12. On Stresses in the Interior and in the Vicinity of a Spherical Inclusion in a Gravitating Semi-infinite Elastic Solid. (I)

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(Read Dec., 20, 1932.—Received March 20, 1933.)

1. It is quite reasonable to suppose that the interior of the earth's crust consists of heterogeneous materials if the stress accumulation is a necessary condition for the occurrence of an earthquake, seeing that the stresses are accumulated mainly in the vicinity of common boundaries of contiguous materials having different physical properties. Thus it may be of some importance to study the effects of heterogeneous inclusions upon stress distribution in the interior of the crust of the earth. For this purpose, we shall investigate the stress distribution in the interior and in the neighbourhood of a spherical inclusion in a gravitating semi-infinite elastic solid, assuming that the state of the gravitating semi-infinite medium is one of plane strain[1], and that the elasticity and the density of the spherical inclusion differ from those of the surrounding medium. The study of stress distribution in the vicinity of a spherical cavity in a gravitating semi-infinite elastic solid, which is a special case of the present study, has already been carried out by us[2].

2. In the present study, we use the spherical coordinates (r, θ, ϕ) and the rectangular (x, y, z), the relations between them being

$$\left.\begin{aligned} x &= r\cos\theta\sin\phi, \\ y &= r\cos\theta\cos\phi, \\ z &= r\sin\theta. \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

Fig. 1 shows this relation. Let the origin o of the coordinates be at the centre of a spherical inclusion of radius a, and the distance from o to the upper horizontal surface of the solid be ξ. u, v, w indicate the

1) The problem of plane stress will be discussed on another occasion.

2) G. NISHIMURA and T. TAKAYAMA, "On the Effect of a Spherical Cavity on the Equilibrium of the Gravitating Semi-infinite Elastic Solid," *Bull. Earthq. Res. Inst.*, **10** (1932), 352–383.

components of displacement in the directions of radius r, colatitude θ, and azimuth ϕ respectively, and $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{\phi\phi}$, the normal components of traction, $\widehat{r\theta}$, $\widehat{r\phi}$, $\widehat{\theta\phi}$ the shearing components of stress separately in regard to the spherical coordinates in the interior of the semi-infinite elastic solid of which the density is ρ and the gravity constant is g. Again, let u', v', w' be the components of displacement and $\widehat{rr}'$, $\widehat{\theta\theta}'$, $\widehat{\phi\phi}'$, $\widehat{r\theta}'$, $\widehat{\theta\phi}'$ the components of stress in the interior of the spherical inclusion of which the density is ρ'.

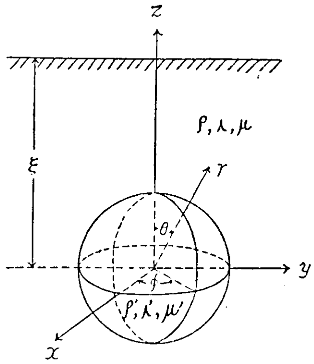

Fig. 1.

Now the top surface ($z=\xi$) of the solid is horizontal, and therefore there is no variation of azimuthal components of displacement and stress. Then the stress equations of the equilibrium of gravitating semi-infinite solid and that of gravitating spherical inclusion are respectively expressed by

$$\left.\begin{aligned}&\frac{\partial\widehat{rr}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{r\theta}}{\partial\theta}+\frac{1}{r}\left\{2\widehat{rr}-\widehat{\theta\theta}-\widehat{\phi\phi}+\widehat{r\theta}\cot\theta\right\}=\rho g\cos\theta,\\&\frac{\partial\widehat{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{\theta\theta}}{\partial\theta}+\frac{1}{r}\left\{(\widehat{\theta\theta}-\widehat{\phi\phi})\cot\theta+3\widehat{r\theta}\right\}=-\rho g\sin\theta,\end{aligned}\right\}\quad\ldots.(2)$$

and

$$\left.\begin{aligned}&\frac{\partial\widehat{rr}'}{\partial r}+\frac{1}{r}\frac{\partial\widehat{r\theta}'}{\partial\theta}+\frac{1}{r}\left\{2\widehat{rr}'-\widehat{\theta\theta}'-\widehat{\phi\phi}'+\widehat{r\theta}'\cot\theta\right\}=\rho' g\cos\theta,\\&\frac{\partial\widehat{r\theta}'}{\partial r}+\frac{1}{r}\frac{\partial\widehat{\theta\theta}'}{\partial\theta}+\frac{1}{r}\left\{(\widehat{\theta\theta}'-\widehat{\phi\phi}')\cot\theta+3\widehat{r\theta}'\right\}=-\rho' g\sin\theta.\end{aligned}\right\}\quad\ldots.(3)$$

The components of stresses of the medium and the inclusion are respectively expressed in the following forms:

$$\left.\begin{aligned}\widehat{rr}&=\frac{\lambda}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2u\sin\theta)+\frac{\partial}{\partial\theta}(rv\sin\theta)\right\}+2\mu\frac{\partial u}{\partial r},\\\widehat{\theta\theta}&=\frac{\lambda}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2u\sin\theta)+\frac{\partial}{\partial\theta}(rv\sin\theta)\right\}+2\mu\left\{\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r}\right\},\\\widehat{\phi\phi}&=\frac{\lambda}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2u\sin\theta)+\frac{\partial}{\partial\theta}(rv\sin\theta)\right\}+2\mu\left\{\frac{v}{r}\cot\theta+\frac{u}{r}\right\},\end{aligned}\right\}\quad(4)$$

$$\left.\begin{aligned}
\widehat{r\theta}&=\mu\left\{\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}\right\},\\
\widehat{rr}'&=\frac{\lambda'}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2u'\sin\theta)+\frac{\partial}{\partial\theta}(rv'\sin\theta)\right\}+2\mu'\frac{\partial u'}{\partial r},\\
\widehat{\theta\theta}'&=\frac{\lambda'}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2u'\sin\theta)+\frac{\partial}{\partial\theta}(rv'\sin\theta)\right\}+2\mu'\left\{\frac{1}{r}\frac{\partial v'}{\partial\theta}+\frac{u'}{r}\right\},\\
\widehat{\phi\phi}'&=\frac{\lambda'}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2u'\sin\theta)+\frac{\partial}{\partial\theta}(rv'\sin\theta)\right\}+2\mu'\left\{\frac{v}{r}\cot\theta+\frac{u'}{r}\right\},\\
\widehat{r\theta}'&=\mu'\left(\frac{\partial v'}{\partial r}-\frac{v'}{r}+\frac{1}{r}\frac{\partial u'}{\partial\theta}\right\},
\end{aligned}\right\}\quad(5)$$

where λ, μ are Lame's elastic constants of the gravitating semiinfinite solid and λ', μ' are those of the spherical inclusion.

In equations (2) and (3), we substitute for the components of stresses the eqations (4) and (5) respectively; and we thus obtain the following equations of equlibrium of solid and inclusion:

$$\left.\begin{aligned}
(\lambda+2\mu)\frac{\partial\Delta}{\partial r}-\frac{2\mu}{r}\frac{\partial\varpi}{\partial\theta}-\frac{2\mu}{r}\varpi\cot\theta&=\rho g\cos\theta,\\
(\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta}+2\mu\frac{\partial\varpi}{\partial r}+2\mu\frac{\varpi}{r}&=-\rho g\sin\theta,
\end{aligned}\right\}\quad\cdots\cdots\quad(6)$$

$$\left.\begin{aligned}
(\lambda'+2\mu)\frac{\partial\Delta'}{\partial r}-\frac{2\mu'}{r}\frac{\partial\varpi'}{\partial\theta}-\frac{2\mu'}{r}\varpi'\cot\theta&=\rho' g\cos\theta,\\
(\lambda'+2\mu')\frac{1}{r}\frac{\partial\Delta'}{\partial\theta}+2\mu'\frac{\partial\varpi'}{\partial r}+2\mu'&=-\rho' g\sin\theta,
\end{aligned}\right\}\quad\cdots\cdots\quad(7)$$

where

$$\left.\begin{aligned}
\Delta&=\frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2u\sin\theta)+\frac{\partial}{\partial\theta}(rv\sin\theta)\right\},\\
2\varpi&=\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{1}{r}\frac{\partial u}{\partial\theta},
\end{aligned}\right\}\quad\cdots\cdots\quad(8)$$

$$\left.\begin{aligned}
\Delta'&=\frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2u'\sin\theta)+\frac{\partial}{\partial\theta}(rv'\sin\partial)\right\},\\
2\varpi'&=\frac{\partial v'}{\partial r}+\frac{v'}{r}-\frac{1}{r}\frac{\partial u'}{\partial\theta}.
\end{aligned}\right\}\quad\cdots\cdots\quad(9)$$

Among the particular solutions satisfying (6) and (7), we take the following solutions which are favourable to the present study:

$$\left.\begin{aligned} \Delta &= \frac{\rho g r}{(\lambda+2\mu)} P_1(\cos\theta), \\ 2\varpi &= 0, \end{aligned}\right\} \qquad \text{..............(10)}$$

$$\left.\begin{aligned} \Delta' &= \frac{\rho' g r}{(\lambda'+2\mu')} P_1(\cos\theta), \\ 2\varpi' &= 0, \end{aligned}\right\} \qquad \text{..............(11)}$$

where $P_1(x)$ is the zonal harmonics of the first order.

Next we must obtain the complementary solutions of (6) and (7) which are necessary to satisfy the boundary conditions of the solids. Now we obtain the following equations, using equations (6) and (7), after some reductions:

$$\frac{\partial^2 \Delta}{\partial r^2}+\frac{2}{r}\frac{\partial \Delta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Delta}{\partial \theta^2}+\frac{\cot\theta}{r}\frac{\partial \Delta}{\partial \theta}=0, \qquad \text{............(12)}$$

$$\frac{\partial^2 \Delta'}{\partial r^2}+\frac{2}{r}\frac{\partial \Delta'}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Delta'}{\partial \theta^2}+\frac{\cot\theta}{r}\frac{\partial \Delta'}{\partial \theta}=0. \qquad \text{..........(13)}$$

We obtain the following solutions of (12) and (13):

$$\Delta = \left\{A_0+\frac{A_0'}{r}\right\} P_0(\cos\theta)+\left\{A_1 r+\frac{A_1'}{r^2}\right\} P_1(\cos\theta)$$

$$+\left\{A_2 r^2+\frac{A_2'}{r^3}\right\} P_2(\cos\theta)+\left\{A_3 r^3+\frac{A_3'}{r^4}\right\} P_3(\cos\theta), \qquad \text{........(14)}$$

$$\Delta' = D_0 P_0(\cos\theta)+D_1 r P_1(\cos\theta)+D_2 r^2 P_2(\cos\theta)+D_3 r^3 P_3(\cos\theta), \quad (15)$$

where A_0, A_0', A_1, A_1', A_2, A_2', A_3', A_3', D_0, D_1, D_2, D_3, are the arbitrary constants, and $P_0(\cos\theta)$, $P_2(\cos\theta)$, $P_3(\cos\theta)$ are the zonal harmonics of zero, second and third order.

Using (14) and (15) respectively, we obtain the following forms of 2ϖ and $2\varpi'$ which are particular solutions of (6) and (7):

$$\begin{aligned} 2\varpi = &-\frac{(\lambda+2\mu)}{\mu}\left\{\frac{A_1 r}{2}-\frac{A_1'}{r^2}\right\}\frac{\partial P_1(\cos\theta)}{\partial\theta} \\ &-\frac{(\lambda+2\mu)}{\mu}\left\{\frac{A_2 r^2}{3}-\frac{A_2'}{2r^3}\right\}\frac{\partial P_2(\cos\theta)}{\partial\theta} \\ &-\frac{(\lambda+2\mu)}{\mu}\left\{\frac{A_3 r^3}{4}-\frac{A_3'}{3r^3}\right\}\frac{\partial P_3(\cos\theta)}{\partial\theta}, \end{aligned} \qquad \text{..........(16)}$$

$$2\varpi' = -\frac{(\lambda'+2\mu')}{\mu'}\frac{D_1 r}{2}\frac{\partial P_1(\cos\theta)}{\partial\theta}$$

$$-\frac{(\lambda'+2\mu')}{\mu'}\frac{D_2r^2}{3}\frac{\partial P_2(\cos\theta)}{\partial\theta}$$

$$-\frac{(\lambda'+2\mu')}{\mu'}\frac{D_3r^2}{4}\frac{\partial P_3(\cos\theta)}{\partial\theta}. \qquad \ldots\ldots\ldots(17)$$

Now we find the following two differential equations in relation to u, v, Δ and 2ϖ after some reductions from relations expressed by (8):

$$\frac{\partial^2(r^2u)}{\partial r^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial(r^2u)}{\partial\theta}\right\}=\frac{\partial}{\partial r}(r^2\Delta)-\frac{2r}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\varpi), \quad ..(18)$$

$$\frac{\partial}{\partial r}\left\{\frac{r^2\partial(rv\sin\theta)}{\partial r}\right\}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\frac{\partial(rv\sin\theta)}{\partial\theta}\right\}+2\cot\theta\frac{\partial(r^2u)}{\partial r}$$

$$=\frac{r^2}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta\Delta)+2\sin\theta\frac{\partial}{\partial r}(r^3\varpi), \quad \ldots\ldots\ldots\ldots(19)$$

and also we have the following equations in relation to u', v', Δ' and $2\varpi'$ from the relations expressed by (9):

$$\frac{\partial^2(r^2u')}{\partial r^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial(r^2u')}{\partial\theta}\right\}=\frac{\partial}{\partial r}(r^2\Delta')-\frac{2r}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\varpi'), \ (20)$$

$$\frac{\partial}{\partial r}\left\{\frac{r^2\partial(rv'\sin\theta)}{\partial r}\right\}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial(rv'\sin\theta)}{\partial\theta}\right\}+2\cot\theta\frac{\partial(r^2u')}{\partial r}$$

$$=\frac{r^2}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta\Delta')+2\sin\theta\frac{\partial}{\partial r}(r^3\varpi'). \quad \ldots\ldots\ldots(21)$$

