13. *On Stresses in the Interior and in the Vicinity of a Horizontal Cylindrical Inclusion of Circular Section in a Gravitating Semi-infinite Elastic Solid. (I)*

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1. For the same object of investigation stated in the previous paper,[1] we shall study the stress distribution in the interior and in the vicinity of a horizontal cylindrical matter of circular section included in a gravitating semi-infinite elastic solid. The investigation of the stress distribution around a horizontal circular hole in a semi-infinite gravitating solid has already been carried out by Professor N. Yamaguti[2] who solved the problem by the usage of stress function. The problem of a cylindrical matter of heterogeneity included in a semi-infinite gravitating solid has not yet been solved.

In the present study, as in the previous paper,[3] the state of the gravitating semi-infinite elastic solid is also one of *plane strain*.

2. Cylindrical coordinates (r, θ, z) are used. The axis z is coincident with the axis Oz of the cylindrical inclusion of which the radius is a, and the centre O is at the point $(x=-\xi,\ y=0)$ as shewn in Fig. 1. The azimuthal angle θ is taken counterclockwise from the axis ox which is taken to be vertically upward positive, and the axis oy is taken to be horizontal. The plane $x=\xi$ is the horizontal surface of the semi-infinite gravitating elastic solid of which the density and the gravity constant are respectively ρ and g. Let the density of the cylindrical inclusion be ρ', and also the gravity constant be g.

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1) G. NISHIMURA and T. TAKAYAMA, "On Stresses in the Interior and in the Vicinity of a Spherical Inclusion in a Gravitating Semi-infinite Elastic Solid," *Bull. Earthq. Res. Intst.* **11** (1933).

2) N. YAMAGUTI," On the stresses Around a Horizontal Circular Hole in a Gravitating Solid," *Jour. Civil Eng.*, Tokyo **15** (1929), 291.

3) G. NISHIMURA and T. TAKAYAMA, *loc. cit.*

4) The problem treated as plane stress was read in the Meeting of this Institute, Feb. 21, 1933, but is not yet pressed. N. Yamaguti also treated the problem as *plane strain* problem.

Now the top surface ($x=\xi$) of the solid is horizontal, and therefore there is no variation of the axial components of displacement and stress. Then the stress equations of equilibrium of the gravitating semi-infinite solid and the equilibrium of the gravitating cylindrical matter included in the solid are respectively expressed by the following equations:

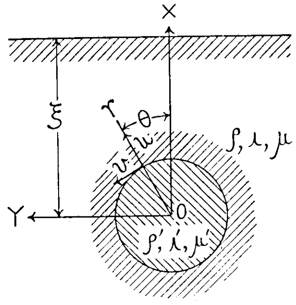

Fig. 1.

$$\left.\begin{aligned}\frac{\partial\widehat{rr}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{r\theta}}{\partial\theta}+\frac{\widehat{rr}-\widehat{\theta\theta}}{r}&=\rho g\cos\theta,\\ \frac{\partial\widehat{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{\theta\theta}}{\partial\theta}+2\frac{\widehat{r\theta}}{r}&=-\rho g\sin\theta,\end{aligned}\right\}\quad\cdots\cdots\cdots(1)$$

$$\left.\begin{aligned}\frac{\partial\widehat{rr}'}{\partial r}+\frac{1}{r}\frac{\partial\widehat{r\theta}'}{\partial\theta}+\frac{\widehat{rr}'-\widehat{\theta\theta}'}{r}&=\rho' g\cos\theta,\\ \frac{\partial\widehat{r\theta}'}{\partial r}+\frac{1}{r}\frac{\partial\widehat{\theta\theta}'}{\partial\theta}+2\frac{\widehat{r\theta}'}{r}&=-\rho' g\sin\theta.\end{aligned}\right\}\quad\cdots\cdots\cdots(2)$$

The normal components of stress $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$ and the shearing components of stress $\widehat{r\theta}$ of the gravitating semi-infinite solid are expressed by the radial and tangential components of displacement u, v in the following forms:

$$\left.\begin{aligned}\widehat{rr}&=\lambda\left\{\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\right\}+2\mu\frac{\partial u}{\partial r},\\ \widehat{\theta\theta}&=\lambda\left\{\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\right\}+2\mu\left\{\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r}\right\},\\ \widehat{zz}&=\lambda\left\{\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\right\},\\ \widehat{r\theta}&=\mu\left\{\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}\right\},\end{aligned}\right\}\quad\cdots\cdots\cdots(3)$$

where λ and μ are the Lame's elastic constants of the solid.

