},\\
v_{2U}''&=\frac{8r''s'}{fk'^2\left\{\left(\dfrac{k'^2}{f^2}-2\right)^2-\dfrac{4r''s''}{f^2}\right\}}B_U e^{s''(y-H)+i(pt-fx-s'H)}.
\end{aligned}\right\}\quad\ldots\ldots(91)$$

The results obtained in this section are summarised as follows:

When the transmitted distortional wave expressed by (79) is primarily incident to the free surface $y=H$, the following waves are reflected at $y=H$.

$$\left.\begin{aligned}
u_{2U}'&=-\frac{is'}{k'^2}B_U e^{i(pt-fx+s'y-2s'H)},\\
v_{2U}'&=-\frac{if}{k'^2}B_U e^{i(pt-fx+s'y-2s'H)},
\end{aligned}\right\}\text{distortional type}\ldots\ldots\ldots\ldots\ldots\ldots(86)$$

$$\left.\begin{aligned}
u_{1U}''&=-\frac{4is'\left(\dfrac{k'^2}{f^2}-2\right)}{k'^2\left\{\left(\dfrac{k'^2}{f^2}-2\right)^2-\dfrac{4r''s''}{f^2}\right\}}B_U e^{r''(y-H)+i(pt-fx-s'H)},\\
v_{1U}''&=\frac{4r''s'\left(\dfrac{k'^2}{f^2}-2\right)}{fk'^2\left\{\left(\dfrac{k'^2}{f^2}-2\right)^2-\dfrac{4r''s''}{f^2}\right\}}B_U e^{r''(y-H)+i(pt-fx-s'H)},
\end{aligned}\right\}\text{dilatational type}\ldots(90)$$

$$\left.\begin{aligned}
u_{2U}''&=-\frac{8ir''s's''}{f^2k'^2\left\{\left(\dfrac{k'^2}{f^2}-2\right)^2-\dfrac{4r''s''}{f^2}\right\}}B_U e^{s''(y-H)+i(pt-fx-s'H)},\\
v_{2U}''&=\frac{8r''s'}{fk'^2\left\{\left(\dfrac{k'^2}{f^2}-2\right)^2-\dfrac{4r''s''}{f^2}\right\}}B_U e^{s''(y-H)+i(pt-fx-s'H)}.
\end{aligned}\right\}\text{distortional type}\ (91)$$

A. Initial Motion of the Free Surface of a Stratified Solid When a Dilatational Wave is Incident to the Lower Boundary.

8. Using the results obtained in Section 6 and 7 as the preliminary calculations, we shall study the problem of reflection of waves at the free surface of the upper layer which resides on a semi-infinite solid. Of course, at the boundary surface the two elastic solids slide upon each other without any friction. (Fig. 7.)

Using the results of Section 3 of Part I, and those of Section 6 and 7, we obtain the following results after some reductions.

When the dilatational wave expressed by the following form is primarily incident to the boundary surface $y=0$;

$$\Delta=\mathfrak{A}e^{i(pt-fx-ry)}, \qquad \text{(92)}$$

of which the displacements are

$$\left.\begin{aligned} u_1&=\frac{if}{h^2}\mathfrak{A}e^{i(pt-fx-ry)},\\ v_1&=\frac{ir}{h^2}\mathfrak{A}e^{i(pt-fx-ry)}, \end{aligned}\right\} \qquad \text{(93)}$$

the transmitted[5] waves are expressed by

$$\Delta''=\frac{2rh'^2k^2\left(\frac{k'^2}{f^2}-2\right)\left(\frac{k^2}{f^2}-2\right)}{h^2\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\mathfrak{A}e^{i(pt-fx-r'y)}, \quad \text{(94)}$$

$$2\varpi_z''=\frac{4rr'k'^2k^2\left(\frac{k^2}{f^2}-2\right)}{h^2f\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\mathfrak{A}e^{i(pt-fx-s'y)}. \quad \text{(95)}$$

The displacement corresponding to Δ'' expressed by (94) are written in the forms of

$$\left.\begin{aligned} u_{1U}&=\frac{2ifrk^2\left(\frac{k'^2}{f^2}-2\right)\left(\frac{k^2}{f^2}-2\right)}{h^2\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\mathfrak{A}e^{i(pt-fx-r'y)},\\ v_{1U}&=\frac{2irr'k^2\left(\frac{k'^2}{f^2}-2\right)\left(\frac{k^2}{f^2}-2\right)}{h^2\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\mathfrak{A}e^{i(pt-fx-r'y)}. \end{aligned}\right\} \quad \text{(96)}$$

Therefore we obtain the following waves reflected at $y=H$ when the wave (93) is incident to this plane:

5) We omit the expressions of reflected wave of which the details are discussed in Section 3 minutely.

$$\left.\begin{aligned}
u_{1U}' &= \frac{2ifrk^2\left(\frac{k^2}{f^2}-2\right)\left(\frac{k'^2}{f^2}-2\right)}{h^2\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\mathfrak{A}e^{i(pt-fx+r'y-2r'H)},\\
v_{1U}' &= -\frac{2irr'k^2\left(\frac{k'^2}{f^2}-2\right)\left(\frac{k^2}{f^2}-2\right)}{h^2\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\mathfrak{A}e^{i(pt-fx+r'y-2r'H)},
\end{aligned}\right\} \qquad \text{.............(97)}$$

$$\left.\begin{aligned}
u_{1U}'' &= -\frac{4ifrk^2\left(\frac{k'^2}{f^2}-2\right)^3\left(\frac{k^2}{f^2}-2\right)}{h^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\\
&\qquad\times\mathfrak{A}e^{r''(y-H)+i(pt-fx-r'H)},\\
v_{1U}'' &= \frac{4rr''k^2\left(\frac{k'^2}{f^2}-2\right)^3\left(\frac{k^2}{f^2}-2\right)}{h^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2-4\frac{r''s''}{f^2}\right\}\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\\
&\qquad\times\mathfrak{A}e^{r''(y-H)+i(pt-fx-r'H)},
\end{aligned}\right\} \qquad \text{.................(98)}$$

$$\left.\begin{aligned}
u_{2U}'' &= -\frac{8irr''s''k^2\left(\frac{k'^2}{f^2}-2\right)^2\left(\frac{k^2}{f^2}-2\right)}{fh^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\\
&\qquad\times\mathfrak{A}e^{s''(y-H)+i(pt-fx-r'H)},\\
v_{2U}'' &= \frac{8rr''k^2\left(\frac{k'^2}{f^2}-2\right)^2\left(\frac{k^2}{f^2}-2\right)}{h^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\\
&\qquad\times\mathfrak{A}e^{s''(y-H)+i(pt-fx-r'H)}.
\end{aligned}\right\} \qquad \text{................(99)}$$

The displacements corresponding to $2\varpi_z''$ expressed by (95) are written by

$$\left.\begin{aligned} u_{2U}&=-\frac{4irr's'k^2\left(\frac{k^2}{f^2}-2\right)}{h^2f\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\mathfrak{R}e^{i(pt-fx-s'y)},\\ v_{2U}&=\frac{4irr'k^2\left(\frac{k^2}{f^2}-2\right)}{h^2\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\mathfrak{R}e^{i(pt-fx-s'y)},\end{aligned}\right\} \qquad (100)$$

and therefore

$$\left.\begin{aligned} u_{2U}'&=-\frac{4irr's'k^2\left(\frac{k^2}{f^2}-2\right)}{h^2f\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\mathfrak{R}e^{i(pt-fx+s'y-2s'H)},\\ v_{2U}'&=-\frac{4irr'k^2\left(\frac{k^2}{f^2}-2\right)}{h^2\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\mathfrak{R}e^{i(pt-fx+s'y-2s'H)},\end{aligned}\right\} \qquad (101)$$

$$\left.\begin{aligned} u_{1U}''&=-\frac{16irr's'k^2\left(\frac{k'^2}{f^2}-2\right)\left(\frac{k^2}{f^2}-2\right)}{h^2f\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\\ &\qquad\times\mathfrak{R}e^{r''(y-H)+i(pt-fx-s'H)},\\ v_{1U}''&=\frac{16rr'r''s'k^2\left(\frac{k'^2}{f^2}-2\right)\left(\frac{k^2}{f^2}-2\right)}{h^2f^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\\ &\qquad\times\mathfrak{R}e^{r''(y-H)+i(pt-fx-s'H)},\end{aligned}\right\} \qquad (102)$$

$$u_{2U}''=-\frac{32irr'r''s's''k^2\left(\frac{k^2}{f^2}-2\right)}{h^2f^3\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]}\times\mathfrak{R}e^{s''(y-H)+i(pt-fx-s'H)},$$

$$v_{2U}''=\frac{32rr'r''s'k^2\left(\frac{k^2}{f^2}-2\right)}{h^2f^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}+r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}\right]} \times \mathfrak{A}e^{s''(y-H)+i(pt-fx-s'H)}. \qquad \text{.................. (103)}$$

When the transmitted dilatational wave (94) is incident to the surface $y=H$ primarily, the waves such as dilatational and distortional ones expressed by (97), (98) and (99) are generated at and reflected at the free surface $y=H$. When the transmitted rotational wave expressed by (95) is incident to the free surface $y=H$ secondarily, the waves of distortional and dilatational types expressed by (101), (103) and (102) are generated at and reflected at the free surface $y=H$. In this case, there is no wave predominating on the free surface which is ordinarily generated at the free surface by the incidence of distortional wave.