Substituting (10), (14) and (16) for the expressions of Δ and 2ϖ in the equation (18), we obtaion the following particular solutions of (18):

$$r^2u=\frac{3}{10}\frac{\rho gr^4}{(\lambda+2\mu)}P_1(\cos\theta)+\frac{1}{5}\frac{\rho gr^4}{(\lambda+2\mu)}P_3(\cos\theta)$$

$$+\left[\frac{1}{3}A_0r^3+\frac{(\lambda+3\mu)}{2\mu}A_0'r^2\right]P_0(\cos\theta)$$

$$+\left[\frac{(-\lambda+\mu)}{10\mu}r^4A_1-\frac{(\lambda+2\mu)}{\mu}A_1'r\right]P_1(\cos\theta)$$

$$-\left[\frac{\lambda}{7\mu}A_2r^5+\frac{(3\lambda+5\mu)}{6\mu}A_2'\right]P_2(\cos\theta)$$

$$-\left[\frac{(3\lambda+\mu)}{18\mu}A_3r^6+\frac{(2\lambda+3\mu)}{5\mu}\frac{A_3'}{r}\right]P_3(\cos\theta). \quad \ldots\ldots\ldots(22)$$

Next substituting (10), (14), (16) and (22) for the expression of Δ, 2ϖ and r^2u in equation (19), we obtain the particular solution of (19) in the following forms:

$$r v \sin\theta = \frac{1}{10}\frac{\rho g r^3}{(\lambda+2\mu)}\sin\frac{\partial P_1(\cos\theta)}{\partial\theta}+\frac{1}{15}\frac{\rho g r^3}{(\lambda+2\mu)}\sin\theta\frac{\partial P_3(\cos\theta)}{\partial\theta}$$
$$-\left[\frac{(2\lambda+3\mu)}{10\mu}A_1 r^3+\frac{(\lambda+3\mu)}{2\mu}A_1'\right]\sin\theta\frac{\partial P_1(\cos\theta)}{\partial\theta}$$
$$-\left[\frac{(5\lambda+7\mu)}{42\mu}A_2 r^4+\frac{A_2'}{6r}\right]\sin\theta\frac{\partial P_2(\cos\theta)}{\partial\theta}$$
$$-\left[\frac{(3\lambda+4\mu)}{36\mu}A_3 r^5+\frac{(-\lambda+\mu)}{30\mu}\frac{A_3'}{r^2}\right]\sin\theta\frac{\partial P_3(\cos\theta)}{\partial\theta}. \quad (23)$$

Also substituting (11), (15) and (17) in the right-hand of the equation (20), we obtain the particular solutions $r^2 u'$ which are favourable for the spherical inclusion in the following forms:

$$r^2 u' = \frac{3}{10}\frac{\rho' g r^4}{(\lambda'+2\mu')}P_1(\cos\theta)+\frac{1}{5}\frac{\rho' g r^4}{(\lambda'+2\mu')}P_3(\cos\theta)$$
$$+\frac{1}{3}D_0 r^3 P_0(\cos\theta)+\frac{(-\lambda+\mu)}{10\mu}r^4 D_1 P_1(\cos\theta)$$
$$-\frac{\lambda}{7\mu}D_2 r^5 P_2(\cos\theta)-\frac{(3\lambda+\mu)}{18\mu}D_3 r^6 P_3(\cos\theta). \;\ldots\ldots\ldots (24)$$

Using (11), (15), (17) and (24), we obtain the following particular solutions of $rv'\sin\theta$ from the equation (21):

$$r v' \sin\theta = \frac{1}{10}\frac{\rho' g r^3}{(\lambda'+2\mu')}\sin\theta\frac{\partial P_1(\cos\theta)}{\partial\theta}+\frac{1}{15}\frac{\rho' g r^3}{(\lambda'+2\mu')}\sin\theta\frac{\partial P_3(\cos\theta)}{\partial\theta}$$
$$-\frac{(2\lambda'+3\mu')}{10\mu'}D_1 r^3\sin\theta\frac{\partial P_1(\cos\theta)}{\partial\theta}$$
$$-\frac{(5\lambda'+7\mu')}{42\mu'}D_2 r^4\sin\theta\frac{\partial P_2(\cos\theta)}{\partial\theta}$$
$$-\frac{3\lambda'+4\mu'}{36\mu}P_3 r^5\sin\theta\frac{\partial P_3(\cos\theta)}{\partial\theta}. \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

The complementary solutions of (18) and (19) and (20), (21), are easily obtained in the following forms:

$$r^2 u = -C_0' P_0(\cos\theta)+\left[C_1 r^2-\frac{2}{r}C_1'\right]P_1(\cos\theta)$$
$$+\left[2r^3 C_2-\frac{3}{r^2}C_2'\right]P_2(\cos\theta)+\left[3r^4 C_3-\frac{3C_4'}{r^3}\right]P_3(\cos\theta), \;\ldots (26)$$

$$rv\sin\theta=\left[C_1r+\frac{C_1'}{r^2}\right]\sin\theta\frac{\partial P_1(\cos\theta)}{\partial\theta}+\left[C_2r^2+\frac{C_2'}{r^3}\right]\sin\theta\frac{\partial P_2(\cos\theta)}{\partial\theta}$$

$$+\left[C_3r^3+\frac{C_3'}{r^4}\right]\sin\theta\frac{\partial P_3(\cos\theta)}{\partial\theta}, \quad\ldots\ldots(27)$$

and

$$r^2u'=E_1r^2P_1(\cos\theta)+2E_2r^3P_2(\cos\theta)+3E_3r^4P_3(\cos\theta), \quad\ldots\ldots(28)$$

$$rv'\sin\theta=E_1r\sin\theta\frac{\partial P_1(\cos\theta)}{\partial\theta}+E_2r^2\sin\theta\frac{\partial P_2(\cos\theta)}{\partial\theta}$$

$$+E_3r^3\sin\theta\frac{\partial P_3(\cos\theta)}{\partial\theta}. \quad\ldots\ldots(29)$$

The general solutions of u, v and u', v' expressed by (22), (23), (26), (27) and (24), (25), (28), (29) satisfy, of course, the equations of equilibrium of elastic gravitating solid (6) and (7) respectively. Using these general expressions of displacements, we find the general expressions of components of stresses $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{\phi\phi}$, $\widehat{r\theta}$ and $\widehat{rr}'$, $\widehat{\theta\theta}'$, $\widehat{\phi\phi}'$, $\widehat{r\theta}'$ as in the following forms by the relations of (4) and (5):

$r\geq a$:—

$$\widehat{rr}=\left[\left(\lambda+\frac{2\mu}{3}\right)A_0+\frac{\lambda A_0'}{r}+\frac{4\mu}{r^3}C_0'\right]P_0(\cos\theta)$$

$$+\left[\frac{(5\lambda+6\mu)}{5(\lambda+2\mu)}\rho gr+\frac{(3\lambda+2\mu)}{5}A_1r+\frac{(3\lambda+4\mu)}{r^2}A_1'+\frac{12\mu}{r^4}C_1'\right]P_1(\cos\theta)$$

$$+\left[\frac{\lambda}{7}A_2r^2+\frac{(9\lambda+10\mu)}{3r^3}A_2'+4\mu C_2+\frac{24\mu}{r^5}C_2'\right]P_2(\cos\theta)$$

$$+\left[\frac{4\mu}{5(\lambda+2\mu)}\rho gr-\frac{(3\lambda+4\mu)}{9}A_3r^3\right.$$

$$\left.+\frac{(17\lambda+18\mu)}{5r^4}A_3'+12\mu C_3r+\frac{40\mu}{r_6}C_3'\right]P_3(\cos\theta), \quad\ldots(30)$$

$$\widehat{\theta\theta}=\left[\left(\lambda+\frac{2\mu}{3}\right)A_0+\frac{(2\lambda+3\mu)}{r}A_0'-\frac{2\mu}{r^3}C_0'\right]P_0(\cos\theta)$$

$$+\left[\frac{(5\lambda+3\mu)}{5(\lambda+2\mu)}\rho gr-\frac{(\lambda+4\mu)}{r^2}A_1'+\frac{(4\lambda+\mu)}{5}A_1r+\frac{2\mu}{r}C_1-\frac{4\mu}{r^4}C_1'\right]P_1(\cos\theta)$$

$$+\left[\frac{5}{7}\lambda A_2r^2-\frac{5\mu}{3r^3}A_2'+4\mu C_2-\frac{6\mu}{r^5}C_2'\right]P_2(\cos\theta)$$

$$+\left[\frac{2\mu}{5(\lambda+2\mu)}\rho gr+\frac{(6\lambda-\mu)}{9}A_3r^3+\frac{(\lambda-6\mu)}{5r^4}A_3'+6\mu C_3r-\frac{8\mu}{r^6}C_3'\right]P_3(\cos\theta)$$

$$+\left[\frac{\mu}{5(\lambda+2\mu)}\rho g r-\frac{(2\lambda+3\mu)}{5}A_1 r-\frac{(\lambda+3\mu)}{r^2}A_1'+\frac{2\mu}{r}C_1+\frac{2\mu}{r^4}C_1'\right]\frac{\partial^2 P_1(\cos\theta)}{\partial\theta^2}$$

$$+\left[-\frac{(5\lambda+7\mu)}{21}A_2 r^2-\frac{\mu}{3}\frac{A_2'}{r^2}+2\mu C_2+\frac{2\mu}{r^5}C_2'\right]\frac{\partial^2 P_2(\cos\theta)}{\partial\theta^2}$$

$$+\left[\frac{2\mu}{15(\lambda+2\mu)}\rho g r-\frac{(3\lambda+4\mu)}{18}A_3 r^3+\frac{(\lambda-\mu)}{15r^4}A_3'+2\mu C_3 r\right.$$

$$\left.+\frac{2\mu}{r^6}C_3'\right]\frac{\partial^2 P_3(\cos\theta)}{\partial\theta^2}, \quad \ldots\ldots(31)$$

$$\widehat{\phi\phi}=\left[\left(\lambda+\frac{2\mu}{3}\right)A_0+\frac{(2\lambda+3\mu)}{r}A_0'-\frac{2\mu}{r^3}C_0'\right]P_0(\cos\theta)$$

$$+\left[\frac{(5\lambda+3\mu)}{5(\lambda+2\mu)}\rho g r+\frac{(4\lambda+\mu)}{5}A_1 r-\frac{(\lambda+4\mu)}{r^2}A_1'+\frac{2\mu}{r}C_1-\frac{4\mu}{r^4}C_1'\right]P_1(\cos\theta)$$

$$+\left[\frac{5}{7}\lambda A_2 r^2-\frac{5\mu}{3r^2}A_2'+4\mu C_2-\frac{6\mu}{r^5}C_2'\right]P_2(\cos\theta)$$

$$+\left[\frac{2}{5}\frac{\mu\rho g r}{(\lambda+2\mu)}+\frac{(6\lambda-\mu)}{9}A_3 r^3+\frac{(\lambda-6\mu)}{5r^4}A_3'+6\mu C_3 r-\frac{8\mu}{r^6}C_3'\right]P_3(\cos\theta)$$

$$+\left[\frac{\mu\rho g r}{5(\lambda+2\mu)}-\frac{(2\lambda+3\mu)}{5}A_1 r-\frac{(\lambda+3\mu)}{r^2}A_1'+\frac{2\mu}{r}C_1+\frac{2\mu}{r^4}C_1'\right]\cot\theta\frac{\partial P_1(\cos\theta)}{\partial\theta}$$

$$+\left[-\frac{(5\lambda+7\mu)}{21}A_2 r^2-\frac{\mu}{3r^3}A_2'+2\mu C_2+\frac{2\mu}{r^5}C_2'\right]\cot\theta\,\frac{\partial P_2(\cos\theta)}{\partial\theta}$$

$$+\left[\frac{2}{15}\frac{\mu\rho g r}{(\lambda+2\mu)}-\frac{(3\lambda+4\mu)}{18}A_3 r^3+\frac{(\lambda-\mu)}{15r^4}A_3'+2\mu C_3 r\right.$$

$$\left.+\frac{2\mu}{r^6}C_3'\right]\cot\theta\frac{\partial P_3(\cos\theta)}{\partial\theta}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32)$$

$$\widehat{r\theta}=\left[\frac{2}{5}\frac{\mu\rho g r}{(\lambda+2\mu)}-\frac{(3\lambda+2\mu)}{10}A_1 r+\frac{\mu A_1'}{r^2}-\frac{6\mu}{r^4}C_1'\right]\frac{\partial P_1(\cos\theta)}{\partial\theta}$$

$$+\left[-\frac{(8\lambda+7\mu)}{21}A_2 r^2-\frac{8\mu}{r^5}C_2'-\frac{(3\lambda+2\mu)}{6r^3}A_2'+2C_2\mu\right]\frac{\partial P_2(\cos\theta)}{\partial\theta}$$

$$+\left[\frac{4}{15}\frac{\mu\rho g r}{(\lambda+2\mu)}-\frac{(15\lambda+14\mu)}{36}A_3 r^3-\frac{(8\lambda+7\mu)}{15}\frac{A_3'}{r^4}\right.$$

$$\left.+4\mu C_3 r-\frac{10\mu}{r^6}C_3'\right]\frac{\partial P_3(\cos\theta)}{\partial\theta}. \quad \ldots\ldots\ldots\ldots(33)$$