The normal components of stress $\widehat{rr}'$, $\widehat{\theta\theta}'$ and the shearing components of stress $\widehat{r\theta}'$ of the cylindrical matter of which the Lame's elastic constants are λ', μ' have the following expressions with respect to the radial and tangential components of displacements u', v' as follows:

$$
\left.\begin{aligned}
\widehat{rr}' &= \lambda'\left\{\frac{1}{r}\frac{\partial(ru')}{\partial r}+\frac{1}{r}\frac{\partial v'}{\partial\theta}\right\}+2\mu'\frac{\partial u'}{\partial r'},\\
\widehat{\theta\theta}' &= \lambda'\left\{\frac{1}{r}\frac{\partial(ru')}{\partial r}+\frac{1}{r}\frac{\partial v'}{\partial\theta}\right\}+2\mu'\left\{\frac{1}{r}\frac{\partial v'}{\partial\theta}+\frac{u'}{r}\right\},\\
\widehat{zz}' &= \lambda'\left\{\frac{1}{r}\frac{\partial(ru')}{\partial r}+\frac{1}{r}\frac{\partial v'}{\partial\theta}\right\}.\\
\widehat{r\theta}' &= \mu'\left\{\frac{\partial v'}{\partial r}-\frac{v'}{r}+\frac{1}{r}\frac{\partial u'}{\partial\theta}\right\},
\end{aligned}\right\}\quad\cdots\cdots\cdots(4)
$$

Eliminating $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{r\theta}$ and $\widehat{rr}'$, $\widehat{\theta\theta}'$, $\widehat{r\theta}'$ from the equations (1) and (2) separately by the usage of (3) and (4), we obtain the following equations of equilibrium of the solid and the cylindrical matter included in the medium:

$$
\left.\begin{aligned}
&(\lambda+2\mu)\frac{\partial\Delta}{\partial r}-\frac{2\mu}{r}\frac{\partial\varpi}{\partial\theta}=\rho g\cos\theta,\\
&(\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta}+2\mu\frac{\partial\varpi}{\partial r}=-\rho g\sin\theta,
\end{aligned}\right\}\quad\cdots\cdots\cdots(5)
$$

$$
\left.\begin{aligned}
&(\lambda'+2\mu')\frac{\partial\Delta'}{\partial r}-\frac{2\mu'}{r}\frac{\partial\varpi'}{\partial\theta}=\rho' g\cos\theta,\\
&(\lambda'+2\mu')\frac{1}{r}\frac{\partial\Delta'}{\partial\theta}+2\mu'\frac{\partial\varpi'}{\partial r}=-\rho' g\sin\theta,
\end{aligned}\right\}\quad\cdots\cdots\cdots(6)
$$

where

$$
\left.\begin{aligned}
\Delta &= \frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \partial},\\
2\varpi &= \frac{\partial v}{\partial r}+\frac{v}{r}-\frac{1}{r}\frac{\partial u}{\partial\theta},
\end{aligned}\right\}\quad\cdots\cdots\cdots(7)
$$

$$
\left.\begin{aligned}
\Delta' &= \frac{\partial u'}{\partial r}+\frac{u'}{r}+\frac{1}{r}\frac{\partial v'}{\partial\theta},\\
2\varpi' &= \frac{\partial v'}{\partial r}+\frac{v'}{r}-\frac{1}{r}\frac{\partial u'}{\partial\theta}.
\end{aligned}\right\}\quad\cdots\cdots\cdots(8)
$$

Among the particular solutions satisfying (5) and (6) respectively, we take the following solutions which are useful for the present study:

$$
\left.\begin{aligned}
\Delta &= 0,\\
2\varpi &= -\frac{\rho g}{\mu}r\sin\theta,
\end{aligned}\right\}\quad\cdots\cdots\cdots(9)
$$

and

$$\left.\begin{aligned} \Delta' &= \frac{\rho' g r}{(\lambda'+2\mu')}\cos\theta, \\ 2\varpi' &= 0. \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Now the equations (5) and (6) give us the following equations:

$$\frac{\partial^2 \Delta}{\partial r^2}+\frac{1}{r}\frac{\partial \Delta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Delta}{\partial \theta^2}=0, \qquad \ldots\ldots\ldots\ldots(11)$$

$$\frac{\partial^2 \Delta'}{\partial r^2}+\frac{1}{r}\frac{\partial \Delta'}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Delta'}{\partial \theta^2}=0. \qquad \ldots\ldots\ldots\ldots(12)$$