When $\rho=\rho'$ and the Poisson's ratios of both touched solids are respectively $\frac{1}{4}$, the waves generated in the second medium due to the transmitted waves of (94) and (95) are expressed by the following formulae:

$$\left.\begin{aligned} u_{1U}&=\frac{2i\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-2\right]\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\Phi}\mathfrak{A}e^{i(pt-fx-r'y)},\\ v_{1U}&=\frac{2i\left[\frac{\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-2\right]\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\Phi}\mathfrak{A}e^{i(pt-fx-r'y)}, \end{aligned}\right\} \qquad \text{.................. (104)}$$

$$\left.\begin{aligned} u_{1U}'&=\frac{2i\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-2\right]\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\Phi}\mathfrak{A}e^{i(pt-fx+r'y-2r'H)},\\ v_{1U}'&=-\frac{2i\left[\frac{\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-2\right]\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\Phi} \times \mathfrak{A}e^{i(pt-fx+r'y-2r'H)}, \end{aligned}\right\} \quad (105)$$

$$u_{1U}''=-\frac{4i\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-2\right]^3\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-2\right\}^2+4\left\{\frac{\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right]\Phi}\mathfrak{A}e^{r''(y-H)+i(pt-fx-r'H)},$$

$$v_{1U}''=\frac{4i\left[\frac{\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-2\right]^3\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-2\right\}^2+4\left\{\frac{\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right]\Phi}\mathfrak{A}e^{r''(y-H)+i(pt-fx-r'H)},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots(106)$$

$$u_{2U}''=\frac{8i\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-2\right]^2\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-2\right\}^2+4\left\{\frac{\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right]\Phi}\mathfrak{A}e^{s''(y-H)+i(pt-fx-r'H)},$$

$$v_{2U}''=\frac{8i\left[\frac{\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-2\right]^2\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-2\right\}^2+4\left\{\frac{\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right]\Phi}\mathfrak{A}e^{s''(y-H)+i(pt-fx-r'H)},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots(107)$$

where

$$\Phi=\frac{\mu'}{\mu}\left[\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-2\right\}^2+4\left\{\frac{\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right]$$

$$+\frac{f}{r}\frac{\mu}{\mu'}\left\{\frac{\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+\frac{4r}{f}\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]. \quad (108)$$

And

$$\left.\begin{aligned}
u_{2U}&=-\frac{4i\left[\frac{\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\Phi}\mathfrak{A}e^{i(pt-fx-s'y)},\\
v_{2U}&=\frac{4i\left[\frac{\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\Phi}\mathfrak{A}e^{i(pt-fx-s'y)},
\end{aligned}\right\}$$

$$\dots\dots\dots\dots\dots\dots(109)$$

$$\left.\begin{aligned}
u_{2U}'&=-\frac{4i\left[\frac{\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\Phi}\\
&\qquad\times\mathfrak{A}e^{i(pt-fx+s'y-2s'H)},\\
v_{2U}'&=-\frac{4i\left[\frac{\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\Phi}\mathfrak{A}e^{i(pt-fx+s'y+2s'H)},
\end{aligned}\right\}(110)$$

$$u_{1U}''=-\frac{16i\left[\frac{\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-2\right\}^2\right.}$$

$$\frac{\left\{3\left(\frac{r}{f}\right)^2+1\right\}\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-2\right]}{\left.+4\left\{\frac{\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right]\Phi}\mathfrak{A}e^{r''(y-H)+i(pt-fx-s'H)},$$

$$v_{1U}''=\frac{16i\left[\frac{\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left\{3\left(\frac{r}{f}\right)^2+1\right\}\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-2\right]}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-2\right\}^2+4\left\{\frac{\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right]\Phi}\,\mathfrak{R}e^{r''(y-H)+i(pt-fx-s'H)},$$

$$\dots\dots\dots\dots(111)$$

$$u_{2U}''=\frac{32i\left[\frac{\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-2\right\}^2+4\left\{\frac{\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right]\Phi}\,\mathfrak{R}e^{s''(y-H)+i(pt-fx-s'H)},$$

$$v_{2U}''=\frac{32i\left[\frac{\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]\left[\frac{3\mu}{\mu'}\left\{\left(\frac{r}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-2\right\}^2+4\left\{\frac{\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{3\mu}{\mu'}\left(\left(\frac{r}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right]\Phi}\,\mathfrak{R}e^{s''(y-H)+i(pt-fx-s'H)}.$$

$$\dots\dots\dots\dots(112)$$

When $\frac{\mu'}{\mu}=1$, expressions (104), (105), (106), (107) and (109), (110), (111), (112) are written by

$$u_{1U}=\frac{i\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]}\,\mathfrak{R}e^{i(pt-fx-ry)},$$

$$v_{1U}=\frac{i\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]}\mathfrak{A}e^{i(pt-fx-ry)},\quad \Bigg\} \quad(113)$$

$$\left.\begin{aligned}u_{1U}'&=\frac{i\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]}\mathfrak{A}e^{i(pt-fx+ry-2rH)},\\ v_{1U}'&=-\frac{i\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]}\mathfrak{A}e^{i(pt-fx+ry-2rH)},\end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots(114)$$

$$\left.\begin{aligned}u_{1U}''&=-\frac{2i\left\{3\left(\frac{r}{f}\right)^2+1\right\}^4}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]^2}\mathfrak{A}e^{i(pt-fx+ry-2rH)},\\ v_{1U}''&=\frac{2i\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+1\right\}^4}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]^2}\mathfrak{A}e^{i(pt-fx+ry-2rH)},\end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots(115)$$

$$\left.\begin{aligned}u_{2U}''&=\frac{4i\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\left\{3\left(\frac{r}{f}\right)^2+1\right\}^3}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]^2}\mathfrak{A}e^{i(pt-fx+sy-rH-sH)},\\ v_{2U}''&=\frac{4i\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+1\right\}^3}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]^2}\mathfrak{A}e^{i(pt-fx+sy-rH-sH)},\end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots(116)$$

and

$$\left.\begin{aligned}
u_{2U}&=-\frac{2i\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]}\mathfrak{A}e^{i(pt-fx-sy)},\\
v_{2U}&=\frac{2i\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]}\mathfrak{A}e^{i(pt-fx-sy)},
\end{aligned}\right\}\quad(117)$$

$$\left.\begin{aligned}
u_{2U}'&=-\frac{2i\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]}\mathfrak{A}e^{i(pt-fx+sy-2sH)},\\
v_{2U}'&=-\frac{2i\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]}\mathfrak{A}e^{i(pt-fx+sy-2sH)},
\end{aligned}\right\}$$

$$\dots\dots\dots\dots\dots (118)$$

$$\left.\begin{aligned}
u_{1U}''&=-\frac{8i\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]^2}\mathfrak{A}e^{i(pt-fx+ry-rH-sH)},\\
v_{1U}''&=\frac{8i\left(\frac{r}{f}\right)^2\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]^2}\mathfrak{A}e^{i(pt-fx+ry-rH-sH)},
\end{aligned}\right\}$$

$$\dots\dots\dots\dots\dots (119)$$

$$\left.\begin{aligned}
u_{2U}''&=\frac{16i\left(\frac{r}{f}\right)^2\left\{3\left(\frac{r}{f}\right)^2+2\right\}\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]^2}\mathfrak{A}e^{i(pt-fx+sy-2sH)},\\
v_{2U}''&=\frac{16i\left(\frac{r}{f}\right)^2\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\left\{3\left(\frac{r}{f}\right)^2+1\right\}}{f\left\{\left(\frac{r}{f}\right)^2+1\right\}\left[\left\{3\left(\frac{r}{f}\right)^2+1\right\}^2+4\left(\frac{r}{f}\right)\left\{3\left(\frac{r}{f}\right)^2+2\right\}^{\frac{1}{2}}\right]^2}\mathfrak{A}e^{i(pt-fx+sy-2sH)}.
\end{aligned}\right\}\quad(120)$$