$r \leq a$:—

$$\widehat{rr}'=\left[\left(\lambda'+\frac{2\mu'}{3}\right)D_0 P_0(\cos\theta)+\left[\frac{5\lambda'+6\mu'}{5(\lambda'+2\mu')}\rho' g r+\frac{(3\lambda'+2\mu')}{5}D_1 r\right]P_1(\cos\theta)\right.$$

$$+\left[\frac{\lambda'}{7}D_2r^2+4\mu' E_2\right]P_2(\cos\theta)+\left[\frac{4\mu'}{5(\lambda'+2\mu')}\rho' gr-\frac{(3\lambda'+4\mu')}{9}D_3r^3\right.$$

$$\left.+12\mu' E_3r\right]P_3(\cos\theta), \quad\ldots\ldots\ldots\ldots(34)$$

$$\widehat{\theta\theta}'=\left[\left(\lambda'+\frac{2\mu'}{3}\right)D_0\right]P_0(\cos\theta)+\left[\frac{5\lambda'+3\mu'}{5(\lambda'+2\mu')}\rho' gr+\frac{(4\lambda'+\mu')}{5}D_1r\right]P_1(\cos\theta)$$

$$+\left[\frac{5}{7}\lambda' D_2r^2+4\mu' C_2\right]P_2(\cos\theta)+\left[\frac{2\mu'}{5(\lambda'+2\mu')}\rho' gr\right.$$

$$\left.+\frac{(6\lambda'+\mu')}{9}D_3r^3+6\mu' E_3r\right]P_3(\cos\theta)$$

$$+\left[\frac{\mu'}{5(\lambda'+2\mu')}\rho' gr-\frac{(2\lambda'+3\mu')}{5}D_1r\right]\frac{\partial^2 P_1(\cos\theta)}{\partial\theta^2}$$

$$+\left[-\frac{(5\lambda'+7\mu')}{21}P_2r^2+2\mu' E_2\right]\frac{\partial^2 P_2(\cos\theta)}{\partial\theta^2}$$

$$+\left[\frac{2\mu'}{15(\lambda'+2\mu')}\rho' gr-\frac{(3\lambda'+4\mu')}{18}D_3r^3+2\mu' E^3r\right]\frac{\partial^2 P_3(\cos\theta)}{\partial\theta^2}, \quad\ldots(35)$$

$$\widehat{\phi\phi}'=\left[\lambda'+\frac{2\mu'}{3}\right]D_0P_0(\cos\theta)+\left[\frac{5\lambda'+3\mu'}{5(\lambda'+2\mu')}\rho' gr+\frac{(4\lambda'+\mu')}{5}D_1r\right]P_1(\cos\theta)$$

$$+\left[\frac{5}{7}\lambda' D_2r^2+4\mu' E_2\right]P_2(\cos\theta)+\left[\frac{2}{5}\frac{\mu'\rho' gr}{(\lambda'+2\mu')}+\frac{(6\lambda'-\mu')}{9}D_3r^3\right.$$

$$\left.+6\mu' E_3r\right]P_3(\cos\theta)$$

$$+\left[\frac{\mu'\rho' gr}{5(\lambda'+2\mu')}-\frac{(2\lambda'+3\mu')}{5}D_1r\right]\cot\theta\frac{\partial P_1(\cos\theta)}{\partial\theta}+\left[-\frac{(5\lambda'+7\mu')}{21}D_2r^2\right.$$

$$\left.+2\mu' E^2\right]\cot\theta\frac{\partial P_2(\cos\theta)}{\partial\theta}$$

$$+\left[\frac{2}{15}\frac{\mu'\rho gr}{(\lambda'+2\mu')}-\frac{(3\lambda'+4\mu')}{18}D_3r^3+2\mu' E_3r\right]\cot\theta\frac{\partial P_3(\cos\theta)}{\partial\theta}, \quad\ldots(36)$$

$$\widehat{r\theta}'=\left[\frac{2}{5}\frac{\mu'\rho gr}{(\lambda'+2\mu')}-\frac{(3\lambda'+2\mu')}{10}D_1r\right]\frac{\partial P_1(\cos\theta)}{\partial\theta}$$

$$+\left[-\frac{(8\lambda'+7\mu')}{21}D_2r^2+2\mu' E_2\right]\frac{\partial P_2(\cos\theta)}{\partial\theta}$$

$$+\left[\frac{4}{15}\frac{\mu'\rho' gr}{(\lambda'+2\mu')}-\frac{(15\lambda'+14\mu')}{36}D_3r^3+4\mu' E_3r\right]\frac{\partial P_3(\cos\theta)}{\partial\theta}. \quad\ldots\ldots(37)$$

The general expressions of the components of stresses thus obtained satisfy, of course, the equations of equilibrium of elastic bodies under gravitating field expressed by (2) and (3).

Using these general expressions of displacements and stresses, we shall study the problem of the effect of an internal spherical heterogeneous matter on the stress distributions in the interior of gravitating semi-infinite elastic solid.

Referring to Fig. 1, we obtain the following expressions of the components of displacement and stresses which are the solutions of the gravitating semi-infinite elastic solid having no heterogeneous matter in its interior: .

$$u=-\frac{\rho g r}{3(\lambda+2\mu)}P_0(\cos\theta)+\left[\frac{\rho g\xi^2}{2(\lambda+2\mu)}+\frac{3}{10}\frac{\rho g r^2}{(\lambda+2\mu)}\right]P_1(\cos\theta)$$
$$-\frac{2}{3}\frac{\rho g\xi r}{(\lambda+2\mu)}P_2(\cos\theta)+\frac{1}{5}\frac{\rho g r^2}{(\lambda+2\mu)}P_3(\cos\theta) \quad\ldots\ldots\ldots\ldots(38)$$

$$v=\left[\frac{\rho g\xi^2}{2(\lambda+2\mu)}+\frac{1}{10}\frac{\rho g r^2}{(\lambda+2\mu)}\right]\frac{\partial P_1(\cos\theta)}{\partial\theta}$$
$$-\frac{\rho g\xi r}{3(\lambda+2\mu)}\frac{\partial P_2(\cos\theta)}{\partial\theta}+\frac{1}{15}\frac{\rho g r^2}{(\lambda+2\mu)}\frac{\partial P_3(\cos\theta)}{\partial\theta}, \quad\ldots\ldots\ldots\ldots(39)$$

$$\widehat{rr}=-\frac{(3\lambda+2\mu)}{3(\lambda+2\mu)}\rho g\xi P_0(\cos\theta)+\frac{(5\lambda+6\mu)\rho g r}{5(\lambda+2\mu)}P_1(\cos\theta)$$
$$-\frac{4}{3}\frac{\mu\rho g\xi}{(\lambda+2\mu)}P_2(\cos\theta)+\frac{4}{5}\frac{\mu\rho g r}{(\lambda+2\mu)}P_3(\cos\theta), \quad\ldots\ldots\ldots\ldots(40)$$

$$\widehat{\theta\theta}=-\frac{(3\lambda+2\mu)}{3(\lambda+2\mu)}\rho g\xi P_0(\cos\theta)+\frac{5\lambda+3\mu}{5(\lambda+2\mu)}\rho g r P_1(\cos\theta)$$
$$-\frac{4}{3}\frac{\mu\rho g\xi}{(\lambda+2\mu)}P_2(\cos\theta)+\frac{2}{5}\frac{\mu\rho g r}{(\lambda+2\mu)}P_3(\cos\theta)$$
$$+\frac{\mu\rho g r}{5(\lambda+2\mu)}\frac{\partial^2 P_1(\cos\theta)}{\partial\theta^2}-\frac{2}{3}\frac{\mu\rho g\xi}{(\lambda+2\mu)}\frac{\partial^2 P_2(\cos\theta)}{\partial\theta^2}$$
$$+\frac{2}{15}\frac{\mu\rho g r}{(\lambda+2\mu)}\frac{\partial^2 P_3(\cos\theta)}{\partial\theta^2}, \quad\ldots\ldots\ldots\ldots(41)$$

$$\widehat{\phi\phi}=-\frac{(3\lambda+2\mu)\rho g\xi}{3(\lambda+2\mu)}P_0(\cos\theta)+\frac{5\lambda+3\mu}{5(\lambda+2\mu)}\rho g r P_1(\cos\theta)$$
$$-\frac{4}{3}\frac{\mu\rho g\xi}{(\lambda+2\mu)}P_2(\cos\theta)+\frac{2}{5}\frac{\mu\rho g r}{(\lambda+2\mu)}P_3(\cos\theta)$$
$$+\frac{1}{5}\frac{\mu\rho g r}{(\lambda+2\mu)}\cot\theta\frac{\partial P_1(\cos\theta)}{\partial\theta}-\frac{2}{3}\frac{\mu\rho g\xi}{(\lambda+2\mu)}\cot\theta\frac{\partial P_2(\cos\theta)}{\partial\theta}$$

$$+\frac{2}{15}\frac{\mu\rho g r}{(\lambda+2\mu)}\cot\theta\frac{\partial P_3(\cos\theta)}{\partial\theta}, \quad\ldots\ldots(42)$$

$$\widehat{r\theta}=\frac{2}{5}\frac{\mu\rho g r}{(\lambda+2\mu)}\frac{\partial P_1(\cos\theta)}{\partial\theta}-\frac{2}{3}\frac{\mu\rho g\xi}{(\lambda+2\mu)}\frac{\partial P_2(\cos\theta)}{\partial\theta}$$

$$+\frac{4}{15}\frac{\mu\rho g r}{(\lambda+2\mu)}\frac{\partial P_3(\cos\theta)}{\partial\theta}. \quad\ldots\ldots(43)$$

These expressions satisfy, of course, the following conditions: the normal and the shearing components of stress vanish at the free surface $z=\xi$, and the stress at any point in the gravitating solid increases according to the increase of the position from the upper surface of the solid. And the state of the semi-infinite solid is one of plane strain.

Now the boundary conditions of the present problem are as follows:

1. The heterogeneous gravitating inclusion and the gravitating semi-infinite elastic medium are perfectly cemented at the contact surface and they are forbidden to slide upon each other on this contact surface.

2. The displacement and the stress at the whole spaces in the medium far from the spherical inclusion are equal to the ones expressed by (38), (39), (40), (41), (42) and (43) respectively.

These are denoted by

$$r=a;\quad u=u',\ v=v',\ \widehat{rr}=\widehat{rr}',\ \widehat{r\theta}=\widehat{r\theta}'. \quad\ldots\ldots(44)$$

$$\left.\begin{array}{lll} r\to\infty: & u=u & \text{expressed by (38)},\\ & v=v & \quad,, \quad ,,\ (39),\\ & \widehat{rr}=\widehat{rr} & \quad,, \quad ,,\ (40),\\ & \widehat{\theta\theta}=\widehat{\theta\theta} & \quad,, \quad ,,\ (41),\\ & \widehat{\phi\phi}=\widehat{\phi\phi} & \quad,, \quad ,,\ (42),\\ & \widehat{r\theta}=\widehat{r\theta} & \quad,, \quad ,,\ (43). \end{array}\right\} \quad\ldots\ldots(45)$$

Using the expressions of displacements (22), (23), (24), (25), (26), (27), (28), (29) and those of stresses (30), (31), (32), (33), (34), (35), (36), (37), we find the following values of the arbitrary constants by the conditions expressed by (44) and (45):

$$\left.\begin{array}{l} A_0=-\dfrac{\rho g\xi}{(\lambda+2\mu)},\ C_0'=\dfrac{\rho g\xi a^3(3\lambda+2\mu-3\lambda'-2\mu')}{3(\lambda+2\mu)(4\mu+3\lambda'+2\mu')},\\ A_0'=0,\ D_0'=0,\ D_0=-\dfrac{3\rho g\xi}{(4\mu+3\lambda'+2\mu')}, \end{array}\right\} (46)$$

$$
\left.\begin{aligned}
&A_1=0,\ C_1=\frac{\rho g\xi^2}{2(\lambda+2\mu)},\quad A_1'=\frac{ga^3(\rho'-\rho)}{3(\lambda+2\mu)},\\
&D_1=-\frac{2}{3}\,\frac{g\mu'\}\rho'(6\mu+5\lambda'+4\mu')-5\rho(\lambda'+2\mu')\}}{(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}},\\
&C_1'=-\frac{\rho' ga^5\{6\mu'(\lambda+2\mu)-5\mu(\lambda'+2\mu')\}}{90\mu(\lambda+2\mu)(\lambda'+2\mu')}\\
&\qquad-\frac{\rho' ga^5\mu'(3\lambda'+2\mu')(6\mu+5\lambda'+4\mu')}{90\mu(\lambda'+2\mu)\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\\
&\qquad+\frac{\rho ga^5}{90(\lambda+2\mu)}+\frac{5\rho ga^5\mu'(3\lambda'+2\mu')}{90\mu\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}},\\
&E_1=\frac{\rho g\xi^2}{2(\lambda+2\mu)}+\frac{3}{10}\,\frac{\rho ga^2}{(\lambda+2\mu)}-\frac{3}{10}\,\frac{\rho' ga^2}{(\lambda'+2\mu')}-\frac{ga^2(\rho'-\rho)}{3\mu}\\
&\qquad+\frac{ga^2\mu'(-\lambda'+\mu')\{\rho'(6\mu+5\lambda'+4\mu')-5\rho(\lambda'+2\mu')\}}{15\mu'(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\\
&\qquad+\frac{\rho' ga^2[\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}\{6\mu'(\lambda+2\mu)-5\mu(\lambda'+2\mu')\}}{45\mu(\lambda+2\mu)(\lambda'+2\mu')}\\
&\qquad\qquad\frac{+\mu'(\lambda+2\mu)(3\lambda'+2\mu')(6\mu+5\lambda'+4\mu')]}{\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\\
&\qquad-\frac{\rho ga^2[\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}\{6\mu(\lambda'+2\mu')-5\mu(\lambda'+2\mu')\}}{(\lambda+2\mu)(\lambda'+2\mu')}\\
&\qquad\qquad\frac{+5\mu'(\lambda+2\mu)(\lambda'+2\mu')(3\lambda'+2\mu')]}{\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}},
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(47)
$$

$$
\left.\begin{aligned}
&A_2=0,\quad C_2=-\frac{\rho g\xi}{3(\lambda+2\mu)},\quad D_2=0,\\
&A_2'=\frac{20\,\rho ga^3\xi\mu(\mu-\mu')}{(\lambda+2\mu)\{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')\}},\\
&C_2'=\frac{2\rho ga^5\xi(\lambda+\mu)(\mu'-\mu)}{(\lambda+2\mu)\{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')\}},\\
&E_2=-\frac{5\rho g\xi\mu}{\{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')\}},
\end{aligned}\right\}\ldots\ldots\ldots(48)
$$

$$
\left.\begin{aligned}
&A_3=0,\quad C_3=0,\quad D_3=0,\\
&A_3'=\frac{28\rho ga^5\mu(\mu'-\mu)}{(\lambda+2\mu)\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}},\\
&C_3'=-\frac{2\rho ga^7(\lambda+\mu)(\mu'-\mu)}{(\lambda+2\mu)\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}},\\
&E_3=\frac{7}{3}\,\frac{\rho g\mu}{\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}-\frac{\rho' g}{15(\lambda'+2\mu')}.
\end{aligned}\right\}\ldots\ldots\ldots(49)
$$

Substituting these values (46), (47), (48) and (49) in the expressions (22), (23), (24), (25), (26), (27), (28), (29) and (30), (31), (32), (33), (34), (45), (36), (37), we obtain the final results which are favourable to the present study in the following forms: (The expressions of the displacements u, v, u', v' are omitted here.)

