11. *Amplitude of Rayleigh Waves on the Surface of a Stratified Medium.*

By Takeo SUZUKI,

Seismological Institute.

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1. The propagation of Rayleigh-type waves on the stratified surface has been already investigated by J.I'A. Bromwich[1] and A. E. H. Love[2] under the assumption that the media are incompressible. However, the general relation between the thickness of the layer and the wave velocity has been completely investigated by K. Sezawa[3] without any assumption as mentioned above.

It is well known that the ratio of the horizontal displacement to the vertical on the surface is always constant and independent of the wave length for the Rayleigh waves propagated along the surface of semi-infinite elastic body. But, when the medium is covered with a superficial layer, the matter may become somewhat different. After K. Sezawa's method, the writer intends to study how the ratio of the horizontal displacement to the vertical on the surface is affected by the existence of a superficial layer.

Recently A. W. Lee[4] has investigated the effect of the layer upon the amplitude of Rayleigh waves on the surface in order to discuss the relation between geological structure and microseismic disturbances. But, the case which he has dealt with is that where the thickness of the layer is small compared with the wave length.

2. The writer will study the problem by using the solution given by Sezawa and here the method will be shown in the following. The origin of coordinates is taken on the lower boundary of the layer of thickness H and let the axis of y vertically upwards and the axis of x in the direction of propagation of the waves. Quantities in the upper layer shall be distinguished with (') from those in the lower. The equations of motion in the lower medium are expressed by

1) J. I'A. Bromwich, *London Math. Soc. Proc.*, **30** (1898).

2) A. E. H. Love, *Some Problems of Geodynamics*, (Cambridge, 1926), 165.

3) K. Sezawa, *Bull. Earthq. Res. Inst.*, **3** (1927), 1-18; *Handb. d. Geophys.*, 4 (1929), 92-94.

4) A. W. Lee, *M. N. R. A. Soc.*, *Geophys. Suppl.*, **3** (1932), 83-105.

$$\left.\begin{aligned} \rho\frac{\partial^2\Delta}{\partial t^2}&=(\lambda+2\mu)\left(\frac{\partial^2\Delta}{\partial x^2}+\frac{\partial^2\Delta}{\partial y^2}\right), \\ \rho\frac{\partial^2\varpi}{\partial t^2}&=\mu\left(\frac{\partial^2\varpi}{\partial x^2}+\frac{\partial^2\varpi}{\partial y^2}\right), \end{aligned}\right\}\quad\cdots\cdots\cdots(1)$$

where

$$\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y},$$

$$2\varpi=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}.$$

u, v=horizontal and vertical components of displacement respectively, ρ=density and λ, μ=Lamé's elastic constants.

The solutions of (1) can be written in the forms:—

$$\left.\begin{aligned} \Delta&=Ae^{ry+i(pt-fx)}, \\ 2\varpi&=Be^{sy+i(pt-fx)}, \end{aligned}\right\}\quad\cdots\cdots\cdots(2)$$

where

$$r^2=f^2-h^2, \qquad s^2=f^2-k^2,$$

$$h^2=\frac{\rho p^2}{\lambda+2\mu}, \qquad k^2=\frac{\rho p^2}{\mu}.$$

$\frac{2\pi}{f}$=wave length, and $\frac{2\pi}{p}$=period of the motion.

Let u_1, v_1 the displacement answering to Δ in (2) and satisfying $\varpi=0$, then

$$\left.\begin{aligned} u_1&=\frac{if}{h^2}Ae^{ry+i(pt-fx)}, \\ v_1&=-\frac{r}{h^2}Ae^{ry+i(pt-fx)}. \end{aligned}\right\}\quad\cdots\cdots\cdots(3)$$

Displacements (u_2, v_2) derived from 2ϖ in (2) and satisfying $\Delta=0$ are given by

$$\left.\begin{aligned} u_2&=\frac{s}{k^2}Be^{sy+i(pt-fx)}, \\ v_2&=\frac{if}{k^2}Be^{sy+i(pt-fx)}. \end{aligned}\right\}\quad\cdots\cdots\cdots(4)$$