Let the velocity, period and emergency angle of the primary dilatational wave which is incident to the boundary surface $y=0$ be V_1, T_1 and e respectively. Then the waves on the free surface $y=H$ are expressed by

$$\left.\begin{aligned} u_{1U}&=\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{\cos e(2\sin^2 e+1)^2}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}}e^{i\left(pt-fx-\frac{2\pi H}{T_1V_1}\sin e\right)},\\ v_{1U}&=\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{\sin e(2\sin^2 e+1)^2}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}}e^{i\left(pt-fx-\frac{2\pi H}{T_1V_1}\sin e\right)},\end{aligned}\right\}\quad(113)'$$

$$\left.\begin{aligned} u_{1U}'&=\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{\cos e(2\sin^2 e+1)^2}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}}e^{i\left(pt-fx-\frac{2\pi H}{T_1V_1}\sin e\right)},\\ v_{1U}'&=-\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{\sin e(2\sin^2 e+1)^2}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}}e^{i\left(pt-fx-\frac{2\pi H}{T_1V_1}\sin e\right)},\end{aligned}\right\}\quad(114)'$$

$$\left.\begin{aligned} u_{1U}''&=-\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{2\cos e(2\sin^2 e+1)^4}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}^2}e^{i\left(pt-fx-\frac{2\pi H}{T_1V_1}\sin e\right)},\\ v_{1U}''&=\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{2\sin e(2\sin^2 e+1)^4}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}^2}e^{i\left(pt-fx-\frac{2\pi H}{T_1V_1}\sin e\right)},\end{aligned}\right\}\quad(115)'$$

$$\left.\begin{aligned} u_{2U}''&=\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{4\sin e\cos e(\sin^2 e+2)^{\frac{1}{2}}(2\sin^2 e+1)^3}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}^2}e^{i\left(pt-fx-\frac{2\pi H}{T_1V_1}\sin e\right)},\\ v_{2U}''&=\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{4\sin e\cos^2 e(2\sin^2 e+1)^3}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}^2}e^{i\left(pt-fx-\frac{2\pi H}{T_1V_1}\sin e\right)},\end{aligned}\right\}\quad(116)'$$

and

$$\left.\begin{aligned} u_{2U}&=-\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{2\sin e\cos e(\sin^2 e+2)^{\frac{1}{2}}(2\sin^2 e+1)}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}}e^{i\left(pt-fx-\frac{2\sqrt{3}\pi H}{T_1V_1}\sin e\right)},\\ v_{2U}&=\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{2\sin e\cos^2 e(2\sin^2 e+1)}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}}e^{i\left(pt-fx-\frac{2\sqrt{3}\pi H}{T_1V_1}\sin e\right)},\end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots(117)'$$

$$\left.\begin{aligned} u_{2U}'&=-\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{2\sin e\cos e(\sin^2 e+2)^{\frac{1}{2}}(2\sin^2 e+1)}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}}e^{i\left(pt-fx-\frac{2\sqrt{3}\pi H}{T_1V_1}\sin e\right)},\\ v_{2U}'&=-\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{2\sin e\cos^2 e(2\sin^2 e+1)}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}}e^{i\left(pt-fx-\frac{2\sqrt{3}\pi H}{T_1V_1}\sin e\right)},\end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots(118)'$$

$$\left.\begin{aligned} u_{1U}''&=-\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{8\sin e\cos^3 e(\sin^2 e+2)^{\frac{1}{2}}(2\sin^2 e+1)^2}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}^2}e^{i\left(pt-fx-\frac{2\sqrt{3}\pi H}{T_1V_1}\sin e\right)},\\ v_{1U}''&=\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{8\sin^2 e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}(2\sin^2 e+1)^2}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}^2}e^{i\left(pt-fx-\frac{2\sqrt{3}\pi H}{T_1V_1}\sin e\right)}, \end{aligned}\right\} \qquad \text{(119)}'$$

$$\left.\begin{aligned} u_{2U}''&=\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{16\sin^2 e\cos^3 e(\sin^2 e+2)(2\sin^2 e+1)}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}^2}e^{i\left(pt-fx-\frac{2\sqrt{3}\pi H}{T_1V_1}\sin e\right)},\\ v_{2U}''&=\frac{iT_1V_1\mathfrak{A}}{2\pi}\frac{16\sin^2 e\cos^4 e(\sin^2 e+2)^{\frac{1}{2}}(2\sin^2 e+1)}{\{(2\sin^2 e+1)^2+4\sin e\cos^2 e(\sin^2 e+2)^{\frac{1}{2}}\}^2}e^{i\left(pt-fx-\frac{2\sqrt{3}\pi H}{T_1V_1}\sin e\right)}. \end{aligned}\right\} \text{(120)}'$$

Table VI.

Max. Amp. \ e	0°	10°	20°	30°	45°	60°	70°	80°	90°
u_{1U}	1	0·5313	0·4353	0·4335	0·4535	0·4065	0·3115	0·1696	0
u_{1U}'	1	0·5313	0·4353	0·4335	0·4535	0·4065	0·3115	0·1696	0
u_{1U}''	−2	−0·5733	−0·4035	−0·4345	−0·5825	−0·6620	−0·5685	−0·3313	0
u_{2U}''	0	0·2675	0·3263	0·4340	0·6505	0·7595	0·6555	0·3826	0

Table VII.

Max. Amp. \ e	0°	10°	20°	30°	45°	60°	70°	80°	90°
v_{1U}	0	0·0937	0·1587	0·2505	0·4535	0·7032	0·8562	0·9620	1
v_{1U}'	0	−0·0937	−0·1587	−0·2505	−0·4535	−0·7032	−0·8562	−0·9620	−1
v_{1U}''	0	0·1008	0·1472	0·2510	0·5825	1·145	1·56	1·88	2
v_{2U}''	0	0·1849	0·2107	0·2505	0·2910	0·2290	0·1322	0·03854	0

Table VIII.

Max. Amp. \ e	0°	10°	20°	30°	45°	60°	70°	80°	90°
u_{2U}	0	−0·2479	−0·3525	−0·4328	−0·5065	−0·4665	−0·3585	−0·1958	0
u_{2U}'	0	−0·2479	−0·3525	−0·4328	−0·5065	−0·4665	−0·3585	−0·1958	0
u_{1U}''	0	−0·4893	−0·4680	−0·4330	−0·3255	−0·1520	−0·0554	−0·00784	0
u_{2U}''	0	0·2282	0·3780	0·4335	0·3635	0·1747	0·0640	0·00906	0

Table IX.

Max. Amp. \ e	0°	10°	20°	30°	45°	60°	70°	80°	90°
v_2U	0	0·1714	0·2275	0·2500	0·2265	0·1408	0·0724	0·0197	0
v_2U'	0	−0·1714	−0·2275	−0·2500	−0·2265	−0·1408	−0·0724	−0·0197	0
v_1U''	0	0·08626	0·1708	0·2505	0·3255	0·2633	0·1520	0·0445	0
v_2U''	0	0 1577	0·2445	0·2505	0·1630	0·0526	0·0129	0·00091	0

To see the properties of these waves expressed by (113)′, (114)′, (115)′, (116)′, (117)′, (118)′, (119′, (120)′ minutely, we obtain the following tables (Table VI, VII, VIII, IX) in which the maximum amplitudes of these waves corresponding to the magnitude of the emergency angle e are tabulated. The results in these tables are graphically shewn in Fig. 8, 9, 10, 11.

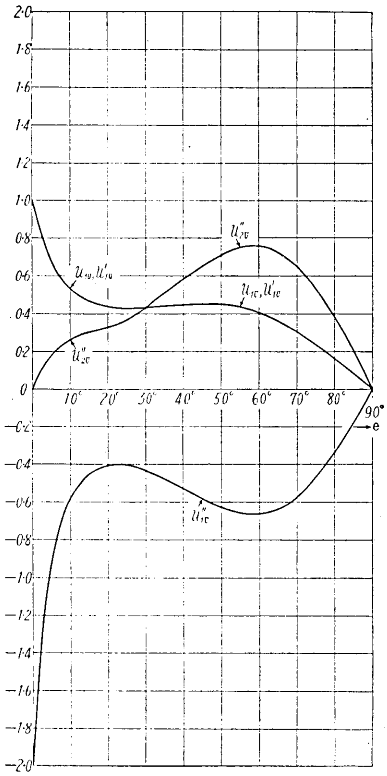

Fig. 8.