$r \geqq a$:—

$$\widehat{rr}=\rho g\xi\Big[\Big\{-\frac{(3\lambda+2\mu)}{3(\lambda+2\mu)}+\frac{4a^3\mu(3\lambda+2\mu-3\lambda'-2\mu')}{3r^3(\lambda+2\mu)(4\mu+3\lambda'+2\mu')}\Big\}P_0(\cos\theta)$$
$$-\frac{4\mu}{3(\lambda+2\mu)}P_2(\cos\theta)\Big]$$
$$+\rho ga\Big[\Big\{\frac{2a^4\mu}{15r^4(\lambda+2\mu)}+\frac{2a^4\mu\mu'(3\lambda'+2\mu')}{3r^4\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\Big\}P_1(\cos\theta)$$
$$+\Big\{\frac{4r\mu}{25a(\lambda+2\mu)}+\frac{28a^4\mu(17\lambda+18\mu)(\mu'-\mu)}{5r^4(\lambda+2\mu)\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}$$
$$-\frac{80a^6\mu(\lambda+\mu)(\mu'-\mu)}{r^6(\lambda+2\mu)\{\lambda(16\mu'+19\mu)+2\mu(22\mu'13\mu)\}}\Big\}P_3(\cos\theta)\Big]$$
$$+\rho'ga\Big[-\frac{2a^4\{6\mu'(\lambda+2\mu)-5\mu(\lambda'+2\mu')\}}{15r^4(\lambda+2\mu)(\lambda'+2\mu')}$$
$$-\frac{2a^4\mu'(3\lambda'+2\mu')(6\mu+5\lambda'+4\mu')}{15r^4(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\Big]P_1(\cos\theta), \qquad \ldots\ldots(50)$$

$$\widehat{\theta\theta}=\rho g\xi\Big[\Big\{-\frac{(3\lambda+2\mu)}{3(\lambda+2\mu)}-\frac{2a^3\mu(3\lambda+2\mu-3\lambda'-2\mu')}{3r^3(\lambda+2\mu)(4\mu+3\lambda'+2\mu')}\Big\}P_0(\cos\theta)$$
$$+\Big\{-\frac{100a^3\mu^2(\mu-\mu')}{3r^3(\lambda+2\mu)\{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')\}}$$
$$-\frac{4\mu}{3(\lambda+2\mu)}-\frac{12a^5\mu(\lambda+\mu)(\mu'-\mu)}{r^5(\lambda+2\mu)\{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')\}}\Big\}P_2(\cos\theta)$$
$$+\Big\{-\frac{20a^3\mu^2(\mu-\mu')}{3r^3(\lambda+2\mu)\{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')\}}$$
$$-\frac{2\mu}{3(\lambda+2\mu)}+\frac{4a^5\mu(\lambda+\mu)(\mu'-\mu)}{r^5(\lambda+2\mu)\{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')\}}\Big\}\frac{\partial^2P_2(\cos\theta)}{\partial\theta^2}\Big]$$
$$+\rho ga\Big[\Big\{\frac{r(5\lambda+3\mu)}{5a(\lambda+2\mu)}+\frac{a^2(\lambda+4\mu)}{3r^2(\lambda+2\mu)}-\frac{4a^4\mu}{90r^4(\lambda+2\mu)}$$
$$-\frac{2a^4\mu'(3\lambda'+2\mu')}{9r^4\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\Big\}P_1(\cos\theta)$$

$$+\left\{\frac{2r\mu}{5a(\lambda+2\mu)}+\frac{28a^4\mu(\mu'-\mu)(\lambda-6\mu)}{5r^4(\lambda+2\mu)\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}\right.$$
$$\left.+\frac{16a^6\mu(\lambda+\mu)(\mu'-\mu)}{r^6(\lambda+22\mu)\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}\right\}P_3(\cos\theta)$$
$$+\left\{\frac{r\mu}{5a(\lambda+2\mu)}+\frac{a^2(\lambda+3\mu)}{3r^2(\lambda+2\mu)}+\frac{a^4\mu}{45r^4(\lambda+2\mu)}\right.$$
$$\left.+\frac{a^4\mu'(3\lambda'+2\mu')}{9r^4\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\right\}\frac{\partial^2 P_1(\cos\theta)}{\partial\theta^2}$$
$$+\left\{\frac{2r\mu}{15a(\lambda+2\mu)}+\frac{28a^4\mu(\lambda-\mu)(\mu'-\mu)}{15r^4(\lambda+2\mu)\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}\right.$$
$$\left.\left.-\frac{4a^6\mu(\lambda+\mu)(\mu'-\mu)}{r^6(\lambda+2\mu)\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}\right\}\frac{\partial^2 P_3(\cos\theta)}{\partial\theta^2}\right]$$
$$+\rho' ga\left[\left\{-\frac{a^2(\lambda+4\mu)}{3r^2(\lambda+2\mu)}+\frac{4a^4\{6\mu'(\lambda+2\mu)-5\mu(\lambda'+2\mu')\}}{90r^4(\lambda+2\mu)(\lambda'+2\mu')}\right.\right.$$
$$\left.+\frac{2a^4\mu'(3\lambda'+2\mu')(6\mu+5\lambda'+4\mu')}{45r^4(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\right\}P_1(\cos\theta)$$
$$+\left\{-\frac{a^2(\lambda+3\mu)}{3r^2(\lambda+2\mu)}-\frac{a^4\{6\mu'(\lambda+2\mu)-5\mu(\lambda'+2\mu')\}}{45r^4(\lambda+2\mu)(\lambda'+2\mu')}\right.$$
$$\left.\left.-\frac{2a^4\mu'(3\lambda'+2\mu')(6\mu+5\lambda'+4\mu')}{90r^4(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\right\}\frac{\partial^2 P_1(\cos\theta)}{\partial\theta^2}\right],$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\ldots\ldots\ldots\ldots(51)$$

$$\widehat{\phi\phi}=\rho g\xi\left[\left\{-\frac{3\lambda+2\mu}{3(\lambda+2\mu)}-\frac{2a^3(3\lambda+2\mu-3\lambda'-2\mu')}{3r^3(\lambda+2\mu)(4\mu+3\lambda'+2\mu')}\right\}P_0(\cos\theta)\right.$$
$$+\left\{-\frac{100a^3\mu(\mu-\mu')}{3r^3(\lambda+2\mu)\{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')\}}\right.$$
$$\left.-\frac{4\mu}{3(\lambda+2\mu)}-\frac{12a^5\mu(\lambda+\mu)(\mu'-\mu)}{r^5(\lambda+2\mu)\{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')\}}\right\}P_2(\cos\theta)$$
$$+\left\{\frac{-20a^3\mu^2(\mu-\mu')}{3r^3(\lambda+2\mu)\{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')\}}-\frac{2\mu}{3(\lambda+2\mu)}\right.$$
$$\left.\left.+\frac{4a^5\mu(\lambda+\mu)(\mu'-\mu)}{r^5(\lambda+2\mu)\{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')\}}\right\}\cot\theta\frac{\partial P_2(\cos\theta)}{\partial\theta}\right]$$
$$+\rho ga\left[\left\{\frac{r(5\lambda+3\mu)}{5a(\lambda+2\mu)}+\frac{a^2(\lambda+4\mu)}{3r^2(\lambda+2\mu)}-\frac{2a^4\mu}{45r^4(\lambda+2\mu)}\right.\right.$$
$$\left.-\frac{2a^4\mu'(3\lambda'+2\mu')}{9r^4\{2\mu(\mu'+4\mu')+\mu'(3\lambda'+2\mu')\}}\right\}P_1(\cos\theta)$$

$$
+\left\{\frac{2r\mu}{5a(\lambda+2\mu)}+\frac{28a^4\mu(\lambda-6\mu)(\mu'-\mu)}{5r^4(\lambda+2\mu)\{\lambda(16\mu'+19\mu)+2\mu)22\mu'+13\mu)}\right.
$$

$$
\left.+\frac{16a^6\mu(\lambda+\mu)(\mu'-\mu)}{r^6(\lambda+2\mu)\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}\right\}P_3(\cos\theta)
$$

$$
+\left\{\frac{r\mu}{5a(\lambda+2\mu)}+\frac{a^2(\lambda+3\mu)}{3r^2(\lambda+2\mu)}+\frac{a^4\mu}{45r^4(\lambda+2\mu)}\right.
$$

$$
\left.+\frac{a^4\mu'(3\lambda'+2\mu')}{9r^4\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\right\}\cot\theta\frac{\partial P_1(\cos\theta)}{\partial\theta}
$$

$$
+\left\{\frac{2r\mu}{15a(\lambda+2\mu)}+\frac{28a^4\mu(\lambda-\mu)(\mu'-\mu)}{15r^4(\lambda+2\mu)\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}\right.
$$

$$
\left.\left.-\frac{4a^6\mu(\lambda+\mu)(\mu'-\mu)}{r^6(\lambda+2\mu)\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}\right\}\frac{\partial P_3(\cos\theta)}{\partial\theta}\right]
$$

$$
+\rho' ga\left[\left\{-\frac{a^2(\lambda+4\mu)}{3r^2(\lambda+2\mu)}+\frac{2a^4\{6\mu'(\lambda+2\mu)-5\mu(\lambda'+2\mu')\}}{45r^4(\lambda+2\mu)(\lambda'+2\mu')}\right.\right.
$$

$$
\left.+\frac{2a^4\mu'(3\lambda'+2\mu')(6\mu+5\lambda'+4\mu')}{45r^4(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3'\lambda+2\mu')\}}\right\}P_1(\cos\theta)
$$

$$
+\left\{-\frac{a^2(\lambda+3\mu)}{3r^2(\lambda+2\mu)}-\frac{a^4\{6\mu'(\lambda+2\mu)-5\mu(\lambda'+2\mu')\}}{45r^4(\lambda+2\mu)(r'+2\mu')}\right.
$$

$$
\left.\left.-\frac{a^4\mu'(3\lambda'+2\mu')(6\mu+5\lambda'+4\mu')}{45r^4(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\right\}\cot\theta\frac{\partial P_1(\cos\theta)}{\partial\theta}\right]
$$

$$
\dots\dots\dots\dots(52)
$$

$$
\widehat{r\theta}=\rho g\xi\left[\left\{-\frac{16a^5\mu(\lambda+\mu)(\mu'-\mu)}{r^5(\lambda+2\mu)\{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')\}}\right.\right.
$$

$$
-\frac{10a^3\mu(3\lambda+2\mu)(\mu-\mu')}{3r^3(\lambda+2\mu)\{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')\}}
$$

$$
\left.\left.-\frac{2\mu}{3(\lambda+2\mu)}\right\}\frac{\partial P_2(\cos\theta)}{\partial\theta}\right]
$$

$$
+\rho ga\left[\left\{\frac{2r\mu}{5a(\lambda+2\mu)}-\frac{a^2\mu}{3r(\lambda+2\mu)}-\frac{2\mu a^4}{30r^4(\lambda+2\mu)}\right.\right.
$$

$$
\left.-\frac{a^4\mu'(3\lambda'+2\mu')}{3r^4\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\right\}\frac{\partial P_1(\cos\theta)}{\partial\theta}
$$

$$
+\left\{\frac{4r\mu}{15a(\lambda+2\mu)}-\frac{28a^4\mu(8\lambda+7\mu)(\mu'-\mu)}{15r^4(\lambda+2\mu)\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}\right.
$$

$$
\left.\left.+\frac{20a^6\mu(\lambda+\mu)(\mu'-\mu)}{r^6(\lambda+2\mu)\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}\right\}\frac{\partial P_3(\cos\theta)}{\partial\theta}\right]
$$

$$+\rho' g a\left[\left\{\frac{a^2\mu}{3r^2(\lambda+2\mu)}+\frac{a^4\mu'(3\lambda'+2\mu')(6\mu+5\lambda'+4\mu')}{15r^4(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\right.\right.$$
$$\left.\left.+\frac{a^4\{6\mu'(\lambda+2\mu)-5\mu(\lambda'+2\mu')\}}{15r^4(\lambda+2\mu)(\lambda'+2\mu')}\right\}\frac{\partial P_1(\cos\theta)}{\partial\theta}\right] \quad \ldots\ldots\ldots(53)$$