By using the similar notations, the equations of motion in the layer are expressed by

$$\left.\begin{aligned}\rho'\frac{\partial^2\Delta'}{\partial t^2}&=(\lambda'+2\mu')\left(\frac{\partial^2\Delta'}{\partial x^2}+\frac{\partial^2\Delta'}{\partial y^2}\right),\\ \rho'\frac{\partial^2\varpi'}{\partial t^2}&=\mu'\left(\frac{\partial^2\varpi'}{\partial x^2}+\frac{\partial^2\varpi'}{\partial y^2}\right),\end{aligned}\right\}\cdots\cdots\cdots\cdots(5)$$

in which

$$\Delta'=\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y},$$

$$2\varpi'=\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}.$$

The solutions of (5), especially in case of $f^2>k'^2$, are expressed by

$$\left.\begin{aligned}\Delta'&=(C\cosh r'y+D\sinh r'y)\,e^{i(pt-fx)},\\ 2\varpi'&=(E\cosh s'y+F\sinh s'y)\,e^{i(pt-fx)},\end{aligned}\right\}\cdots\cdots\cdots\cdots(6)$$

where

$$r'^2=f^2-h'^2\gtrless 0,\qquad s'^2=f^2-k'^2\gtrless 0,$$

$$h'^2=\frac{\rho'p^2}{\lambda'+2\mu'},\qquad k'^2=\frac{\rho'p^2}{\mu'},$$

and A, B, C, D, E, F are arbitrary constants to be determined by the boundary conditions.

From (6) the following expressions will be obtained:—

$$\left.\begin{aligned}u_1'&=\frac{if}{h'^2}(C\cosh r'y+D\sinh r'y)\,e^{i(pt-fx)},\\ v_1'&=-\frac{r'}{h'^2}(C\sinh r'y+D\cosh r'y)e^{i(pt-fx)},\\ u_2'&=\frac{s'}{k'^2}(E\sinh s'y+F\cosh s'y)\,e^{i(pt-fx)},\\ v_2'&=\frac{if}{k'^2}(E\cosh s'y+F\sinh s'y)\,e^{i(pt-fx)}.\end{aligned}\right\}\cdots\cdots\cdots(7)$$

The boundary conditions are expressed by at $y=0$

$$\left.\begin{aligned}&u_1+u_2=u_1'+u_2',\\ &v_1+v_2=v_1'+v_2',\\ &\lambda\Delta+2\mu\frac{\partial}{\partial y}(v_1+v_2)=\lambda'\Delta'+2\mu'\frac{\partial}{\partial y}(v_1'+v_2'),\\ &\mu\left[\frac{\partial}{\partial y}(u_1+u_2)+\frac{\partial}{\partial x}(v_1+v_2)\right]=\mu'\left[\frac{\partial}{\partial y}(u_1'+u_2')+\frac{\partial}{\partial x}(v_1'+v_2')\right],\end{aligned}\right\}\quad(8)$$

at $y=H$

$$\left.\begin{aligned}&\lambda'\Delta'+2\mu'\frac{\partial}{\partial y}(v_1'+v_2')=0,\\&\frac{\partial}{\partial y}(u_1'+u_2')+\frac{\partial}{\partial x}(v_1'+v_2')=0.\end{aligned}\right\}\qquad\cdots\cdots(9)$$

On substitution of (3), (4), and (7), assuming that $\lambda=\mu$, $\lambda'=\mu'$ and $\rho=\rho'$, the conditions of (8) and (9) become

$$\left.\begin{aligned}&\frac{if}{h^2}A+\frac{s}{k^2}B=\frac{if}{h'^2}C+\frac{s'}{k'^2}F,\\&-\frac{r}{h^2}A+\frac{if}{k^2}B=-\frac{r'}{h'^2}D+\frac{if}{k'^2}E,\\&\frac{f^2-3r^2}{h^2}A+2\frac{ifs}{k^2}B=\frac{\mu'}{\mu}\left[\frac{f^2-3r'^2}{h'^2}C+\frac{2ifs'}{k'^2}F\right],\\&\frac{2ifr}{h^2}A+\frac{s^2+f^2}{k^2}B=\frac{\mu'}{\mu}\left[\frac{2ifr'}{h'^2}D+\frac{f^2+s'^2}{k'^2}E\right].\end{aligned}\right\}\qquad\cdots(10)$$