The respective solutions of (11) and (12) are expressed by

$$\Delta = B_0 + \left\{B_1 r + \frac{C_1}{r}\right\}\cos\theta + \left\{B_2 r^2 + \frac{C_2}{r^2}\right\}\cos 2\theta$$

$$+\left\{B_3 r^3 + \frac{C_3}{r^3}\right\}\cos 3\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

$$\Delta' = D_0 + D_1 r\cos\theta + D_2 r^2\cos 2\theta + D_3 r^3 \cos 3\theta, \quad \ldots\ldots(14)$$

where B_0, B_1, B_2, B_3, C_1, C_2, C_3, D_0, D_1, D_2, D_3 are arbitrary constants to be determined by elasticity conditions.

Using (13) and (14) respectively, we obtain the following forms of 2ϖ and $2\varpi'$ which are particular solutions of (5) and (6):

$$2\varpi = \left\{\frac{(\lambda+2\mu)}{\mu}B_1 r + \frac{(\lambda+2\mu)}{\mu}\frac{C_1}{r}\right\}\sin\theta$$

$$+\left\{\frac{(\lambda+2\mu)}{\mu}B_2 r^2 + \frac{(\lambda+2\mu)}{\mu}\frac{C_2}{r^2}\right\}\sin 2\theta$$

$$+\left\{\frac{(\lambda+2\mu)}{\mu}B_3 r^3 + \frac{(\lambda+2\mu)}{\mu}\frac{C_3}{r^3}\right\}\sin 3\theta, \quad \ldots\ldots\ldots(15)$$

$$2\varpi' = \frac{(\lambda'+2\mu')}{\mu'}D_1 r\sin\theta + \frac{(\lambda'+2\mu')}{\mu}D_2 r^2 \sin 2\theta$$

$$+\frac{(\lambda'+2\mu')}{\mu'}D_3 r^3 \sin 3\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots(16)$$

After some reductions we obtain the following equations in relation to u, v and Δ, 2ϖ from the relations expressed by (7):

$$\frac{\partial^2(ru)}{\partial r^2}+\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{1}{r^2}\frac{\partial^2(ru)}{\partial\theta^2}=\frac{1}{r}\frac{\partial(r^2\Delta)}{\partial r}-2\frac{\partial\varpi}{\partial\theta}, \quad \ldots\ldots\ldots(17)$$

$$\frac{\partial^2(rv)}{\partial r^2}+\frac{1}{r}\frac{\partial(rv)}{\partial r}+\frac{1}{r^2}\frac{\partial^2(rv)}{\partial\theta^2}=\frac{\partial\Delta}{\partial\theta}+\frac{2}{r}\frac{\partial(r\varpi)}{\partial r}. \quad \ldots\ldots\ldots(18)$$

And also we have the following equations in relation to u', v', Δ' and $2\varpi'$ from the equations (8):

$$\frac{\partial^2(ru')}{\partial r^2}+\frac{1}{r}\frac{\partial(ru')}{\partial r}+\frac{1}{r^2}\frac{\partial^2(ru')}{\partial\theta^2}=\frac{1}{r}\frac{\partial(r^2\Delta')}{\partial r}-2\frac{\partial\varpi'}{\partial\theta}, \quad \ldots\ldots(19)$$

$$\frac{\partial^2(rv')}{\partial r^2}+\frac{1}{r}\frac{\partial(rv')}{\partial r}+\frac{1}{r^2}\frac{\partial^2(rv')}{\partial\theta^2}=\frac{\partial\Delta'}{\partial\theta}+\frac{2}{r}\frac{\partial(r\varpi')}{\partial r}. \quad \ldots\ldots(20)$$