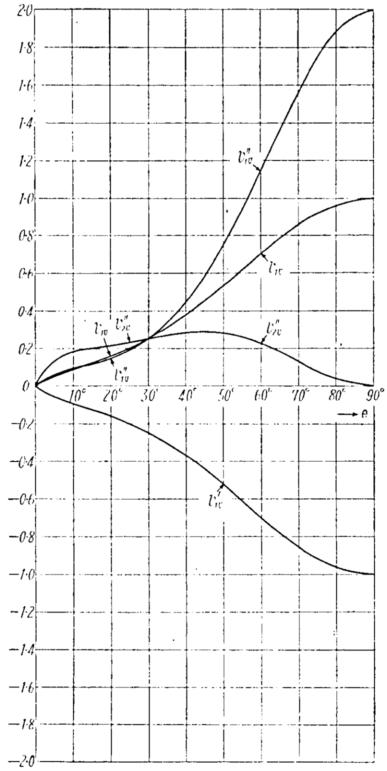

Fig. 9.

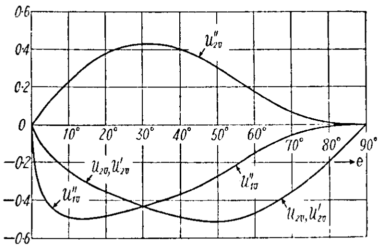

Fig. 10.

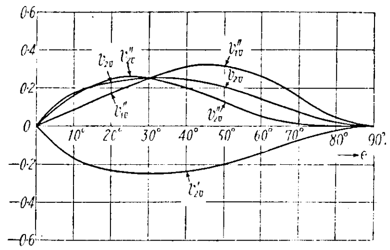

Fig. 11.

These tables and figures shew us many important properties of the waves, and some of them are summarised as follows:

1. When the transmitted dilatational wave is incident to the free surface, the reflected waves of dilatational and distortional types are only generated at the free surface.

And when the transmitted distortional wave is incident to the free surface, the reflected waves of dilatational and distortional types are only generated at the free surface. Moreover there exist no waves of which the energies are accumulated on the free surface.

2. The phases of all reflected waves generated at the free surface by the incidence of transmitted distortional wave differ equally from that of the primary dilatational wave incident to the lower boundary by $\left(\frac{2H\pi}{T_1V_1}\sin e\right)$.

And the phases of all reflected waves generated at the free surface by the incidence of transmitted dilatational wave differ also equally from that of the primary dilatational wave by $\left(\frac{2\sqrt{3}\pi H}{T_1V_1}\sin e\right)$.

3. When $e=0°$, there exist no transmitted wave, and the free surface does not move.

4. When $e=90°$, the free surface does not move when the transmitted distortional wave is incident to the free surface. In this case, however, when the transmitted dilatational wave is incident to the free surface, there exist movement of the free surface of the layer.

B. Initial Motion of the Free Surface of a Stratified Solid When a Distortional Wave is Incident to the Lower Boundary.

9. Using the results of Section 4 of Part I, and the results in Section 6 and 7, we obtain the following results after some reductions:

When the distortional wave expressed by the following form is incident to the boundary surface $y=0$,

$$2\varpi=\mathfrak{B}e^{i(pt-fx-sy)}, \qquad \text{.....................(121)}$$

of which the displacement are

$$\left.\begin{aligned} u_2&=-\frac{is}{k^2}\mathfrak{B}e^{i(pt-fx-sy)},\\ v_2&=\frac{if}{k^2}\mathfrak{B}e^{i(pt-fx-sy)}, \end{aligned}\right\} \qquad \text{.....................(122)}$$

the transmitted[6] waves are expressed by

$$\varDelta''=\frac{4rsh'^2\left(\frac{k'^2}{f^2}-2\right)}{f\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]}\mathfrak{B}e^{i(pt-fx-r'y)}, \quad \text{..(123)}$$

$$2\varpi''=\frac{8rr'sk'^2}{f^2\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]}\mathfrak{B}e^{i(pt-fx-s'y)}. \quad \text{..(124)}$$

The displacement corresponding to $\varDelta''$ expressed by (123) are written as follows;

$$\left.\begin{aligned} u_{1U}&=\frac{4irs\left(\frac{k'^2}{f^2}-2\right)}{\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]}\mathfrak{B}e^{i(pt-fx-r'y)},\\ v_{1U}&=\frac{4irr's\left(\frac{k'^2}{f^2}-2\right)}{f\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]}\mathfrak{B}e^{i(pt-fx-r'y)}. \end{aligned}\right\} \quad (125)$$

Therefore we obtain the following waves reflected at $y=H$ when the wave expressed by (123) is incident to this surface;

$$u_{1U}'=\frac{4irs\left(\frac{k'^2}{f^2}-2\right)}{\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]}\mathfrak{B}e^{i(pt-fx+r'y-2r'H)},$$

6) The discussions of waves which are reflected at the boundary surface are already made in the preceding section.

$$v_{1U}'=-\frac{4irr's\left(\frac{k'^2}{f^2}-2\right)}{f\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]}\mathfrak{B}e^{i(pt-fx+r'y-2r'H)},\qquad\ldots\ldots(126)$$

$$u_{1U}''=-\frac{8irs\left(\frac{k'^2}{f^2}-2\right)^3\mathfrak{B}e^{r''(y-H)+i(pt-fx-r'H)}}{\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]},$$

$$v_{1U}''=\frac{8rr''s\left(\frac{k'^2}{f^2}-2\right)^3\mathfrak{B}e^{r''(y-H)+i(pt-fx-r'H)}}{f\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]},\qquad\ldots\ldots(127)$$

$$u_{2U}''=-\frac{16irsr''s''\left(\frac{k'^2}{f^2}-2\right)^2\mathfrak{B}e^{s''(y-H)+i(pt-fx-r'H)}}{f^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]},$$

$$v_{2U}''=\frac{16rr''s\left(\frac{k'^2}{f^2}-2\right)^2\mathfrak{B}e^{s''(y-H)+i(pt-fx-r'H)}}{f\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]}.\qquad\ldots\ldots(128)$$

The displacement corresponding to $2\varpi''$ expressed by (124) are written by

$$u_{2U}=-\frac{8irr'ss'}{f^2\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]}\mathfrak{B}e^{i(pt-fx-s'y)},$$

$$v_{2U}=\frac{8irr's}{f\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]}\mathfrak{B}e^{i(pt-fx-s'y)}.\qquad\ldots\ldots(129)$$

And therefore the waves, which are generated by the wave (129), on the surface $y=H$ are expressed by

$$\left.\begin{aligned}
u_{2U}' &= -\frac{8irr'ss'}{f^2\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]}\mathfrak{B}e^{i(pt-fx+s'y-2s'H)},\\
v_{2U}' &= -\frac{8irr's}{f\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]}\mathfrak{B}e^{i(pt-fx+s'y-2s'H)},
\end{aligned}\right\}$$

$$\dots\dots\dots\dots\dots\dots(130)$$

$$\left.\begin{aligned}
u_{1U}'' &= -\frac{32irr'ss'\left(\frac{k'^2}{f^2}-2\right)\mathfrak{B}e^{r''(y-H)+i(pt-fx-s'H)}}{f^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]},\\
v_{1U}'' &= \frac{32rr'r''ss'\left(\frac{k'^2}{f^2}-2\right)\mathfrak{B}e^{r''(y-H)+i(pt-fx-s'H)}}{f^3\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]},
\end{aligned}\right\}$$

$$\dots\dots\dots\dots\dots(131)$$

$$\left.\begin{aligned}
u_{2U}'' &= -\frac{64irr'r''ss's''\mathfrak{B}e^{s''(y-H)+i(pt-fx-s'H)}}{f^4\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]},\\
v_{2U}'' &= \frac{64rr'r''ss'\mathfrak{B}e^{s''(y-H)+i(pt-fx-s'H)}}{f^3\left\{\left(\frac{k'^2}{f^2}-2\right)^2-\frac{4r''s''}{f^2}\right\}\left[r'k'^2\left\{\left(\frac{k^2}{f^2}-2\right)^2+\frac{4rs}{f^2}\right\}+\frac{\mu'}{\mu}rk^2\left\{\left(\frac{k'^2}{f^2}-2\right)^2+\frac{4r's'}{f^2}\right\}\right]}.
\end{aligned}\right\}$$

$$\dots\dots\dots\dots\dots\dots(132)$$

When the transmitted dilatational wave expressed by (124) is incident to the free surface $y=H$ primarily, the waves of dilatational and distortional types expressed by (126), (127) and (128) are generated at the free surface $y=H$ and reflected at this plane. When the emergency angle of the distortional wave (121) which is incident to the boundary $y=0$ is smaller than a certain magnitude, these waves are predominated on the free surface and not reflected in the interior of the medium, and there is the phase difference between these waves and the distortional wave which is incident to the boundary $y=0$.