$$\left.\begin{aligned}&(k'^2-2f^2)(C\cosh r'H+D\sinh r'H)+\frac{2ifs'}{3}(E\sinh s'H+F\cosh s'H)=0,\\&\frac{2ifr'}{h'^2}(C\sinh r'H+D\cosh r'H)+\frac{s'^2+f^2}{k'^2}(E\cosh s'H+F\sinh s'H)=0.\end{aligned}\right\}$$

$$\cdots\cdots\cdots\cdots(11)$$

From (10)

$$\left.\begin{aligned}&\frac{\mu'}{\mu}C=PA+\frac{ifs}{3k^2}QB,\\&\frac{s'}{f}\frac{\mu'}{\mu}F=3iRA+\frac{s}{f}SB,\\&-\frac{\mu'}{\mu}E=\frac{3ifr}{k^2}QA-PB,\\&\frac{r'}{f}\frac{\mu'}{\mu}D=\frac{r}{f}SA-i\frac{R}{3}B,\end{aligned}\right\}\qquad\cdots\cdots(12)$$

where

$$P=1-\frac{2f^2}{k^2}\left(1-\frac{\mu'}{\mu}\right),\qquad Q=2\left(1-\frac{\mu'}{\mu}\right),$$

$$R=\frac{2f^2}{k^2}\left(1-\frac{\mu'}{\mu}\right),\qquad S=1+\frac{2f^2}{k^2}\left(1-\frac{\mu'}{\mu}\right).$$

Substituting (12), the first equation of (11) reduces to

$$\xi A + i\eta B = 0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

in which

$$\left.\begin{aligned} \xi &= \left(\frac{k'^2}{f^2}-2\right)\left[P\cosh r'H + \frac{r}{r'}S\sinh r'H\right] \\ &\qquad -2\left[-\frac{s'r}{k^2}Q\sinh s'H + R\cosh s'H\right], \\ 3\eta &= \left(\frac{k'^2}{f^2}-2\right)\left[\frac{fs}{k^2}Q\cosh r'H - \frac{f}{r'}R\sinh r'H\right] \\ &\qquad +2\left[\frac{s'}{f}P\sinh s'H + \frac{s}{f}S\cosh s'H\right]. \end{aligned}\right\} \qquad (14)$$

The horizontal and the vertical component of the displacement on the surface is obtained by putting $y=H$ in (7), that is

$$\left.\begin{aligned} u_1' + u_2' &= \frac{s'}{k'^2-2f^2}(E\sinh s'H + F\cosh s'H)\, e^{i(pt-fx)}, \\ v_1' + v_2' &= -\frac{1}{2if}(E\cosh s'H + F\sinh s'H)\, e^{i(pt-fx)}. \end{aligned}\right\} \ldots\ldots (15)$$

Accordingly the ratio of the horizontal displacement to the vertical on the surface is given by

$$\frac{u_1'+u_2'}{v_1'+v_2'} = i\frac{\sqrt{1-\dfrac{k'^2}{f^2}}}{1-\dfrac{k'^2}{2f^2}}\cdot\frac{\dfrac{E}{F}\tanh s'H + 1}{\dfrac{E}{F}+\tanh s'H}, \qquad \ldots\ldots\ldots\ldots (16)$$

where

$$\frac{E}{F} = \frac{s'}{f}\cdot\frac{\dfrac{3fr}{k^2}Q - P\dfrac{\xi}{\eta}}{-3R - \dfrac{s}{f}S\dfrac{\xi}{\eta}}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