Substituting (9), (13), (15) for the expressions of Δ and 2ϖ in the equations (17), (18), we obtain the particular solutions of (17) and (18) as follows:

$$\left.\begin{aligned}
u=&\frac{1}{2}B_0r''+\left[B_1''+\frac{\rho g}{8\mu}r^2+\frac{(-\lambda+\mu)}{8\mu}B_1r^2+\left\{\frac{1}{2}+\frac{(\lambda+3\mu)}{2\mu}ln\frac{r}{a}\right\}\right]\cos\theta\\
&-\left\{\frac{\lambda}{6\mu}B_2r^3+\frac{(\lambda+2\mu)}{2\mu}\frac{C_2}{r}\right\}\cos 2\theta-\left\{\frac{(3\lambda+\mu)}{16\mu}B_3r^4+\frac{(3\lambda+5\mu)}{8\mu}\frac{C_3}{r^2}\right\}\cos 3\theta,\\
v=&\left[-\frac{3}{8}\frac{\rho g}{\mu}r^2+\frac{(3\lambda+5\mu)}{8\mu}B_1r^2-\left\{\frac{(\lambda+3\mu)}{2\mu}ln\frac{r}{a}+\frac{(\lambda+2\mu)}{2\mu}\right\}F_1-B_1''\right]\sin\theta\\
&+\left[\frac{(2\lambda+3\mu)}{6\mu}B_2r^3+\frac{1}{2}\frac{C_2}{r}\right]\sin 2\theta+\left[\frac{(5\lambda+7\mu)}{16\mu}B_3r^4-\frac{(\lambda-\mu)}{8\mu}\frac{C_3}{r^2}\right]\sin 3\theta,
\end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(21)$$

and we obtain the following expressions of u and v as the complementary solutions of (17), (18):

$$\left.\begin{aligned}
u=&\frac{B_0''}{r}+\frac{C_1''}{r^2}\cos\theta+\left\{B_2''r+\frac{C_2''}{r^3}\right\}\cos 2\theta+\left\{B_3''r^2+\frac{C_3''}{r^4}\right\}\cos 3\theta,\\
v=&\frac{C_1''}{r^2}\sin\theta-\left\{B_2''r-\frac{C_2''}{r^3}\right\}\sin 2\theta-\left\{B_3''r^2-\frac{C_3''}{r^4}\right\}\sin 3\theta.
\end{aligned}\right\}(22)$$

Substituting (10), (14), (16) for the expressions of Δ' and $2\varpi'$ in the equations (19) and (20), we obtain the particular solutions u' and v' which are favourable for the cylindrical inclusion as follows:

$$\left.\begin{aligned}
u'=&\frac{1}{2}D_0r+\left[\frac{3\rho' g}{8(\lambda'+2\mu')}r^2+\frac{(-\lambda'+\mu')}{8\mu'}D_1r^2+D_1''\right]\cos\theta\\
&-\frac{\lambda'}{6\mu'}D_2r^3\cos 2\theta-\frac{(3\lambda'+\mu')}{16\mu'}D_3r^4\cos 3\theta,\\
v'=&\left[-\frac{\rho' g}{8(\lambda'+2\mu')}r^2+\frac{(3\lambda'+5\mu')}{8\mu'}D_1r^2-D_1''\right]\sin\theta\\
&+\frac{(2\lambda'+3\mu')}{6\mu'}D_2r^3\sin 2\theta+\frac{(5\lambda'+7\mu')}{16\mu'}D_3r^4\sin 4\theta,
\end{aligned}\right\}\ldots(23)$$

while the complementary solutions of (19) and (20) are expressed by

$$\left.\begin{aligned} u' &= D_2''r\cos 2\theta + D_3''r^2\cos 3\theta, \\ v' &= -D_2''r\sin 2\theta + D_3''r^2\sin 3\theta. \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(24)$$

The general expressions of the components of displacement thus obtained (21), (22) and (23), (24) satisfy, of course, the equations of equilibrium of the solids (5) and (6) respectively. Using these general expressions we get the general expressions of the components of stresses $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{r\theta}$, $\widehat{zz}$ and $\widehat{rr}'$, $\widehat{\theta\theta}'$, $\widehat{r\theta}'$, $\widehat{zz}'$ as in the following forms by the relations of (3) and (4) separately.