When the transmitted distortional wave expressed by (129) is incident to the free surface $y=H$ secondarily, the waves of dilatational and distortional types expressed by (130), (131) and (132) are generated at the plane $y=H$ and reflected at this plane. In this case, also, these waves are predominated on the free surface, and have also the phase difference when the emergency angle of the primary incident distortional wave (121) is smaller than a certain magnitude.

When $\rho=\rho'$ and the Poisson's ratios of both media are equally $\frac{1}{4}$, the wave generated in the second medium due to the transmitted waves of (123) and (124) are expressed by the following formulae:

$$\left.\begin{aligned} u_{1U}&=\frac{4i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-2\right]}{f\Psi}\mathfrak{B}e^{i(pt-fx-r'y)},\\ v_{1U}&=\frac{4i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left[\frac{\mu}{3\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-2\right]}{f\Psi}\\ &\qquad\times\mathfrak{B}e^{i(pt-fx-r'y)}, \end{aligned}\right\} \qquad \ldots\ldots(133)$$

$$\left.\begin{aligned} u_{1U}'&=\frac{4i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-2\right]}{f\Psi}\mathfrak{B}e^{i(pt-fx+r'y-2r'H)},\\ v_{1U}'&=-\frac{4i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left[\frac{1}{3}\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-2\right]}{f\Psi}\\ &\qquad\times\mathfrak{B}e^{i(pt-fx+r'y-2r'H)}, \end{aligned}\right\} \qquad \ldots\ldots(134)$$

$$u_{1U}''=-\frac{8i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}}{f\left[\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-2\right\}^2\right.}$$

$$\frac{\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-2\right]^3}{\left.+4\left\{\frac{1}{3}\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right]\Psi}\mathfrak{B}e^{r''(y-H)+i(pt-fx-r'H)},$$

$$\left.\begin{aligned}
v_{1U}''=&\frac{8i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left[\frac{1}{3}\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}}{f\left[\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-2\right\}^2\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-2\right]^3+4\left\{\frac{1}{3}\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right]\Psi}\mathfrak{B}e^{r''(y-H)+i(pt-fx-r'H)},
\end{aligned}\right\}\qquad(135)$$

$$\left.\begin{aligned}
u_{2U}''=&\frac{16i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left[\frac{1}{3}\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}}{f\left[\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-2\right\}^2\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-2\right]^2+4\left\{\frac{1}{3}\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right]^{\frac{1}{2}}\right]\Psi}\mathfrak{B}e^{s''(y-H)+i(pt-fx-r'H)},\\
v_{2U}''=&\frac{16i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left[\frac{1}{3}\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}}{f\left[\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-2\right\}^2\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-2\right]^2+4\left\{\frac{1}{3}\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right]\Psi}\mathfrak{B}e^{s''(y-H)+i(pt-fx-r'H)},
\end{aligned}\right\}\qquad(136)$$

where

$$\begin{aligned}
\Psi=&\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\frac{1}{3}\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2\right.\\
&\left.+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]\\
&+\frac{\mu'}{\mu}\left\{\left(\frac{s}{f}\right)^2+1\right\}\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left[\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-2\right\}^2\right.\\
&\left.+4\left\{\frac{1}{3}\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right].\qquad(137)
\end{aligned}$$

And

$$
\left.
\begin{aligned}
u_{2U}&=-\frac{8i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left[\frac{1}{3}\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}}{f\Psi}\\
&\qquad\times\mathfrak{B}e^{i(pt-fx-s'y)},\\
v_{2U}&=\frac{8i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left[\frac{1}{3}\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}}{f\Psi}\mathfrak{B}e^{i(pt-fx-s'y)},
\end{aligned}
\right\}
\qquad\ldots\ldots\ldots\ldots(138)
$$

$$
\left.
\begin{aligned}
u_{2U}'&=-\frac{8i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left[\frac{1}{3}\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}}{f\Psi}\\
&\qquad\times\mathfrak{B}e^{i(pt-fx+s'y-2s'H)},\\
v_{2U}'&=-\frac{8i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left[\frac{1}{3}\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}}{f\Psi}\mathfrak{B}e^{i(pt-fx+s'y-2s'H)},
\end{aligned}
\right\}
\qquad\ldots\ldots\ldots\ldots(139)
$$

$$
\left.
\begin{aligned}
u_{1U}''&=-\frac{32i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left[\frac{1}{3}\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-2\right]}{f\left[\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-2\right\}^2+4\left\{\frac{1}{3}\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right]\Psi}\\
&\qquad\times\mathfrak{B}e^{r'(y-H)+i(pt-fx-s'H)},\\
v_{1U}''&=\frac{32i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left[\frac{1}{3}\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}}\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-2\right]}{f\left[\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-2\right\}^2+4\left\{\frac{1}{3}\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\right]\Psi}\\
&\qquad\times\mathfrak{B}e^{r''(y-H)+i(pt-fx-s'H)},
\end{aligned}
\right\}
\qquad\ldots\ldots\ldots\ldots(140)
$$

$$
\left.
\begin{aligned}
u_{2U}'' &= \frac{64i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}}{f\left[\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-2\right\}^2\right.} \\
&\quad \frac{}{\left[\frac{1}{3}\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right)-1\right]\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right] +4\left\{\frac{1}{3}\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\Bigg]\Psi}\mathfrak{B}e^{s''(y-H)+i(pt-fx-s'H)}, \\
v_{2U}'' &= \frac{64i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}}{f\left[\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-2\right\}^2\right.} \\
&\quad \frac{}{\left[\frac{1}{3}\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]\left[\frac{\mu}{\mu'}\left\{\left(\frac{s}{f}\right)^2+1\right\}-1\right]^{\frac{1}{2}} +4\left\{\frac{1}{3}\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\left\{\frac{\mu}{\mu'}\left(\left(\frac{s}{f}\right)^2+1\right)-1\right\}^{\frac{1}{2}}\Bigg]\Psi}\mathfrak{B}e^{s''(y-H)+i(pt-fx-s'H)}.
\end{aligned}
\right\}
$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(141)$$

When $\frac{\mu'}{\mu}=1$, the expressions (133), (134), (135), (136) and (138), (139), (140), (141) are expressed by

$$
\left.
\begin{aligned}
u_{1U} &= \frac{2i\left(\frac{s}{f}\right)\left\{\left(\frac{s}{f}\right)^2-1\right\}}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]}\mathfrak{B}e^{i(pt-fx-ry)}, \\
v_{1U} &= \frac{2i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left\{\left(\frac{s}{f}\right)^2-1\right\}}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]}\mathfrak{B}e^{i(pt-fx-ry)},
\end{aligned}
\right\}(142)
$$

$$
u_{1U}' = \frac{2i\left(\frac{s}{f}\right)\left\{\left(\frac{s}{f}\right)^2-1\right\}}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]}\mathfrak{B}e^{i(pt-fx+ry-2rH)},
$$

$$v_{1U}'=\frac{2i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left\{\left(\frac{s}{f}\right)^2-1\right\}}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]}\mathfrak{B}e^{i(pt-fx+ry-2rH)},$$

$$\dots\dots\dots\dots(143)$$

$$u_{1U}''=-\frac{4i\left(\frac{s}{f}\right)\left\{\left(\frac{s}{f}\right)^2-1\right\}^3}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]^2}\mathfrak{B}e^{i(pt-fx+ry-2rH)},$$

$$v_{1U}''=\frac{4i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left\{\left(\frac{s}{f}\right)^2-1\right\}^3}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]^2}\mathfrak{B}e^{i(pt-fx+ry-2rH)},$$

$$\dots\dots\dots\dots(144)$$

$$u_{2U}''=\frac{8i\left(\frac{s}{f}\right)^2\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left\{\left(\frac{s}{f}\right)^2-1\right\}^2}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]^2}\mathfrak{B}e^{i(pt-fx+ry-rH-sH)},$$

$$v_{2U}''=\frac{8i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\left\{\left(\frac{s}{f}\right)^2-1\right\}^2}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]^2}\mathfrak{B}e^{i(pt-fx+ry-rH-sH)},$$

$$\dots\dots\dots\dots(145)$$

and

$$u_{2U}=-\frac{4i\left(\frac{s}{f}\right)^2\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]}\mathfrak{B}e^{i(pt-fx-sy)},$$

$$v_{2U}=\frac{4i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]}\mathfrak{B}e^{i(pt-fx-sy)},$$

$$\dots\dots\dots\dots(146)$$

$$
\left.
\begin{aligned}
u_{2U}' &= -\frac{4i\left(\frac{s}{f}\right)^2\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]}\mathfrak{B}e^{i(pt-fx+sy-2sH)},\\
v_{2U}' &= -\frac{4i\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]}\mathfrak{B}e^{i(pt-fx+sy-2sH)},
\end{aligned}
\right\}
$$