When $k'^2 > f^2 > h'^2$, the solutions subjected to the boundary conditions will be obtained if $\cos s'y$ and $\sin s'y$ are taken instead of $\cosh s'y$ and $\sinh s'y$; and when $h'^2 > f^2$, similarly $\cos r'y$ and $\sin r'y$ shall be taken instead of $\cosh r'y$ and $\sinh r'y$. For these cases the ratio is expressed by

$$\frac{u_1'+u_2'}{v_1'+v_2'} = i\frac{\sqrt{\dfrac{k'^2}{f^2}-1}}{1-\dfrac{k'^2}{2f^2}}\cdot\frac{-\dfrac{E}{F}\tan s'H + 1}{\dfrac{E}{F}+\tan s'H},$$

where

$$\left.\frac{E}{F}=\frac{s'}{f}\cdot\frac{\frac{3fr}{k^2}Q-P\frac{\xi}{\eta}}{-3R-\frac{s}{f}S\frac{\xi}{\eta}}.\right\}\quad\ldots\ldots(18)$$

When $\frac{s'}{f}$ becomes zero, (16) and (18) reduce to the same expression.

Thus the value of $\frac{u}{v}$ on the surface is not always constant but a function of wave length.

3. Since the relation between the velocity of Rayleigh waves and $\frac{L}{H}$, L being the wave length, has been already completely investigated by K. Sezawa for the cases $\mu=2\mu'$, $\mu=3\mu'$, $\mu=4\mu'$, and $\mu=5\mu'$, the value of $\frac{u}{v}$ is easily obtained from the formulas expressed in (16) or (18) for each case. The results of numerical calculation are shown in Table I—IV and plotted in Fig. 1. In these tables the value of $\frac{p}{f}$ corresponding to $\frac{L}{H}$ is taken from the curves of dispersion given by K. Sezawa.

Table I. ($\mu=2\mu'$)

$\frac{L}{H}$	$\frac{p}{f}$	$\frac{u}{v}$
0	$0{\cdot}9194\sqrt{\frac{\mu'}{\rho}}$	0·6811
1·0	0·929	0·6657
1·5	0·953	0·6563
2·08	1·000	0·766
2·5	1·035	0·678
3·0	1.075	0·688
4·0	1·155	0·737
6·0	1·215	0·763
10·0	1·255	0·808
15·0	1·2714	0·8135
20·0	1·285	0·784
∞	1·301	0·684

Table II. ($\mu=3\mu'$)

$\frac{L}{H}$	$\frac{p}{f}$	$\frac{u}{v}$
0	$0{\cdot}9194\sqrt{\frac{\mu'}{\rho}}$	0·6811
1·0	0·930	0·665
1·5	0·956	0·659
1·854	1·000	0·731
2·5	1·076	0·678
3·0	1·1445	0·707
4·0	1·261	0·793
6·0	1·373	0·945
10·0	1·470	0·979
15·0	1·510	0·942
20·0	1·535	0·868
∞	1·593	0·682

Table III. ($\mu=4\mu'$)

$\frac{L}{H}$	$\frac{p}{f}$	$\frac{u}{v}$
0	$0{\cdot}9194\sqrt{\frac{\mu'}{\rho}}$	0·6811
1·0	0·930	0·667
1·5	0·956	0·6772
1·840	1·000	0·7166
2·5	1·120	0·6202
3·0	1·216	0·6202
4·0	1·365	0·744
4·48	1·414	0·8933
6·0	1·515	1·071
10·0	1·643	1·126
15·0	1·704	1·0301
18·0	1·726	0·9826
20·0	1·736	0·9588
∞	1·840	0·6830

Table IV. ($\mu=5\mu'$)

$\frac{L}{H}$	$\frac{p}{f}$	$\frac{u}{v}$
0	$0{\cdot}9194\sqrt{\frac{\mu'}{\rho}}$	0·6811
1·0	0·930	0·6682
1·5	0·956	0·6822
1·753	1·000	0·7152
2·5	1·146	0·5822
3·0	1·276	0·4851
3·66	1·414	0·511
4·00	1·465	0·6089
6·00	1·633	1·186
8·152	1·732	1·299
10·0	1·794	1·2533
15·0	1·891	1·0999
20·0	1·920	0·9943
∞	2·057	0·6866