$$\left.\begin{aligned}
\widehat{rr} &= (\lambda+\mu)B_0 - 2\mu\frac{B_0''}{r^2} + \left[\frac{1}{2}\rho g r + \frac{(\lambda+\mu)}{2}B_1 r - \frac{4\mu}{r^3}C_1'' + \frac{(2\lambda+3\mu)}{r}F_1\right]\cos\theta \\
&\quad + \left[2\mu B_2'' + \frac{2(\lambda+\mu)}{r^2}C_2 - \frac{6\mu}{r^4}C_2''\right]\cos 2\theta \\
&\quad + \left[-\frac{(\lambda+\mu)}{2}B_3 r^3 + 4\mu B_3'' r + \frac{5(\lambda+\mu)}{2}\frac{C_3}{r^3} - \frac{8\mu}{r^5}C_3''\right]\cos 3\theta, \\
\widehat{\theta\theta} &= (\lambda+\mu)B_0 + 2\mu\frac{B_0''}{r^2} + \left[-\frac{\rho g}{2}r + \frac{3(\lambda+\mu)}{2}B_1 r + \frac{4\mu}{r^3}C_1'' - \frac{\mu}{r}F_1\right]\cos\theta \\
&\quad + \left[2(\lambda+\mu)B_2 r^2 - 2\mu B_2'' + \frac{6\mu}{r^4}C_2''\right]\cos 2\theta \\
&\quad + \left[\frac{5}{2}(\lambda+\mu)B_3 r^3 - 4\mu B_3'' r - \frac{(\lambda+\mu)}{2}\frac{C_3}{r^3} + \frac{8\mu}{r^5}C_3''\right]\cos 3\theta, \\
\widehat{zz} &= \lambda B_0 + \left[\lambda B_1 r + \frac{\lambda}{r}F_1\right]\cos\theta + \left[\lambda B_2 r^2 + \frac{\lambda}{r^2}C_2\right]\cos 2\theta \\
&\quad + \left[\lambda B_3 r^3 + \frac{\lambda}{r^3}C_3\right]\cos 3\theta, \\
\widehat{r\theta} &= \left[-\frac{\rho g r}{2} + \frac{(\lambda+\mu)}{2}B_1 r - \frac{4\mu}{r^3}C_1'' - \frac{\mu}{r}F_1\right]\sin\theta \\
&\quad + \left[(\lambda+\mu)B_2 r^2 - 2\mu B_2'' + (\lambda+\mu)\frac{C_2}{r^2} - \frac{6\mu}{r^4}C_2''\right]\sin 2\theta \\
&\quad + \left[\frac{3(\lambda+\mu)}{2}B_3 r^3 - 4\mu B_3'' r + \frac{3}{2}\frac{(\lambda+\mu)}{r^3}C_3 - \frac{8\mu}{r^5}C_3''\right]\sin 3\theta.
\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

$$\begin{aligned}
\widehat{rr}' &= (\lambda'+\mu')D_0 + \left[\frac{(2\lambda'+3\mu')}{2(\lambda'+2\mu')}\rho' g r + \frac{(\lambda'+\mu')}{2}D_1 r\right]\cos\theta \\
&\quad + 2\mu' D_2''\cos 2\theta + \left[4\mu' D_3'' r - \frac{(\lambda'+\mu')}{2}D_3 r^3\right]\cos 3\theta,
\end{aligned}$$

$$\left.\begin{aligned}
\widehat{\theta\theta'} &= (\lambda'+\mu')D_0 + \left[\frac{(2\lambda'+\mu')}{2(\lambda'+2\mu')}\rho' g r + \frac{3}{2}(\lambda'+\mu')D_1 r\right]\cos\theta \\
&\quad + [2(\lambda'+\mu')r^2 D_2 - 2\mu' D_2'']\cos 2\theta \\
&\quad + \left[\frac{5}{2}(\lambda'+\mu')D_3 r^3 - 4\mu' D_3'' r\right]\cos 3\theta, \\
\widehat{zz'} &= \lambda' D_0 + \left[\frac{\lambda'}{(\lambda'+2\mu')}\rho' g r + \lambda' D_1 r\right]\cos\theta \\
&\quad + [\lambda' r^2 D_2]\cos 2\theta + \lambda' r^3 D_3 \cos 3\theta, \\
\widehat{r\theta'} &= \left[-\frac{\mu'}{2(\lambda'+2\mu')}\rho' g r + \frac{(\lambda'+\mu')}{2}D_1 r\right]\sin\theta \\
&\quad + [(\lambda'+\mu')D_2 r^2 - 2\mu' D_2'']\sin 2\theta \\
&\quad + \left[\frac{3(\lambda'+\mu')}{2}D_3 r^3 - 4\mu' D_3'' r\right]\sin 3\theta.
\end{aligned}\right\} \quad \ldots (26)$$

These stress expressions satisfy, of course, the stress equations of the gravitating elastic solids expressed by (1) and (2).