..............(147)

$$
\left.
\begin{aligned}
u_{1U}'' &= -\frac{16i\left(\frac{s}{f}\right)^2\left\{\left(\frac{s}{f}\right)^2-1\right\}\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]^2}\mathfrak{B}e^{i(pt-fx+ry-rH-sH)},\\
v_{1U}'' &= \frac{16i\left(\frac{s}{f}\right)^2\left\{\left(\frac{s}{f}\right)^2-1\right\}\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right)^{\frac{1}{2}}\right]^2}\mathfrak{B}e^{i(pt-fx+ry-rH-sH)},
\end{aligned}
\right\}
$$

..............(148)

$$
\left.
\begin{aligned}
u_{2U}'' &= \frac{32i\left(\frac{s}{f}\right)^3\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]^2}\mathfrak{B}e^{i(pt-fx+sy-2sH)},\\
v_{2U}'' &= \frac{32i\left(\frac{s}{f}\right)^2\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}}{f\left\{\left(\frac{s}{f}\right)^2+1\right\}\left[\left\{\left(\frac{s}{f}\right)^2-1\right\}^2+4\left(\frac{s}{f}\right)\left\{\frac{1}{3}\left(\frac{s}{f}\right)^2-\frac{2}{3}\right\}^{\frac{1}{2}}\right]^2}\mathfrak{B}e^{i(pt-fx+sy-2sH)}.
\end{aligned}
\right\}
$$

..............(149)

Let the velocity, period and emergency angle of the primary distortional wave (121) which is incident to the boundary surface $y=0$ be V_2, T_2 and e respectively. Then the waves on the free surface $y=H$ are expressed by

$$\left.\begin{aligned}
u_{1U}&=\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{2\sin e\cos^2 e(\sin^2 e-\cos^2 e)}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}\,e^{i\left(pt-fx-\frac{2\pi H}{\sqrt{3}T_2V_2}\sin e\right)},\\
v_{1U}&=\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{2\sin e\cos e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}(\sin^2 e-\cos^2 e)}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}\,e^{i\left(pt-fx-\frac{2\pi H}{\sqrt{3}T_2V_2}\sin e\right)},
\end{aligned}\right\}$$

$$\cdots\cdots\cdots\cdots(142)'$$

$$\left.\begin{aligned}
u_{1U}'&=\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{2\sin e\cos^2 e(\sin^2 e-\cos^2 e)}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}\,e^{i\left(pt-fx-\frac{2\pi H}{\sqrt{3}T_2V_2}\sin e\right)},\\
v_{1U}'&=-\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{2\sin e\cos e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}(\sin^2 e-\cos^2 e)}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}\,e^{i\left(pt-fx-\frac{2\pi H}{\sqrt{3}T_2V_2}\sin e\right)},
\end{aligned}\right\}$$

$$\cdots\cdots\cdots\cdots(143)'$$

$$\left.\begin{aligned}
u_{1U}''&=-\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{4\sin e\cos^2 e(\sin^2 e-\cos^2 e)^3}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}^2}\,e^{i\left(pt-fx-\frac{2\pi H}{\sqrt{3}T_2V_2}\sin e\right)},\\
v_{1U}''&=\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{4\sin e\cos e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}(\sin^2 e-\cos^2 e)^3}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}^2}\,e^{i\left(pt-fx-\frac{2\pi H}{\sqrt{3}T_2V_2}\sin e\right)},
\end{aligned}\right\}$$

$$\cdots\cdots\cdots\cdots(144)'$$

$$\left.\begin{aligned}
u_{2U}''&=\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{8\sin^2 e\cos^2 e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}(\sin^2 e-\cos^2 e)^2}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}^2}\,e^{i\left(pt-fx-\frac{2\pi H}{\sqrt{3}T_2V_2}\sin e\right)},\\
v_{2U}''&=\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{8\sin^2 e\cos e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}(\sin^2 e-\cos^2 e)^2}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\frac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}^2}\,e^{i\left(pt-fx-\frac{2\pi H}{\sqrt{3}T_2V_2}\sin e\right)},
\end{aligned}\right\}$$

$$\cdots\cdots\cdots\cdots(145)'$$

and

$$
\left.
\begin{aligned}
u_{2U} &= -\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{4\sin^2 e\cos^2 e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}\,e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e\right)},\\
v_{2U} &= \frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{4\sin e\cos^3 e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}\,e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e\right)},
\end{aligned}
\right\}
\qquad\ldots\ldots\ldots\ldots\ldots\ldots(146)'
$$

$$
\left.
\begin{aligned}
u_{2U}' &= -\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{4\sin^2 e\cos^2 e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}\,e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e\right)},\\
v_{2U}' &= -\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{4\sin e\cos^3 e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}}\,e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e\right)},
\end{aligned}
\right\}
\qquad\ldots\ldots\ldots\ldots\ldots\ldots(147)'
$$

$$
\left.
\begin{aligned}
u_{1U}'' &= -\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{16\sin^2 e\cos^4 e(\sin^2 e-\cos^2 e)\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}^2}\,e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e\right)},\\
v_{1U}'' &= \frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{16\sin^2 e\cos^3 e(\sin^2 e-\cos^2 e)\left(\dfrac{1}{3}-\cos^2 e\right)}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}^2}\,e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e\right)},
\end{aligned}
\right\}
\qquad\ldots\ldots\ldots\ldots\ldots\ldots(148)'
$$

$$
u_{2U}'' = \frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{32\sin^3 e\cos^4 e\left(\dfrac{1}{3}-\cos^2 e\right)}{\left\{(\sin^2 e-\cos^2 e)^2+4\sin e\cos^2 e\left(\dfrac{1}{3}-\cos^2 e\right)^{\frac{1}{2}}\right\}^2}\,e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e\right)},
$$

$$v_{2U}''=\frac{iT_2V_2\mathfrak{B}}{2\pi}\frac{32\sin^2e\cos^5e\left(\frac{1}{3}-\cos^2e\right)}{\left\{(\sin^2e-\cos^2e)^2+4\sin e\cos^2e\left(\frac{1}{3}-\cos^2e\right)^{\frac{1}{2}}\right\}^2}e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e\right)}. \quad \ldots\ldots\ldots\ldots\ldots(149)'$$

When the emergency angle is smaller than 54°44′, the expressions (142)′, (143)′, (144)′, (145)′, (146)′, (147)′, (148)′ and (149)′ should be replaced by the following formulae:

$$\left.\begin{aligned}
u_{1U}&=\frac{iT_2V_2\mathfrak{B}}{2\pi}\frac{2\sin e\cos^2e(\sin^2e-\cos^2e)}{\left\{(\sin^2e-\cos^2e)^4+16\sin^2e\cos^4e\left(\cos^2e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}}\times e^{i\left(pt-fx-\frac{2\pi}{\sqrt{3}T_2V_2}H\sin e-\varphi\right)},\\
v_{1U}&=\frac{iT_2V_2\mathfrak{B}}{2\pi}\frac{2\sin e\cos e\left(\cos^2e-\frac{1}{3}\right)^{\frac{1}{2}}(\sin^2e-\cos^2e)}{\left\{(\sin^2e-\cos^2e)^4+16\sin^2e\cos^4e\left(\cos^2e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}}\times e^{i\left(pt-fx-\frac{2\pi}{\sqrt{3}T_2V_2}H\sin e-\varphi+\frac{\pi}{2}\right)},
\end{aligned}\right\}\ (142)''$$

$$\left.\begin{aligned}
u_{1U}'&=\frac{iT_2V_2\mathfrak{B}}{2\pi}\frac{2\sin e\cos^2e(\sin^2e-\cos^2e)}{\left\{(\sin^2e-\cos^2e)^4+16\sin^2e\cos^4e\left(\cos^2e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}}\times e^{i\left(pt-fx-\frac{2\pi}{\sqrt{3}T_2V_2}H\sin e-\varphi\right)},\\
v_{1U}'&=-\frac{iT_2V_2\mathfrak{B}}{2\pi}\frac{2\sin e\cos e\left(\cos^2e-\frac{1}{3}\right)^{\frac{1}{2}}(\sin^2e-\cos^2e)}{\left\{(\sin^2e-\cos^2e)^4+16\sin^2e\cos^4e\left(\cos^2e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}}\times e^{i\left(pt-fx-\frac{2\pi}{\sqrt{3}T_2V_2}H\sin e-\varphi+\frac{\pi}{2}\right)},
\end{aligned}\right\}\ (143)''$$