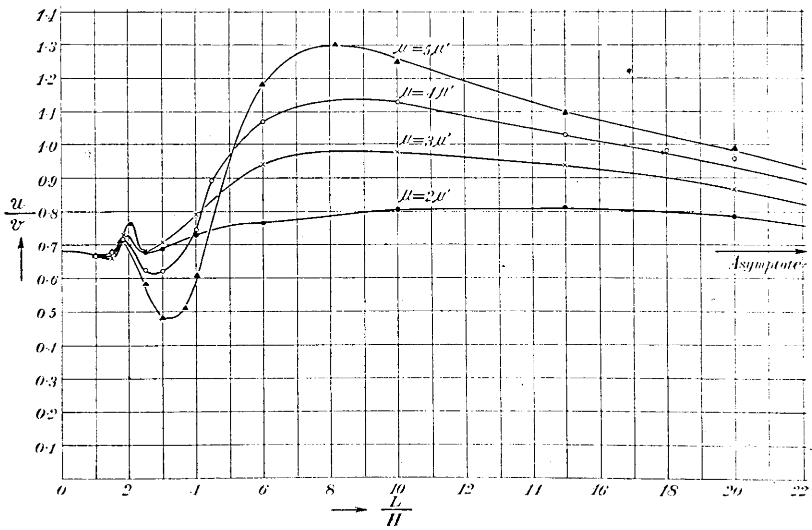

Fig. 1.

In the case where the wave length is very small compared with the thickness, $\frac{u}{v}$ approaches that of simple Rayleigh waves, that is 0·6811, as $\tanh s'H$ tends to unity in the expression of (16). When the wave length becomes very large compared with the thickness, again $\frac{u}{v}$ tends asymptotically to the value of simple Rayleigh waves. In the intermediate case where the wave length is comparable with the thickness, the velocity of propagation and the ratio of amplitude are much affected by the existence of a superficial layer.

According to the observations obtained hitherto, the ratio of the horizontal movement to the vertical of Rayleigh waves described in the seismographic records is in most cases larger than the theoretical value for simple Rayleigh waves. It may be remarked that this discrepancy may be to a certain degree explained under the plausible hypothesis of the stratified structure of the earth crust.

Next, it is still in question whether the microseismic movements with a period of a few seconds which are often observed at various places, consist mainly of progressive disturbances or of stationary vibrations proper to each locality[5]. Hence, at the present it may be safe to refrain from discussing the problems of microseisms as surface waves transmitted through the outer layers of the earth crust.

Dispersion of a solitary Rayleigh wave propagated on a surface of a stratified medium may be somewhat different from that of a kind of shocks, displacements of which have a constant $\frac{u}{v}$, what was discussed by K. Sezawa[6] and G. Nishimura, because in this case the ratio $\frac{u}{v}$ is not constant but a function of the wave length.

In conclusion the writer wishes to express his heartfelt thanks to Professor T. Matuzawa and Dr. H. Kawasumi for their kind advices bestowed on him.

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5) For example, F. OMORI, *Bull. Imp. Earthq. Invest. Commit.*, **3** (1909). T. MATUZAWA, *Journ. Fac. Sci., Imp. Univ.*, Tôkyô, Sec. II, **2** (1927), 5.

6) K. SEZAWA and G. NISHIMURA, *Bull. Earthq. Res. Inst.*, **8** (1930), 330-335.

11. 表面層のある場合の表面に於けるレーリー波の振幅

地震學教室 鈴 木 武 夫

半無限彈性體に表面層がある場合に於けるレーリー波の傳播に就ては旣にブロムウィッチ，ラブ等により論ぜられ最近には妹澤博士によりて詳しく研究せられた．筆者は妹澤博士によるレーリー波の分散の結果を用ひて表面に於ける變位の水平成分と鉛直成分との比を波長と層の厚さとの函數として計算した．この比は表面層が下層より軟い場合には波長が層の厚さに比して非常に短い場合に於ても又非常に長い場合に於ても共に普通のレーリー波の値に漸近的に近い値を示すが，波長と層の厚さとが同程度である場合には概して云へば普通のレーリー波のそれよりも大なる値となる．