$r\leq a$:

$$\widehat{rr}'=\rho g\xi\left[-\frac{(3\lambda'+2\mu')}{(4\mu+3\lambda'+2\mu')}P_0(\cos\theta)-\frac{20\mu\mu'}{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')}P_2(\cos\theta)\right]$$
$$+\rho g a\left[\frac{2r(3\lambda'+2\mu')(\lambda'+2\mu')}{3a(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}P_1(\cos\theta)\right.$$
$$\left.+\frac{28r\mu\mu'}{a\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}P_3(\cos\theta)\right]$$
$$+\rho' g a\left[\left\{\frac{r(5\lambda'+6\mu')}{5a(\lambda'+2\mu')}-\frac{2r(3\lambda'+2\mu')\mu'(6\mu+5\lambda'+4\mu')}{3a(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\right\}P_1(\cos\theta)\right.$$
$$\left.-\frac{4r\mu'}{5(\lambda'+2\mu')}P_3(\cos\theta)\right],\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(54)$$

$$\widehat{\theta\theta}'=\rho g\xi\left[-\frac{3\lambda'+2\mu'}{4\mu+3\lambda'+2\mu'}P_0(\cos\theta)-\frac{4\mu'}{3(\lambda+2\mu)}P_2(\cos\theta)\right.$$
$$\left.-\frac{10\mu\mu'}{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')}\frac{\partial^2P_2(\cos\theta)}{\partial\theta^2}\right]$$
$$+\rho g a\left[\frac{2r\mu'(4\lambda'+\mu')(\lambda'+2\mu')}{3a(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}P_1(\cos\theta)\right.$$
$$+\frac{14r\mu}{a\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}P_3(\cos\theta)$$
$$-\frac{2r(2\lambda'+3\mu')(\lambda'+2\mu')}{3a(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\frac{\partial^2P_1(\cos\theta)}{\partial\theta^2}$$
$$\left.+\frac{14r\mu\mu'}{3a\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)}\frac{\partial^2P_3(\cos\theta)}{\partial\theta^2}\right]$$
$$+\rho' g a\left[\left\{\frac{r(5\lambda'+3\mu')}{5a(\lambda'+2\mu')}-\frac{2r\mu'(4\lambda'+\mu')(6\mu+5\lambda'+4\mu')}{15a(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\right\}P_1(\cos\theta)\right.$$
$$+\left\{\frac{r\mu'}{5a(\lambda'+2\mu')}+\frac{2r\mu'(2\lambda'+3\mu')(6\mu+5\lambda'+4\mu')}{15a(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\right\}$$
$$\left.\times\frac{\partial^2P_1(\cos\theta)}{\partial\theta^2}\right],$$
$$\ldots\ldots\ldots(55)$$

$$\widehat{\phi\phi}'=\rho g\xi\Big[-\frac{(3\lambda'+2\mu')}{4\mu+3\lambda'+2\mu'}P_0(\cos\theta)-\frac{20\mu\mu'}{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')}P_2(\cos\theta)$$

$$-\frac{10\mu\mu'}{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')}\cot\theta\frac{P_2(\cos\theta)}{\partial\theta}\Big]$$

$$+\rho ga\Big[\frac{2r\mu'(4\lambda'+\mu')(\lambda'+2\mu')}{3(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}P_1(\cos\theta)$$

$$+\frac{14\,r\mu\mu'}{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)}P_3(\cos\theta)$$

$$+\frac{14r\mu\mu'}{3a\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}\cot\theta\frac{\partial P_3(\cos\theta)}{\partial\theta}\Big]$$

$$+\rho' ga\Big[\Big\{\frac{r(5\lambda'+3\mu')}{5a(\lambda'+2\mu')}-\frac{2r\mu'(4\lambda'+\mu')(6\mu+5\lambda'+4\mu')}{15a(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\Big\}P_1(\cos\theta)$$

$$+\Big\{\frac{2r\mu'}{5a(\lambda'+2\mu')}-\frac{2r\mu'}{5(\lambda'+2\mu')}\Big\}P_3(\cos\theta)$$

$$+\Big\{\frac{r\mu'}{5a(\lambda'+2\mu')}+\frac{2r\mu'(2\lambda'+3\mu')(6\mu+5\lambda'+4\mu')}{15a(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\Big\}$$

$$\times\cot\theta\frac{\partial P_1(\cos\theta)}{\partial\theta}\Big], \qquad \ldots\ldots\ldots\ldots(56)$$

$$\widehat{r\theta}'=\rho g\xi\Big[-\frac{10\mu\mu'}{3\lambda(3\mu+2\mu')+2\mu(7\mu+8\mu')}\frac{\partial P_2(\cos\theta)}{\partial\theta}\Big]$$

$$+\rho ga\Big[-\frac{r(3\lambda'+2\mu')(\lambda'+2\mu')}{3a(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\frac{\partial P_1(\cos\theta)}{\partial\theta}$$

$$+\frac{7r\mu\mu'}{3a\{\lambda(16\mu'+19\mu)+2\mu(22\mu'+13\mu)\}}\frac{\partial P_3(\cos\theta)}{\partial\theta}\Big]$$

$$+\rho' ga\Big[\Big\{\frac{2}{5}\frac{r\mu'}{a(\lambda'+2\mu')}+\frac{r\mu'(3\lambda'+2\mu')(6\mu+5\lambda'+4\mu')}{15a(\lambda'+2\mu')\{2\mu(\lambda'+4\mu')+\mu'(3\lambda'+2\mu')\}}\Big\}$$

$$\times\frac{\partial P_1(\cos\theta)}{\partial\theta}\Big]. \qquad \ldots\ldots\ldots\ldots(57)$$

When $\lambda=\mu$, $\lambda'=\mu'$, namely the Poisson's ratio of the semi-infinite solid and that of the heterogeneous inclusion are equally $\frac{1}{4}$, we obtain the expressions of the stress distributions in the solid and those in the inclusion as follows:

$r \geqq a$:--

$$\widehat{rr}=\rho g\xi\Big[-\frac{2}{3}+\Big\{\frac{20(\mu-\mu')}{9(4\mu+5\mu')}+\frac{95(\mu-\mu')}{9(23\mu+22\mu')}\Big\}\Big(\frac{a}{r}\Big)^3+\frac{8(\mu'-\mu)}{23\mu+22\mu'}\Big(\frac{a}{r}\Big)^5$$
$$+\Big\{-\frac{1}{3}+\frac{95}{3}\frac{(\mu-\mu')}{(23\mu+22\mu')}\Big(\frac{a}{r}\Big)^3+\frac{24(\mu'-\mu)}{(23\mu+22\mu')}\Big(\frac{a}{r}\Big)^5\Big\}\cos 2\theta\Big]$$
$$+\rho g a\Big[\Big\{\frac{5}{6}\Big(\frac{r}{a}\Big)-\frac{7}{9}\Big(\frac{a}{r}\Big)^2+\frac{4(\mu+8\mu')}{45(2\mu+\mu')}\Big(\frac{a}{r}\Big)^4$$
$$+\frac{49(\mu'-\mu)}{30(4\mu'+3\mu)}\Big(\frac{a}{r}\Big)^4-\frac{4(\mu'-\mu)}{3(4\mu'+3\mu)}\Big(\frac{a}{r}\Big)^6\Big\}\cos\theta$$
$$+\Big\{\frac{1}{6}\Big(\frac{r}{a}\Big)+\frac{49(\mu'-\mu)}{18(4\mu'+3\mu)}\Big(\frac{a}{r}\Big)^4-\frac{20(\mu'-\mu)}{9(4\mu'+3\mu)}\Big(\frac{a}{r}\Big)^6\Big\}\cos 3\theta\Big]$$
$$+\rho' g a\Big[\frac{7}{9}\Big(\frac{a}{r}\Big)^2-\frac{4(4\mu+5\mu')}{45(2\mu+\mu')}\Big(\frac{a}{r}\Big)^4\cos\theta\Big], \qquad \text{(58)}$$

$$\widehat{\theta\theta}=\rho g\xi\Big[-\frac{2}{3}-\Big\{\frac{10(\mu-\mu')}{9(4\mu+5\mu')}+\frac{25(\mu-\mu')}{9(23\mu+22\mu')}\Big\}\Big(\frac{a}{r}\Big)^3$$
$$-\frac{2(\mu'-\mu)}{(23\mu+22\mu')}\Big(\frac{a}{r}\Big)^5+\Big\{\frac{1}{3}-\frac{5(\mu-\mu')}{3(23\mu+22\mu')}\Big(\frac{a}{r}\Big)^3$$
$$-\frac{14(\mu'-\mu)}{(23\mu+22\mu')}\Big(\frac{a}{r}\Big)^5\Big\}\cos 2\theta\Big]$$
$$+\rho g a\Big[\Big\{\frac{1}{2}\Big(\frac{r}{a}\Big)+\frac{1}{9}\Big(\frac{a}{r}\Big)^2-\Big\{\frac{6(\mu+8\mu')}{135(2\mu+\mu')}+\frac{7(\mu'-\mu)}{30(4\mu'+3\mu)}\Big\}\Big(\frac{a}{r}\Big)^4$$
$$+\frac{(\mu'-\mu)}{3(4\mu'+3\mu)}\Big(\frac{a}{r}\Big)^6\Big\}\cos\theta$$
$$+\Big\{-\frac{1}{6}\Big(\frac{r}{a}\Big)-\frac{7(\mu'-\mu)}{18(4\mu'+3\mu)}\Big(\frac{a}{r}\Big)^4+\frac{13(\mu'-\mu)}{9(4\mu'+3\mu)}\Big(\frac{a}{r}\Big)^6\Big\}\cos 3\theta\Big]$$
$$+\rho' g a\Big[\Big\{-\frac{1}{9}\Big(\frac{a}{r}\Big)^2+\frac{6(4\mu+5\mu')}{135(2\mu+\mu')}\Big(\frac{a}{r}\Big)^4\Big\}\cos\theta\Big], \qquad \text{(59)}$$

$$\widehat{\phi\phi}=\rho g\xi\Big[-\frac{1}{3}+\Big\{-\frac{10(\mu-\mu')}{9(4\mu+5\mu')}+\frac{5(\mu-\mu')}{9(23\mu+22\mu')}\Big\}\Big(\frac{a}{r}\Big)^3$$
$$-\frac{6(\mu'-\mu)}{(23\mu+22\mu')}\Big(\frac{a}{r}\Big)^5+\Big\{-\frac{5\mu(\mu-\mu')}{23\mu+22\mu'}\Big(\frac{a}{r}\Big)^3-\frac{10(\mu'-\mu)}{23\mu+22\mu'}\Big(\frac{a}{r}\Big)^5\Big\}\cos 2\theta\Big]$$

$$+\rho ga\Big[\Big\{\frac{1}{3}\Big(\frac{r}{a}\Big)+\frac{1}{9}\Big(\frac{a}{r}\Big)^2-\frac{7(\mu'-\mu)}{30(4\mu'+3\mu)}\Big(\frac{a}{r}\Big)^4-\frac{6(\mu+8\mu')}{135(2\mu+\mu')}\Big(\frac{a}{r}\Big)^4$$

$$+\frac{\mu'-\mu}{4\mu'+3\mu}\Big(\frac{a}{r}\Big)^6\Big\}\cos\theta$$

$$+\Big\{-\frac{7(\mu'-\mu)}{18(4\mu'+3\mu)}\Big(\frac{a}{r}\Big)^4+\frac{7(\mu'-\mu)}{9(4\mu'+3\mu)}\Big(\frac{a}{r}\Big)^6\Big\}\cos 3\theta\Big]$$

$$+\rho' ga\Big[-\frac{1}{9}\Big(\frac{a}{r}\Big)^2+\frac{6(4\mu+5\mu')}{135(2\mu+\mu')}\Big(\frac{a}{r}\Big)^4\Big]\cos\theta, \quad\ldots\ldots\ldots\ldots\ldots\ldots (60)$$

$$\widehat{r\theta}=\rho g\xi\Big[\frac{1}{3}+\frac{25(\mu-\mu')}{3(23\mu+22\mu')}\Big(\frac{a}{r}\Big)^3+\frac{16(\mu'-\mu)}{(23\mu+22\mu')}\Big(\frac{a}{r}\Big)^5\Big]\sin 2\theta$$

$$+\rho ga\Big[\Big\{-\frac{1}{6}\Big(\frac{r}{a}\Big)+\frac{1}{9}\Big(\frac{a}{r}\Big)^2+\frac{6(\mu+8\mu')}{135(2\mu+\mu')}\Big(\frac{a}{r}\Big)^4$$

$$+\frac{7(\mu'-\mu)}{30(4\mu'+3\mu)}\Big(\frac{a}{r}\Big)^4-\frac{(\mu'-\mu)}{3(4\mu'+3\mu)}\Big(\frac{a}{r}\Big)^6\Big\}\sin\theta$$

$$+\Big\{-\frac{1}{6}\Big(\frac{r}{a}\Big)+\frac{7(\mu'-\mu)}{6(4\mu'+3\mu)}\Big(\frac{a}{r}\Big)^4-\frac{5(\mu'-\mu)}{3(4\mu'+3\mu)}\Big(\frac{a}{r}\Big)^6\Big\}\sin 3\theta\Big]$$

$$+\rho' ga\Big[-\frac{1}{9}\Big(\frac{a}{r}\Big)^2-\frac{6(4\mu+5\mu')}{135(2\mu+\mu')}\Big(\frac{a}{r}\Big)^4\Big]\sin\theta. \quad\ldots\ldots\ldots\ldots\ldots\ldots (61)$$