$$u_{1U}''=-\frac{iT_2V_2\mathfrak{B}}{2\pi}\frac{4\sin e\cos^2e(\sin^2e-\cos^2e)^3}{\left\{(\sin^2e-\cos^2e)^4+16\sin^2e\cos^4e\left(\cos^2e-\frac{1}{3}\right)\right\}}\times e^{i\left(pt-fx-\frac{2\pi}{\sqrt{3}T_2V_2}H\sin e-2\varphi\right)},$$

$$\left.\begin{aligned}
v_{1U}''=\frac{iT_2V_2\mathfrak{B}}{2\pi}\frac{4\sin e\cos e\left(\cos^2 e-\frac{1}{3}\right)^{\frac{1}{2}}(\sin^2 e-\cos^2 e)^3}{\left\{(\sin^2 e-\cos^2 e)^4+16\sin^2 e\cos^4 e\left(\cos^2 e-\frac{1}{3}\right)\right\}} \\
\times e^{i\left(pt-fx-\frac{2\pi}{\sqrt{3}\,T_2V_2}H\sin e-2\varphi+\frac{\pi}{2}\right)},
\end{aligned}\right\}\quad(144)''$$

$$\left.\begin{aligned}
u_{2U}''=\frac{iT_2V_2\mathfrak{B}}{2\pi}\frac{8\sin^2 e\cos^2 e\left(\cos^2 e-\frac{1}{3}\right)^{\frac{1}{2}}(\sin^2 e-\cos^2 e)^2}{\left\{(\sin^2 e-\cos^2 e)^4+16\sin^2 e\cos^4 e\left(\cos^2 e-\frac{1}{3}\right)\right\}} \\
\times e^{i\left(pt-fx-\frac{2\pi}{\sqrt{3}\,T_2V_2}H\sin e-2\varphi+\frac{\pi}{2}\right)}, \\
v_{2U}''=\frac{iT_2V_2\mathfrak{B}}{2\pi}\frac{8\sin e\cos^3 e\left(\cos^2 e-\frac{1}{3}\right)^{\frac{1}{2}}(\sin^2 e-\cos^2 e)^2}{\left\{(\sin^2 e-\cos^2 e)^4+16\sin^2 e\cos^4 e\left(\cos^2 e-\frac{1}{3}\right)\right\}} \\
\times e^{i\left(pt-fx-\frac{2\pi}{\sqrt{3}\,T_2V_2}H\sin e-2\varphi+\frac{\pi}{2}\right)},
\end{aligned}\right\}\quad(145)''$$

and

$$\left.\begin{aligned}
u_{2U}=-\frac{iT_2V_2\mathfrak{B}}{2\pi}\frac{4\sin^2 e\cos^2 e\left(\cos^2 e-\frac{1}{3}\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^4+16\sin^2 e\cos^4 e\left(\cos^2 e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}} \\
\times e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e-\varphi+\frac{\pi}{2}\right)}, \\
v_{2U}=\frac{iT_2V_2\mathfrak{B}}{2\pi}\frac{4\sin e\cos^3 e\left(\cos^2 e-\frac{1}{3}\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^4+16\sin^2 e\cos^4 e\left(\cos^2 e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}} \\
\times e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e-\varphi+\frac{\pi}{2}\right)},
\end{aligned}\right\}\quad(146)''$$

$$\left.\begin{aligned}
u_{2U}'=-\frac{iT_2V_2\mathfrak{B}}{2\pi}\frac{4\sin^2 e\cos^2 e\left(\cos^2 e-\frac{1}{3}\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^4+16\sin^2 e\cos^4 e\left(\cos^2 e-\frac{1}{3}\right)\right\}^{\frac{1}{2}}} \\
\times e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e-\varphi+\frac{\pi}{2}\right)},
\end{aligned}\right\}\quad(147)''$$

$$
\left.
\begin{aligned}
v_{2U}{}' &= -\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{4\sin e\cos^3 e\left(\cos^2 e-\dfrac{1}{3}\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^4+16\sin^2 e\cos^4 e\left(\cos^2 e-\dfrac{1}{3}\right)\right\}^{\frac{1}{2}}} \\
&\qquad\times e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e-\varphi+\frac{\pi}{2}\right)},
\end{aligned}
\right|
$$

$$
\left.
\begin{aligned}
u_{1U}{}'' &= -\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{16\sin^2 e\cos^4 e(\sin^2 e-\cos^2 e)\left(\cos^2 e-\dfrac{1}{3}\right)^{\frac{1}{2}}}{\left\{(\sin^2 e-\cos^2 e)^4+16\sin^2 e\cos^4 e\left(\cos^2 e-\dfrac{1}{3}\right)\right\}} \\
&\qquad\times e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e-2\varphi+\frac{\pi}{2}\right)}, \\
v_{1U}{}'' &= -\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{16\sin^2 e\cos^3 e(\sin^2 e-\cos^2 e)\left(\cos^2 e-\dfrac{1}{3}\right)}{\left\{(\sin^2 e-\cos^2 e)^4+16\sin^2 e\cos^4 e\left(\cos^2 e-\dfrac{1}{3}\right)\right\}} \\
&\qquad\times e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e-2\varphi\right)},
\end{aligned}
\right\} \quad (148)''
$$

$$
\left.
\begin{aligned}
u_{2U}{}'' &= -\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{32\sin^3 e\cos^4 e\left(\cos^2 e-\dfrac{1}{3}\right)}{\left\{(\sin^2 e-\cos^2 e)^4+16\sin^2 e\cos^4 e\left(\cos^2 e-\dfrac{1}{3}\right)\right\}} \\
&\qquad\times e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e-2\varphi\right)}, \\
v_{2U}{}'' &= -\frac{iT_2V_2\mathfrak{B}}{2\pi}\,\frac{32\sin^2 e\cos^5 e\left(\cos^2 e-\dfrac{1}{3}\right)}{\left\{(\sin^2 e-\cos^2 e)^4+16\sin^2 e\cos^4 e\left(\cos^2 e-\dfrac{1}{3}\right)\right\}} \\
&\qquad\times e^{i\left(pt-fx-\frac{2\pi H}{T_2V_2}\sin e-2\varphi\right)},
\end{aligned}
\right\} \quad (149)''
$$

where

$$\varphi=\tan^{-1}\left\{\frac{4\sin e\cos^2 e\left(\cos^2 e-\dfrac{1}{3}\right)^{\frac{1}{2}}}{(\sin^2 e-\cos^2 e)^2}\right\}. \qquad \ldots\ldots\ldots\ldots (150)$$

To see the natures of these waves closely, the maximum amplitudes of them corresponding to the magnitude of emergency angle e are given in Table X, XI, XII and XIII and also are shewn graphically in Fig. 12, 13, 14, 15.

We can see the following facts from these tables and figures:

1. When $0° < e < 54°44'$, the energy of the reflected waves of dilatational and distortional types which are generated by the incidence of the transmitted dilatational and distortional waves are accumulated on the free surface of the layer.

2. When $0° < e < 54°44'$, the phases of the waves generated by the incidence of the transmitted waves at the free surface are all different

Table X.

Max. Amp. \ e	0°	10°	20°	30°	40°	45°	50°	52°30′	53°30′
u_1v	0	−0·3061	−0·4315	−0·3750	−0·1722	0	0·3057	0·6450	0·9030
u_1v'	0	−0·3061	−0·4315	−0·3750	−0·1722	0	0·3057	0·6450	0·9030
u_1v''	0	0·5230	0·4730	0·1873	0·0137	0	−0·0515	−0·3665	−0·8400
u_2v''	0	0·1542	0·3140	0·2420	0·0511	0	0·1285	0·4350	0·6610
Max. Amp. \ e	**54°**	**54°44′**	**56°30′**	**57°30′**	**60°**	**65°**	**70°**	**80°**	**90°**
u_1v	1.106	1·634	0·6115	0·534	0·4330	0·3115	0·2130	0·0588	0
u_1v'	1.106	1·634	0·6115	0·534	0·4330	0·3115	0·2130	0·0588	0
u_1v''	−1·36	−3·275	−0·5760	−0·4963	−0·4330	−0·3850	−0·3165	−0·1094	0
u_2v''	0·7820	0	0·4155	0·4205	0·4330	0·4257	0·3605	0·1266	0

Table XI.