$r\leqq a$:—

$$\widehat{rr'}=\rho g\xi\Big[-\frac{5\mu'}{4\mu+5\mu'}-\frac{5\mu'}{23\mu+22\mu'}-\frac{15\mu'}{23\mu+22\mu'}\cos 2\theta\Big]$$

$$+\rho ga\Big[\Big\{\frac{2\mu'}{3(2\mu+\mu')}\Big(\frac{r}{a}\Big)+\frac{7\mu'}{10(4\mu'+3\mu)}\Big(\frac{r}{a}\Big)\Big\}\cos\theta$$

$$+\frac{7\mu'}{6(4\mu'+3\mu)}\Big(\frac{r}{a}\Big)\cos 3\theta\Big]$$

$$+\rho' ga\Big[\Big\{\frac{11}{15}\Big(\frac{r}{a}\Big)-\frac{2(2\mu+3\mu')}{15(2\mu+\mu')}\Big(\frac{r}{a}\Big)\Big\}\cos\theta\Big], \quad\ldots\ldots\ldots\ldots\ldots\ldots (62)$$

$$\widehat{\theta\theta'}=\rho g\xi\Big[-\frac{5\mu'}{4\mu+5\mu'}-\frac{5\mu'}{23\mu+22\mu'}+\frac{15\mu'}{23\mu+22\mu'}\cos 2\theta\Big]$$

$$+\rho ga\Big[\frac{4\mu'}{3(2\mu+\mu')}\Big(\frac{r}{a}\Big)+\frac{7\mu'}{30(4\mu'+3\mu)}\Big(\frac{r}{a}\Big)\Big\}\cos\theta-\frac{7\mu'}{6(4\mu'+3\mu)}\Big(\frac{r}{a}\Big)\cos 3\theta\Big]$$

$$+\rho' ga\Big[\frac{7}{15}\Big(\frac{r}{a}\Big)-\frac{4}{15}\frac{(2\mu+3\mu')}{(2\mu+\mu')}\Big(\frac{r}{a}\Big)\Big]\cos\theta, \quad\ldots\ldots\ldots\ldots\ldots\ldots (63)$$

$$\widehat{\phi\phi'}=\rho g\xi\Big[-\frac{5\mu'}{4\mu+5\mu'}+\frac{10\mu'}{23\mu+22\mu'}-\frac{15\mu'}{2(23\mu+22\mu')}\cos 2\theta\Big]$$

$$+\rho g a\left[\frac{4\mu'}{3(2\mu+\mu')}\left(\frac{r}{a}\right)-\frac{14\mu'}{15(4\mu'+3\mu)}\left(\frac{r}{a}\right)\right]\cos\theta$$

$$+\rho' g a\left[\frac{7}{15}\left(\frac{r}{a}\right)-\frac{4(2\mu+3\mu')}{15(2\mu+\mu')}\left(\frac{r}{a}\right)\right]\cos\theta, \quad\ldots\ldots(64)$$

$$\widehat{r\theta}'=\rho g\xi\left[\frac{15\mu'}{23\mu+22\mu'}\sin 2\theta\right]$$

$$+\rho g a\left[\left\{\frac{\mu'}{3(2\mu+\mu')}\left(\frac{r}{a}\right)-\frac{7\mu'}{30(4\mu'+3\mu)}\left(\frac{r}{a}\right)\right\}\sin\theta\right.$$

$$\left.-\frac{7\mu'}{6(4\mu'+3\mu)}\left(\frac{r}{a}\right)\sin 3\theta\right]$$

$$+\rho' g a\left[-\frac{2}{15}\left(\frac{r}{a}\right)-\frac{2\mu+3\mu'}{15(2\mu+\mu')}\left(\frac{r}{a}\right)\right]\sin\theta. \quad\ldots\ldots(65)$$

To obtain these expressions (58), (59), (60), (61), (62), (63), (64) and (65), use has been made of the following formulae:

$$\left.\begin{array}{l} P_0(\cos\theta)=1,\ P_1(\cos\theta)=\cos\theta,\ P_2(\cos\theta)=\frac{1}{4}(1+3\cos 2\theta), \\ \qquad P_3(\cos\theta)=\frac{1}{8}(3\cos\theta+5\cos 3\theta). \end{array}\right\}\quad\ldots\ldots(66)$$

To ascertain the distributions of stresses in the medium and those in the spherical inclusion, we take the following numerical examples. Putting $\frac{r}{a}=1$, we tabulated the magnitudes of the separate terms related to $\rho g\xi$, $\rho g a$ and $\rho' g a$ in all stress components expressed by (58), (59), (60), (61), (62), (63), (64), (65) in the annexed tables when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5, and ∞ (Table I, II, III, IV, V, VI, VII, VIII, IX, X, XI, XII, XIII, XIV, XV, XVI, XVII, XVIII), and the following figures (Fig. 2a, 2b, 2c, 3a, 3b, 3c, 4a, 4b, 4c, 5a, 5b, 5c, 6a, 6b, 6c, 7a, 7b, 7c) shew us these results. In these figures, the abscissa is the angle of colatitude θ from 0° to 180°, and the parameter of each curve is the magnitude of $\frac{\mu'}{\mu}$.

From these figures we can see many valuable facts, and some of them are summarised as follows:

1. The terms related to $\rho g\xi$ of the components of stresses $\widehat{rr}'_{r=a}$, $\widehat{\theta\theta}'_{r=a}$, $\widehat{\phi\phi}'_{r=a}$ of the heterogeneous spherical inclusion and those of the components of stresses $\widehat{rr}_{r=a}$, $\widehat{\theta\theta}_{r=a}$, $\widehat{\phi\phi}_{r=a}$ of the surrounding medium have

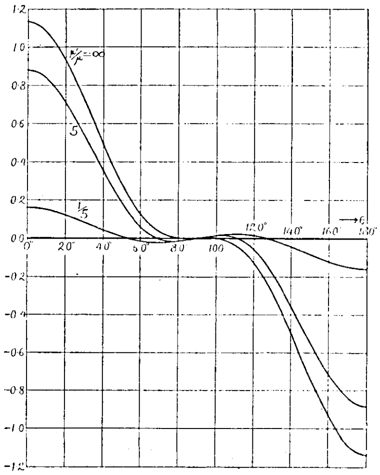

Fig. 2a. Magnitude of the term related to ρga of $\widehat{rr}_{r=a}$ $(=\widehat{rr'}_{r=a})$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

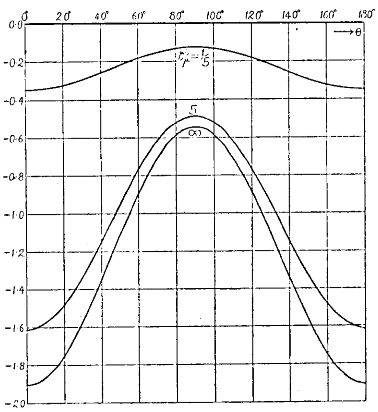

Fig. 2b. Magnitude of the term related to $\rho g\xi$ of $\widehat{rr}_{r=a}$ $(=\widehat{rr'}_{r=a})$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

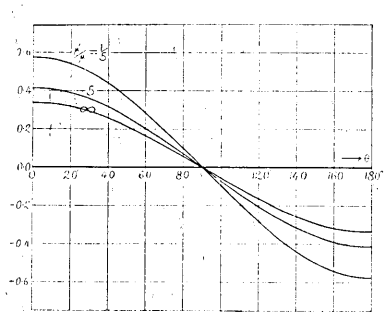

Fig. 2c. Magnitude of the term related to $\rho' ga$ of $\widehat{rr}_{r=a}$ $(=\widehat{rr'}_{r=a})$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

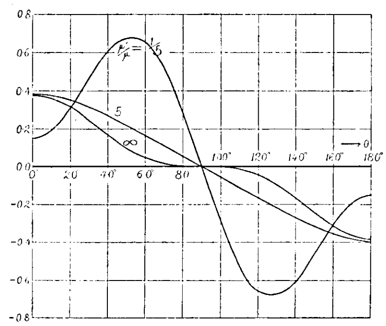

Fig. 3a. Magnitude of the term related to ρga of $\widehat{\theta\theta}_{r=a}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5, ∞.

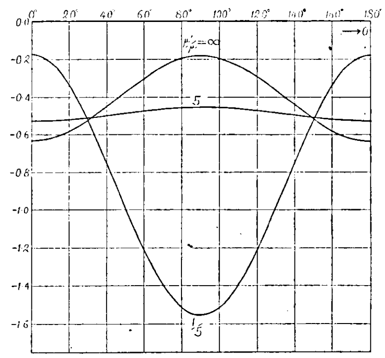

Fig. 3b. Magnitude of term related to $\rho g\xi$ of $\widehat{\theta\theta}_{r=a}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

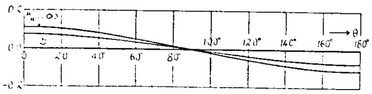

Fig. 3c. Magnitude of term related to $\rho' ga$ of $\widehat{\theta\theta}_{r=a}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 5 and ∞.

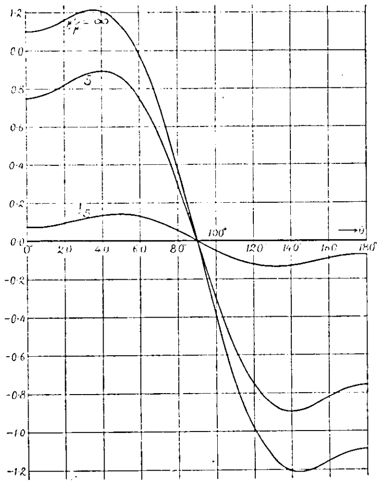

Fig. 4a. Magnitude of term related to ρga of $\widehat{\theta\theta'}_{r=a}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 5 and ∞.

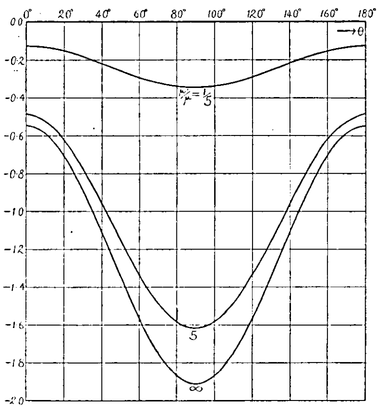

Fig. 4b. Magnitude of term related to $\rho g\xi$ of $\widehat{\theta\theta'}_{r=a}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 5 and ∞.

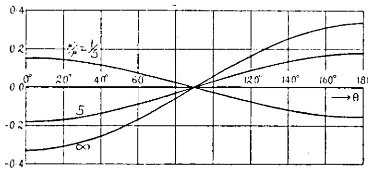

Fig. 4c. Magnitude of term related to $\rho' ga$ of $\widehat{\theta\theta'}_{r=a}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 5 and ∞.

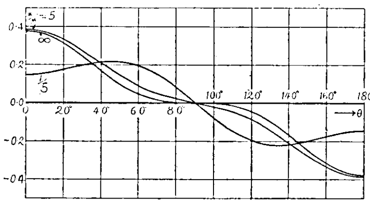

Fig. 5a. Magnitude of term related to ρga of $\widehat{\phi\phi}_{r=a}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 5 and ∞.

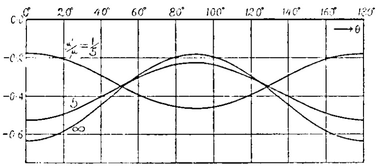

Fig. 5b. Magnitude of term related to $\rho g\xi$ of $\widehat{\phi\phi}_{r=a}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 5 and ∞.

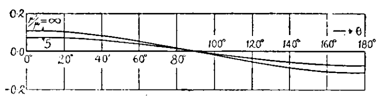

Fig. 5c. Magnitude of term related to $\rho' ga$ of $\widehat{\phi\phi}_{r=a}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 5 and ∞.

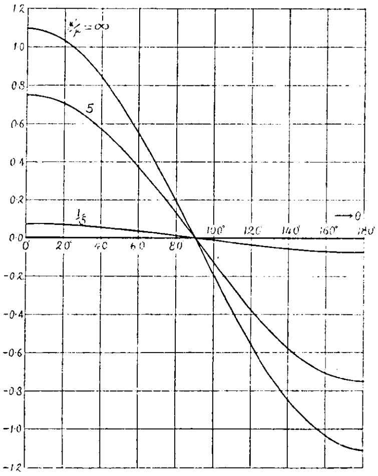

Fig. 6a. Magnitude of term related to ρga of $\widehat{\phi\phi'}_{r=a'}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

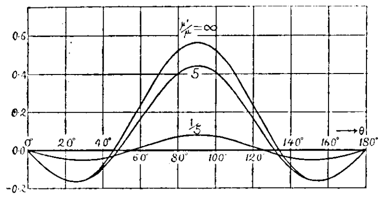

Fig. 7a. Magnitude of term related to ρga of $\widehat{r\theta}_{r=a}$ $(=\widehat{r\theta'}_{r=a})$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

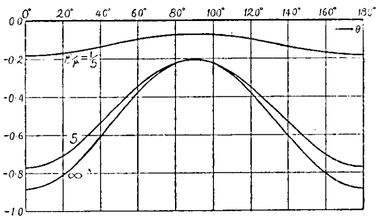

Fig. 6b. Magnitude of term related to $\rho g\xi$ of $\widehat{\phi\phi'}_{r=a}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

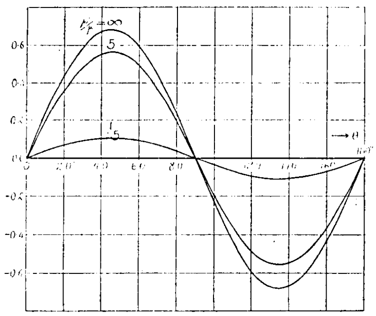

Fig. 7b. Magnitude of term related to $\rho g\xi$ of $\widehat{r\theta}_{r=a}$ $(=\widehat{r\theta'}_{r=a})$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

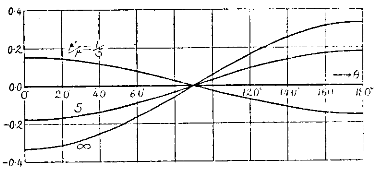

Fig. 6c. Magnitude of term related to $\rho' ga$ of $\widehat{\phi\phi'}_{r=a}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

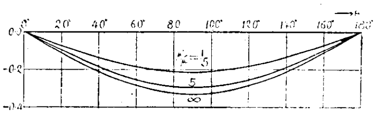

Fig. 7c. Magnitude of term related to $\rho' ga$ of $\widehat{r\theta}_{r=a}$ $(=\widehat{r\theta'}_{r=a})$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

Table I.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{rr}_{r=a}$ $(=\widehat{rr'}_{r=a})$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−0·347	0·160	0·576
10°	−0·340	0·150	0·567
20°	−0·321	0·123	0·541
30°	−0·292	0·085	0·498
45°	−0·236	0·025	0·407
60°	−0·180	−0·013	0·288
70°	−0·151	−0·020	0·196
80°	−0·132	−0·014	0·100
90°	−0·125	0	0
100°	−0·132	0·014	0·100
110°	−0·151	0·020	−0·196
120°	−0·180	0·013	−0·288
135°	−0·236	−0·025	−0·407
150°	−0·292	−0·085	−0·498
160°	−0·321	−0·123	−0·541
170°	−0·340	−0·150	−0·567
180°	−0·347	−0·160	−0·576

Table II.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{rr}_{r=a}$ $(\widehat{rr'}_{r=a})$ when $\dfrac{\mu'}{\mu}=5$.