Max. Amp. \ e	0°	10°	20°	30°	40°	45°	50°	52°30′	53°30′
v_1v	0	−0·2480	−0·3405	−0·2790	−0·1133	0	0·1345	0·2040	0·2170
v_1v'	0	0·2480	0·3405	0·2790	0·1133	0	−0·1345	−0·2040	−0·2170
v_1v''	0	−0·4237	−0·3735	−0·1396	−0·0090	0	0·0226	0·1160	0·2015
v_2v''	0	0·8746	0·8615	0·4190	0·0610	0	0·1080	0·3330	0·4895
Max. Amp. \ e	**54°**	**54°44′**	**56°30′**	**57°30′**	**60°**	**65°**	**70°**	**80°**	**90°**
v_1v	0·2075	0	0·1875	0·2105	0·2500	0·2905	0·2893	0·1867	0
v_1v'	−0·2075	0	−0·1875	−0·2105	−0·2500	−0·2905	−0·2893	−0·1867	0
v_1v''	0·2550	0	0·1760	0·1960	0·2500	0·3580	0·4300	0·3470	0
v_2v''	0·5690	0	0·2750	0·2680	0·2500	0·1987	0·1315	0·0223	0

Table XII.

Max. Amp. \ e	0°	10°	20°	30°	40°	45°	50°	52°30′	53°30′
u_{2U}	0	−0·0902	−0·2865	−0·4850	−0·6430	−0·7071	−0·7630	−0·7600	−0·7110
$u_{2U'}$	0	−0·0902	−0·2865	−0·4850	−0·6430	−0·7071	−0·7630	−0·7600	−0·7110
$u_{1U''}$	0	0·3175	0·7210	0·7285	0·3450	0	−0·6100	−1·235	−1·6
$u_{2U''}$	0	−0·0937	−0·4785	−0·9380	−1·286	−1·4142	−1·517	−1·46	−1·26

Max. Amp. \ e	54°	54°44′	56°30′	57°30′	60°	65°	70°	80°	90°
u_{2U}	−0·6370	0	−0·4420	−0·4535	−0·4330	−0·3455	−0·2420	−0·0679	0
$u_{2U'}$	−0·6370	0	−0·4420	−0·4535	−0·4330	−0·3455	−0·2420	−0·0679	0
$u_{1U''}$	−1·752	0	−0·6480	−0·5730	−0·4330	−0·2300	−0·1102	−0·0081	0
$u_{2U''}$	−1·008	0	0·4690	0·4845	0·4330	0·2630	0·1260	0·00934	0

Table XIII.

Max. Amp. \ e	0°	10°	20°	30°	40°	45°	50°	52°30′	53°30′
v_{2U}	0	0·5119	0·7875	0·8400	0·7660	0·7071	0·6410	0·5820	0·5270
$v_{2U'}$	0	−0·5119	−0·7875	−0·8400	−0·7660	−0·7071	−0·6410	−0·5820	−0·5270
$v_{1U''}$	0	0·2575	0·5690	0·5423	0·2265	0	−0·2680	−0·3910	−0·3845
$v_{2U''}$	0	−0·5310	−1·313	−1·625	−1·532	−1·4142	−1·275	−1·12	−0·9315

Max. Amp. \ e	54°	54°44′	56°30′	57°30′	60°	65°	70°	80°	90°
v_{2U}	0·4630	0	0·2925	0·2885	0·2500	0·1613	0·0883	0·0119	0
$v_{2U'}$	−0·4630	0	−0·2925	−0·2885	−0·2500	−0·1613	−0·0883	−0·0119	0
$v_{1U''}$	−0·3280	0	0·1983	0·2260	0·2500	0·2145	0·1496	0·02165	0
$v_{2U''}$	−0·7325	0	0·3110	0·3085	0·2500	0·1226	0·0459	0·00165	0

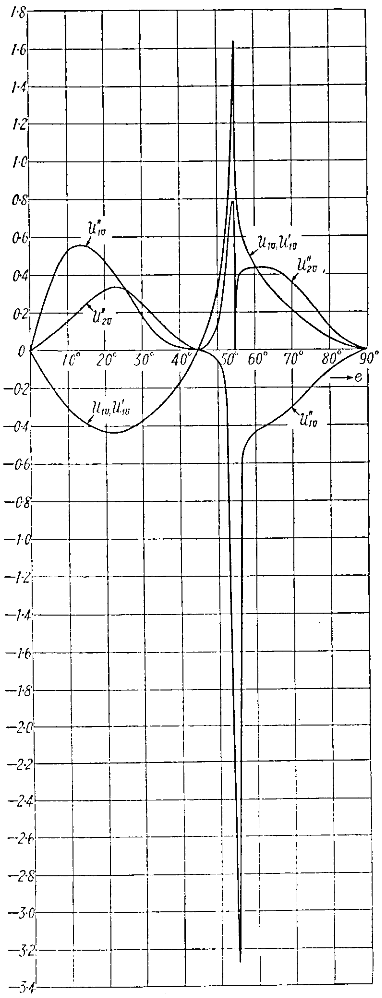

Fig. 12.

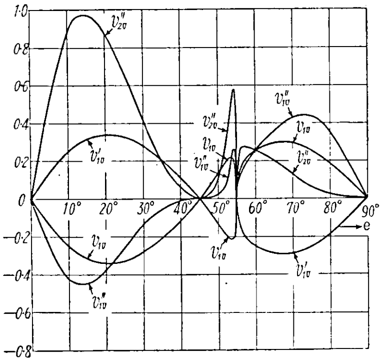

Fig. 13.

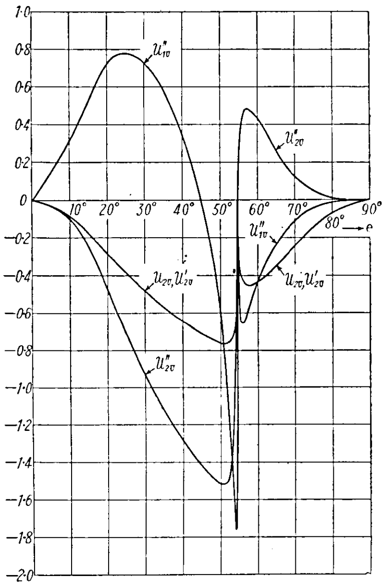

Fig. 14.

from the phase of the primary wave of distortional type incident to the lower boundary of the two solids, and the magnitudes of these differences vary with the magnitudes of e, V_2, T_2 and H.

3. When $e=0°$ and $e=90°$, there exist no transmitted waves in the upper layer, and therefore the free surface does not move.

4. When $e=45°$, no transmitted dilatational wave in the layer exists, and therefore the free surface does not move.

5. When $e=54°44'$, and the transmitted dilatational wave is incident to the free surface, the surface of the layer does only move horizontally

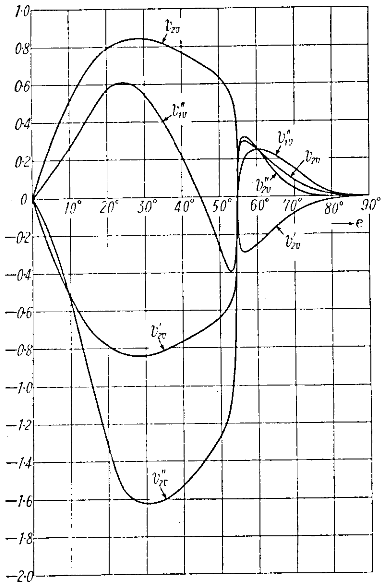

Fig. 15.

in the direction of x, and not move in the direction of y.

6. When $e=54°44'$ and the transmitted distortional wave is incident to the free surface, there is no wave in the upper layer and therefore the free surface does not move.

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10. 不連續面が彈性波の傳播に及ぼす影響 (I)

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彈性體がその不連續面でお互に自由に辷り得るとし，全く摩擦の影響がないとした時，その附近ではどの様な彈性波が生ずるものであるかを先づ研究した．次に一つの表面層を有する半無限彈性體內を傳播する波が表面にあたつた時，表面にはどの樣な波が起されるかを檢べた．(勿論表面層と半無限體との境では全く摩擦はなく自由に辷り得るものとした.)

計算の結果は可なり面白い彈性波生成に關する性質を示してゐるが，殊に二つの物質がその密度彈性係數が全く同じ時でも，その境で辷り得るとした時には，その境で色々彈性波理論上面白い生成波のある事を知つた．實際上地殼內部に不連續面があるとしても，その面で完全に兩物質が密着してゐるか如何かは分らない事であり或は多少の摩擦が存在してゐて辷り得るかも知れない故，この意味から云つても，その極端である全く摩擦がないとして彈性波の生成を研究するのも無益な事でなく又彈性理論上から云つても面白い事と思ひ，敢て本研究を始めてみた．