	$\rho g\xi$	ρga	$\rho' ga$
0°	−1·613	0·882	0·412
10°	−1·579	0·838	0·406
20°	−1·482	0·717	0·387
30°	−1·331	0·544	0·357
45°	−1·050	0·264	0·291
60°	−0·769	0·060	0·206
70°	−0·618	−0·005	0·142
80°	−0·521	−0·018	0·072
90°	−0·487	0	0
100°	−0·521	0·018	−0·072
110°	−0·618	0·005	−0·142
120°	−0·769	−0·060	−0·206
135°	−1·050	−0·264	−0·291
150°	−1·331	−0·544	−0·357
160°	−1·482	−0·717	−0·387
170°	−1·579	−0·838	−0·406
180°	−1·613	−0·882	−0·412

Table III.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{rr}_{r=a}$ $(\widehat{rr'}_{r=a})$ when $\dfrac{\mu'}{\mu}=\infty$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−1·909	1·138	0·335
10°	−1·868	1·083	0·330
20°	−1·749	0·936	0·315
30°	−1·568	0·729	0·290
45°	−1·228	0·388	0·236
60°	−0·888	0·129	0·167
70°	−0·707	0·035	0·114
80°	−0·588	0	0·058
90°	−0·547	0	0
100°	−0·588	0	−0·058
110°	−0·707	−0·035	−0·114
120°	−0·888	−0·129	−0·167
135°	−1·228	−0·388	−0·236
150°	−1·568	−0·729	−0·290
160°	−1·749	−0·936	−0·315
170°	−1·863	−1·083	−0·330
180°	−1·909	−1·138	−0·335

Table IV.

Magnitudes of separate terms related to ρga, $\rho g\xi$ and $\rho' ga$ of $\widehat{\theta\theta}_{r=a}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−0·174	0·149	−0·010
10°	−0·216	0·193	−0·010
20°	−0·337	0·312	−0·010
30°	−0·521	0·466	−0·009
45°	−0·867	0·655	−0·009
60°	−1·213	0·658	−0·007
70°	−1·397	0·521	−0·005
80°	−1·518	0·288	−0·003
90°	−1·560	0	−0·002
100°	−1·518	−0·288	0
110°	−1·397	−0·521	0·002
120°	−1·213	−0·658	0·003
135°	−0·867	−0·655	0·005
150°	−0·521	−0·466	0·007
160°	−0·337	−0·312	0·009
170°	−0·216	−0·193	0·010
180°	−0·174	−0·149	0·010

Table V.

Magnitudes of separate terms related to $\rho g\xi$, $\rho g a$ and $\rho' g a$ of $\widehat{\theta\theta}_{r=a}$ when $\frac{\mu'}{\mu}=5$.

θ	$\rho g\xi$	$\rho g a$	$\rho' g a$
0°	-0·529	0·385	0·073
10°	−0·527	0·377	0·072
20°	−0·520	0·355	0·069
30°	−0·510	0·318	0·063
45°	−0·491	0·248	0·052
60°	−0·472	0·167	0·036
70°	−0·462	0·111	0·025
80°	−0·455	0·055	0·013
90°	−0·453	0	0
100°	−0·455	−0·055	−0·013
110°	−0·462	−0·111	−0·025
120°	−0·472	-0·167	−0·036
135°	−0·491	−0·248	−0·052
150°	−0·510	−0·318	−0·063
160°	−0·520	−0·355	−0·069
170°	−0·527	−0·377	−0·072
180°	-0·529	−0·385	−0·073

Table VI.

Magnitudes of separate terms related to $\rho g\xi$, $\rho g a$ and $\rho' g a$ of $\widehat{\theta\theta}_{r=a}$ when $\frac{\mu'}{\mu}=\infty$.

θ	$\rho g\xi$	$\rho g a$	$\rho' g a$
0°	−0·637	0·378	0·111
10°	−0·623	0·361	0·109
20°	−0·584	0·313	0·104
30°	−0·523	0·243	0·096
45°	−0·410	0·130	0·079
60°	−0·297	0·043	0·056
70°	−0·236	0·012	0·038
80°	−0·197	0	0·019
90°	−0·183	0	0
100°	−0·197	0	−0·019
110°	−0·236	−0·012	−0·038
120°	−0·297	−0·043	−0·056
135°	−0·410	−0·130	−0·079
150°	−0·523	−0·243	−0·096
160°	−0·584	−0·313	−0·104
170°	−0·623	−0·361	−0·109
180°	−0·637	−0·378	−0·111

Table VII.

Magnitudes of separate terms related to $\rho g\xi$, $\rho g a$ and $\rho' g a$ of $\widehat{\theta\theta'}_{r=a}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$.

θ	$\rho g\xi$	$\rho g a$	$\rho' g a$
0°	−0·127	0·072	0·152
10°	−0·134	0·078	0·150
20°	−0·152	0·095	0·143
30°	−0·181	0·115	0·132
45°	−0·236	0·137	0·107
60°	−0·291	0·128	0·076
70°	−0·320	0·099	0·052
80°	−0·338	0·053	0·026
90°	−0·345	0	0
100°	−0·338	−0·053	−0·026
110°	−0·320	−0·099	−0·052
120°	−0·291	−0·128	−0·076
135°	−0·236	−0·137	−0·107
150°	−0·181	−0·115	−0·132
160°	−0·152	−0·095	−0·143
170°	−0·134	−0·078	−0·150
180°	−0·127	−0·072	−0·152

Table VIII.

Magnitudes of separate terms related to $\rho g\xi$, $\rho g a$ and $\rho' g a$ of $\widehat{\theta\theta'}_{r=a}$ when $\frac{\mu'}{\mu}=5$.

θ	$\rho g\xi$	$\rho g a$	$\rho' g a$
0°	−0·488	0·751	−0·180
10°	−0·522	0·772	−0·177
20°	−0·619	0·819	−0·169
30°	−0·770	0·871	−0·156
45°	−1·051	0·890	−0·127
60°	−1·332	0·756	−0·090
70°	−1·483	0·563	−0·062
80°	−1·580	0·301	−0·031
90°	−1·614	0	0
100°	−1·580	−0·301	0·031
100°	−1·483	−0·563	0·062
120°	−1·332	−0·756	0·090
135°	−1·051	−0·890	0·127
150°	−0·770	−0·871	0·156
160°	−0·619	−0·819	0·169
170°	−0·522	−0·772	0·177
180°	−0·488	−0·751	0·180

Table IX.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{\theta\theta'}_{r=a}$ when $\frac{\mu'}{\mu}=\infty$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−0·544	1·098	−0·333
10°	−0·585	1·112	−0·328
20°	−0·704	1·159	−0·313
30°	−0·886	1·203	−0·288
45°	−1·227	1·186	−0·235
60°	−1·568	0·987	−0·166
70°	−1·750	0·733	−0·114
80°	−1·869	0·388	−0·058
90°	−1·910	0	0
100°	−1·869	−0·388	0·058
110°	−1·750	−0·733	0·114
120°	−1·568	−0·987	0·166
135°	−1·227	−1·186	0·235
150°	−0·886	−1·203	0·288
160°	−0·704	−1·159	0·313
170°	−0·585	−1·912	0·328
180°	−0·544	−1·098	0·333

Table X.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{\phi\phi}_{r=a}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−0·173	0·149	−0·010
10°	−0·182	0·157	−0·010
20°	−0·207	0·176	−0·009
30°	−0·246	0·200	−0·009
45°	−0·319	0·221	=0·007
60°	−0·392	0·197	−0·005
70°	−0·431	0·150	−0·003
80°	−0·456	0·081	−0·002
90°	−0·465	0	0
100°	−0·456	−0·081	0·002
110°	−0·431	−0·150	0·003
120°	−0·392	−0·197	0·005
135°	−0·319	−0·221	0·007
150°	−0·246	−0·200	0·009
160°	−0·207	−0·176	0·009
170°	−0·182	−0·157	0·010
180°	−0·173	−0·149	0·010

Table XI.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{\phi\phi}_{r=a}$ when $\frac{\mu'}{\mu}=5$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−0·527	0·385	0·073
10°	−0·518	0·371	0·072
20°	−0·492	0·332	0·069
30°	−0·452	0·274	0·063
45°	−0·377	0·176	0·052
60°	−0·302	0·090	0·036
70°	−0·262	0·049	0·025
80°	−0·236	0·021	0·013
90°	−0·227	0	0
100°	−0·236	−0·021	−0·013
110°	−0·262	−0·049	−0·025
120°	−0·302	−0·090	−0·036
135°	−0·377	−0·176	−0·052
150°	−0·452	−0·274	−0·063
160°	−0·492	−0·332	−0·069
170°	−0·518	−0·371	−0·072
180°	−0·527	−0·385	−0·073

Table XII.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{\phi\phi}_{r=a}$ when $\frac{\mu'}{\mu}=\infty$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−0·636	0·378	0·111
10°	−0·622	0·361	0·109
20°	−0·583	0·313	0·104
30°	−0·522	0·243	0·096
45°	−0·409	0·130	0·079
60°	−0·296	0·043	0·056
70°	−0·235	0·012	0·038
80°	−0·196	0	0·019
90°	−0·182	0	0
100°	−0·196	0	−0·019
110°	−0·235	−0·012	−0·038
120°	−0·296	−0·043	−0·056
135°	−0·409	−0·130	−0·079
150°	−0·522	−0·243	−0·096
160°	−0·583	−0·313	−0·104
170°	−0·622	−0·361	−0·109
180°	−0·636	−0·378	−0·111

Table XIII.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{\phi\phi'}_{r=a}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−0·182	0·072	0·152
10°	−0·179	0·071	0·150
20°	−0·169	0·068	0·143
30°	−0·155	0·062	0·132
45°	−0·127	0·051	0·107
60°	−0·099	0·036	0·076
70°	−0·085	0·025	0·052
80°	−0·075	0·013	0·026
90°	−0·072	0	0
100°	−0·075	−0·013	−0·026
110°	−0·085	−0·025	−0·052
120°	−0·099	−0·036	−0·076
135°	−0·127	−0·051	−0·107
150°	−0·155	−0·062	−0·132
160°	−0·169	−0·068	−0·143
170°	−0·179	−0·071	−0·150
180°	−0·182	−0·072	−0·152

Table XIV.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{\phi\phi}_{r=a}$ when $\frac{\mu'}{\mu}=5$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−0·769	0·750	−0·180
10°	−0·752	0·739	−0·177
20°	−0·703	0·705	−0·169
30°	−0·628	0·649	−0·156
45°	−0·487	0·530	−0·127
60°	−0·346	0·375	−0·090
70°	−0·271	0·256	−0·062
80°	−0·222	0·130	−0·031
90°	−0·205	0	0
100°	−0·222	−0·130	0·031
110°	−0·271	−0·256	0·062
120°	−0·346	−0·375	0·090
135°	−0·487	−0·530	0·127
150°	−0·628	−0·649	0·156
160°	−0·703	−0·705	0·169
170°	−0·752	−0·739	0·177
180°	−0·769	−0·750	0·180

Table XV.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{\phi\phi'}_{r=a}$ when $\frac{\mu'}{\mu}=\infty$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−0·887	1·10	−0·333
10°	−0·866	1·083	−0·328
20°	−0·807	1·033	−0·313
30°	−0·716	6·954	−0·288
45°	−0·546	0·778	−0·235
60°	−0·376	0·550	−0·163
70°	−0·285	0·376	−0·114
80°	−0·226	0·191	−0·058
90°	−0·205	0	0
100°	−0·226	−0·191	0·058
110°	−0·285	−0·376	0·114
120°	−0·376	−0·550	0·166
135°	−0·546	−0·778	0·235
150°	−0·716	−0·954	0·288
160°	−0·807	−0·033	0·313
170°	−0·866	−1·083	0·328
180°	−0·887	−1·10	0·333

Table XVI.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ $\widehat{r\theta}_{r=a}$ $(=\widehat{r\theta'}_{r=a})$ when $\frac{\mu'}{\mu}=\frac{1}{5}$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	0	0	0
10°	0·037	−0·028	−0·037
20°	0·070	−0·048	−0·073
30°	0·094	−0·053	−0·106
45°	0·109	−0·032	−0·150
60°	0·094	0·015	−0·184
70°	0·070	0·047	−0·199
80°	0·037	0·071	−0·209
90°	0	0·079	−0·212
100°	−0·037	0·071	−0·209
110°	−0·070	0·047	−0·199
120°	−0·098	0·015	−0·184
135°	−0·109	−0·032	−0·150
150°	−0·098	−0·053	−0·106
160°	−0·070	−0·048	−0·703
170°	−0·037	−0·028	−0·037
180°	0	0	0

Table XVII.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{r\theta}_{r=a}$ $(=\widehat{r\theta'}_{r=a})$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	0	0	0
10°	0·193	−0·094	−0·051
20°	0·362	−0·156	−0·101
30°	0·490	−0·160	−0·147
45°	0·564	−0·047	−0·208
60°	0·490	0·162	−0·255
70°	0·362	0·303	−0·277
80°	0·193	0·404	−0·290
90°	0	0·441	−0·295
100°	−0·193	0·404	−0·290
110°	−0·362	0·303	−0·277
120°	−0·490	0·162	−0·255
135°	−0·564	−0·047	−0·208
150°	−0·490	−0·160	−0·147
160°	−0·362	−1·156	−0·101
170°	−0·193	−0·094	−0·051
180°	0	0	0

Table XVIII.

Magnitudes of separate terms related to $\rho g\xi$; ρga and $\rho' ga$ of $\widehat{r\theta}_{r=a}$ $(\widehat{r\theta'}_{r=a})$ when $\dfrac{\mu'}{\mu}=\infty$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	0	0	0
10°	0·233	−0·098	−0·058
20°	0·438	−0·159	−0·114
30°	0·590	−0·155	−0·167
45°	0·682	−0·012	−0·236
60°	0·590	0·237	−0·278
70°	0·438	0·404	−0·313
80°	0·233	0·523	−0·328
90°	0	0·564	−0·333
100°	−0·233	0·523	−0·328
110°	−0·438	0·404	−0·313
120°	−0·590	0·237	−0·268
135°	−0·682	−0·012	−3·236
150°	−0·590	−0·155	−0·167
160°	−0·438	−0·159	−0·114
170°	−0·233	−0·098	−0·058
180°	0	0	0

symmetrical distributions with respect to the plane $\theta=90°$. The terms related to $\rho g\xi$ of the shear stresses $\widehat{r\theta}_{r=a}$, $\widehat{r\theta'}_{r=a}$ have, however, unsymmetrical distributions with respect to this plane.

2. The terms related to ρga of the components of stresses $\widehat{rr}_{r=a}$, $\widehat{\theta\theta}_{r=a}$, $\widehat{\phi\phi}_{r=a}$, and $\widehat{rr'}_{r=a}$, $\widehat{\theta\theta'}_{r=a}$, $\widehat{\phi\phi'}_{r=a}$ have unsymmetrical distributions with respect to the plane $\theta=90°$ and those related to ρga of $\widehat{r\theta}_{r=a}$, $\widehat{r\theta'}_{r=a}$ have symmetrical ones with respect to this plane.

3. The terms concerning $\rho' ga$ of $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{\phi\phi}$, $\widehat{rr'}$, $\widehat{\theta\theta'}$, $\widehat{\phi\phi'}$ are also unsymmetry with respect to the plane $\theta=90°$, and those related to of $\widehat{r\theta}$ and $\widehat{r\theta'}$ have symmetrical ones with respect to the plane $\theta=90°$.

4. When the depth ξ of the centre of spherical inclusion is deep, the stress distributions are mainly affected by the terms related to $\rho g\xi$ and the terms connected with ρga and $\rho' ga$ have no much effect upon the stress distributions. When ρ' is very large, however, the terms related to $\rho' ga$ have some effects upon them.

5. When the depth is moderately shallow, the respective terms related to ρga and $\rho' ga$, which are unbalancing force due to the body force, have some effects upon the stress distributions.

6. When the spherical inclusion is more rigid than the surrounding solid, the magnitudes of the respective terms with respect to $\rho g\xi$ and ρga of $\widehat{rr}_{r=a}$, $\widehat{rr'}_{r=a}$, $\widehat{r\theta}_{r=a}$, $\widehat{r\theta'}_{r=a}$ are large, and those of the terms related to $\rho' ga$ of $\widehat{r\vartheta}_{r=a}$ and $\widehat{r\theta'}_{r=a}$ are also large. The magnitudes of the terms related to $\rho' ga$ of $\widehat{rr}_{r=a}$ and $\widehat{rr'}_{r=a}$, however, are small in this case.

7. When the spherical inclusion is more rigid than the surrounding medium, the magnitudes of the respective terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{\theta\theta}_{r=a}$ are generally small, and those of the terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{\theta\theta'}_{r=a}$, $\widehat{\phi\phi'}_{r=a}$, on the contrary, are generally large. The terms related to $\rho g\xi$ and ρga, $\rho' ga$ of the component of stress $\widehat{\phi\phi}_{r=a}$ have special distributions differ from those of all other components of stress when the magnitude of $\frac{\mu'}{\mu}$ is large.

To see the properties of stress distributions along radial directions from the center of spherical inclusion, we obtained the following figures by the usage of the expressions (66), (67), (68), (69), (70), (71), (72), and (73). (Fig. 8a, 8b, 8c, 9a, 9b, 9c, 10a, 10b, 10c, 11a, 11b, 11c.)

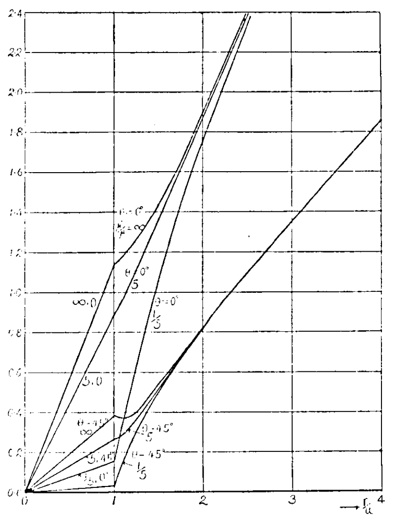

Fig. 8a. Magnitudes of respective terms related to ρga of $\widehat{rr}_{\substack{\theta=0^\circ\\ \theta=45^\circ}}$ and $\widehat{rr'}_{\substack{\theta=0^\circ\\ \theta=45^\circ}}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

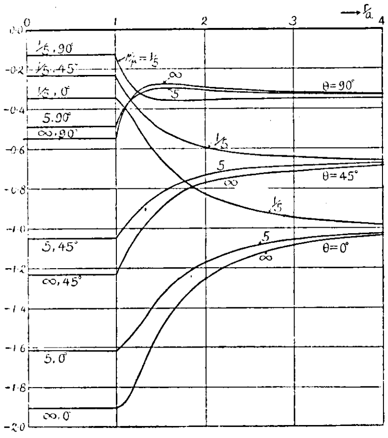

Fig. 8b. Magnitudes of respective terms related to $\rho g\xi$ of $\widehat{rr}_{\substack{\theta=0^\circ\\ \theta=40^\circ\\ \theta=90^\circ}}$ and $\widehat{rr'}_{\substack{\theta=0^\circ\\ \theta=45^\circ\\ \theta=90^\circ}}$ when $\frac{\mu}{\mu}=\frac{1}{5}$, 5 and ∞.

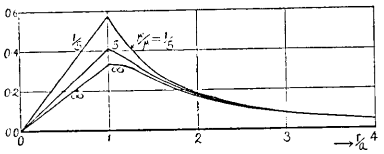

Fig. 8c. Magnitudes of respective terms related to $\rho' ga$ of $\widehat{rr}_{\theta=0°}$ and $\widehat{rr'}_{\theta=0°}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 5 and ∞.

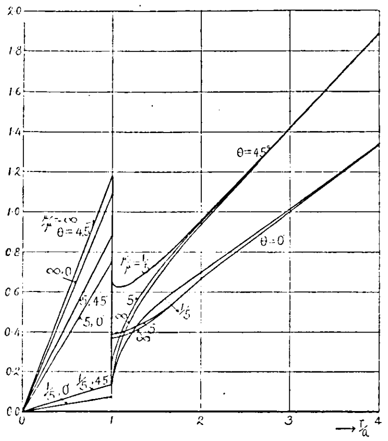

Fig. 9a. Magnitudes of respective terms related to ρga oi $\widehat{\theta\theta}_{\substack{\theta=0°\\ \theta=45°}}$ and $\widehat{\theta\theta'}_{\substack{\theta=0°\\ \theta=45°}}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 5 and ∞.

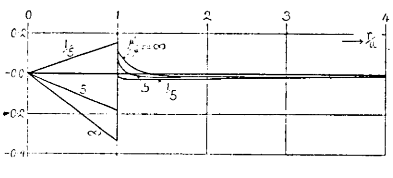

Fig. 9c. Magnitudes of respective terms related to $\rho' ga$ of $\widehat{\theta\theta}_{\theta=0°}$ and $\widehat{\theta\theta'}_{\theta=0°}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 5 and ∞.

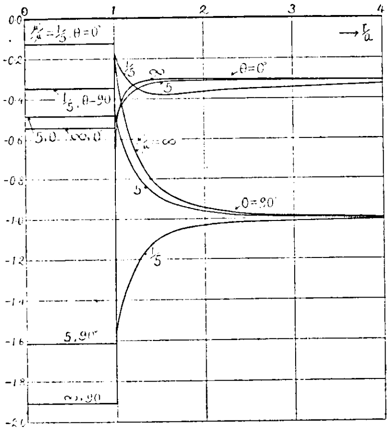

Fig. 9b. Magnitudes of respective terms related to $\rho g\xi$ of $\widehat{\theta\theta}_{\substack{\theta=0°\\ \theta=90°}}$ and $\widehat{\theta\theta'}_{\substack{\theta=0°\\ \theta=90°}}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 5 and ∞.

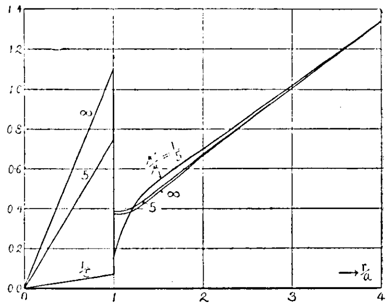

Fig. 10a. Magnitudes of respective terms related to ρga of $\widehat{\phi\phi}_{\theta=0°}$ and $\widehat{\phi\phi'}_{\theta=0°}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 5 and ∞.

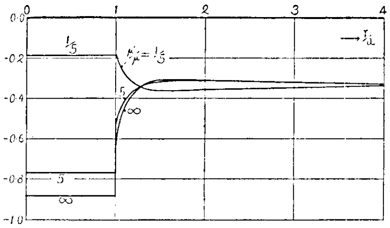

Fig. 10b. Magnitudes of respective terms related to $\rho g\xi$ of $\widehat{\phi\phi}_{\theta=0^\circ}$ and $\widehat{\phi\phi'}_{\theta=0^\circ}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

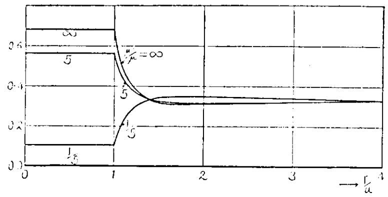

Fig. 11b. Magnitudes of respective terms related to $\rho g\xi$ of $\widehat{r\theta}_{\theta=45^\circ}$ and $\widehat{r\theta'}_{\theta=45^\circ}$ when $\frac{\mu'}{\mu}=\frac{1}{2}$, 5 and ∞.

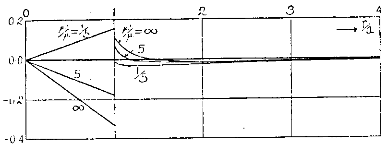

Fig. 10c. Magnitudes of respective terms related to $\rho' ga$ of $\widehat{\phi\phi}_{\theta=0^\circ}$ and $\widehat{\phi\phi'}_{\theta=0^\circ}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

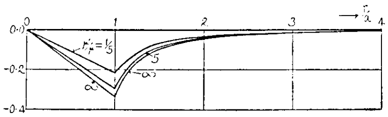

Fig. 11c. Magnitudes of respective terms related to $\rho' ga$ of $\widehat{r\theta}_{\theta=90^\circ}$ and $\widehat{r\theta'}_{\theta=90^\circ}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

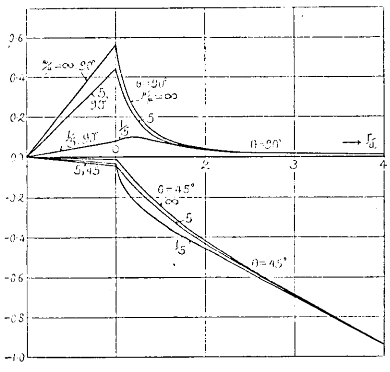

Fig. 11a. Magnitudes of respective terms related to ρga of $\widehat{r\theta}_{\substack{\theta=45^\circ\\ \theta=90^\circ}}$ and $\widehat{r\theta'}_{\substack{\theta=45^\circ\\ \theta=90^\circ}}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 5 and ∞.

From the figures we can see many interesting facts with respect to the radial distributions of stresses, and some of them are summarised as follows:

1. Whether or not the elastic constants of the semi-infinite solid are larger than those of the spherical inclusion, the magnitudes of the terms related to $\rho g\xi$ of all components of stress in the spherical inclusion are constant along radial directions but not along colatitudinal directions. And the magnitudes of the terms related to ρga and $\rho' ga$ of the respective components of stresses of the heterogeneous inclusion increase linearly along radial directions.

2. When the spherical inclusion is more rigid than the surrounding solid, the stresses are mainly accumulated in the spherical inclusion, and the stress components of the surrounding medium in the neighbourhood of the boundary surface are generally more or less released. In the reversed case, when the surrounding medium is more rigid than the included, the stresses are mainly accumulated in the former, especially in the neighbourhood of the boundary surface.

3. The region where the stress distributions in the semi-infinite solid affected by the presence of a spherical inclusion is limited in the boundary region of that spherical inclusion and the effective radius upon the distributions of stresses in the boundary medium is approximately four times the radius of the spherical inclusion.

Concluding this paper some remarkes are written as follows :[3]

The mathematical results obtained in this paper are not applicable to the case where $\frac{\mu'}{\mu}=0$ and the spherical inclusion has no rigidity. In this case the problem should be formally solved anew by the following boundary conditions.

$$r=a: \quad \left.\begin{aligned} &\widehat{rr}=\widehat{rr}', \\ &\widehat{r\theta}=\widehat{r\theta}'=0, \\ &u=u'. \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots\dots (67)$$

$$r\to\infty: \quad \left.\begin{aligned} &\widehat{rr}=\widehat{rr} \text{ expressed by } (40), \\ &\widehat{\theta\theta}=\widehat{\theta\theta} \quad ,, \quad ,, \ (41), \\ &\widehat{\phi\phi}=\widehat{\phi\phi} \quad ,, \quad ,, \ (42), \\ &\widehat{r\theta}=\widehat{r\theta} \quad ,, \quad ,, \ (43), \end{aligned}\right\} \quad \dots\dots\dots\dots (68)$$

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3) The present writers must express their thanks to Dr. C. Tsuboi for his discussions on this point at the 70th coll. meeting of the institute, Dec. 20, 1932.

12. 重力の働ける半無限彈性體内に存在する球狀塡充物附近の應力 (I)

地震研究所 { 西村源六郎 / 高山威雄

重力の作用してゐる半無限彈性體内に，それと密度，彈性常數の異つた球狀な物質が塡充してゐる場合，この物質の内外に於ける應力釣合を論じた．目的は地殼内部に異物質のある場合その附近で地殼内部の釣合がどの様にたもたれてゐるかを知り，地震發生の原因が應力釣合の破れる事にあるとすれば，どの様な關係でやぶれるものであるかを知る爲めに必要な第一の楷段として試みたものである．解き方は半無限彈性體は plane strain に保たれてゐるとしてやり，異物質との硬さの異ひによる應力分布の趣を圖によつて詳しく示した．