Using these general expressions of displacements and stresses we shall study the stresses in the vicinity of a horizontal cylindrical inclusion in the semi-infinite solid.

Referring to Fig. 1, we have the following expressions of the components of stresses which are the solutions of the gravitating semi-infinite elastic solid having no heterogeneous inclusion in its interior:

$$\left.\begin{aligned}
\widehat{rr} &= -\rho g\xi\frac{(\lambda+\mu)}{(\lambda+2\mu)} + \rho g r\frac{(2\lambda+3\mu)}{2(\lambda+2\mu)}\cos\theta \\
&\quad - \rho g\xi\frac{\mu}{(\lambda+2\mu)}\cos 2\theta + \rho g r\frac{\mu}{2(\lambda+2\mu)}\cos 3\theta, \\
\widehat{\theta\theta} &= -\rho g\xi\frac{(\lambda+\mu)}{(\lambda+2\mu)} + \rho g r\frac{(2\lambda+\mu)}{2(\lambda+2\mu)}\cos\theta \\
&\quad + \rho g\xi\frac{\mu}{(\lambda+2\mu)}\cos 2\theta - \rho g r\frac{\mu}{2(\lambda+2\mu)}\cos 3\theta, \\
\widehat{r\theta} &= -\frac{\mu}{2(\lambda+2\mu)}\rho g r\sin\theta + \rho g\xi\frac{\mu}{(\lambda+2\mu)}\sin 2\theta \\
&\quad - \rho g r\frac{\mu}{2(\lambda+2\mu)}\sin 3\theta.
\end{aligned}\right\} \quad \ldots (27)$$

These expressions are obtained by the following conditions: The normal and the shearing components of stress vanish at the horizontal surface $x=\xi$, and the stress at any point in the gravitating solid increases linearly according with the increase of the position form the upper surface of the solid. And the semi-infinite solid is in plane strain.

Now the boundary conditions of the present study are as follows:

$$r=a;\quad \left.\begin{array}{l} u=u',\quad v=v', \\ \widehat{rr}=\widehat{rr}',\ \widehat{r\theta}=\widehat{r\theta}', \end{array}\right\} \cdots\cdots(28)$$

$$r\to\infty;\quad \left.\begin{array}{l} \widehat{rr}=\widehat{rr} \text{ expressed by (27)}, \\ \widehat{\theta\theta}=\widehat{\theta\theta} \text{ expressed by (27)}, \\ \widehat{r\theta}=\widehat{r\theta} \text{ expressed by (27)}. \end{array}\right\} \cdots\cdots(29)$$

Condition (28) indicates us that the heterogeneous inclusion and the outer medium are perfectly cemented at the contact surface and they are forbidden to slide upn each other on this contact surface. And the condition expressed by (29) shews that the stress distribution at the whole spaces in the medium far from the inclusion are equal to the ones expressed by (27).

By the expressions of (21), (22), (23), (24), (25), (26) to satisfy (28), (29), we find the following values of the arbitrary constants:

$$\left.\begin{array}{l} B_0=-\dfrac{\rho g\xi}{(\lambda+2\mu)},\quad D_0=-\dfrac{\rho g\xi}{(\mu+\lambda'+\mu')}, \\ B_0''=\dfrac{\rho g\xi a^2}{2}\dfrac{(\lambda'+\mu'-\lambda-\mu)}{(\lambda+2\mu)(\mu+\lambda'+\mu')}, \end{array}\right\} \cdots\cdots(30)$$

$$\left.\begin{array}{l} B_1=\dfrac{\rho g}{(\lambda+2\mu)},\quad F_1=\dfrac{(\rho'-\rho)ga^2}{2(\lambda+2\mu)}, \\ D_1=\dfrac{\rho g\mu'}{\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}}-\dfrac{\rho' g\mu'(\mu+\lambda'+\mu')}{(\lambda'+2\mu')\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}}, \\ C_1=-\dfrac{\rho g a^4\mu'(\lambda'+\mu')}{8\mu\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}} \\ \qquad -\dfrac{\rho' g a^4}{8\mu(\lambda'+2\mu')}\left[\dfrac{\{(\lambda+2\mu)(2\lambda'+3\mu')+(2\lambda+3\mu)(\lambda'+2\mu')\}}{(\lambda+2\mu)}\right. \\ \qquad\qquad \left.-\dfrac{\mu'(\lambda'+\mu')(\mu+\lambda'+\mu')}{\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}}\right], \end{array}\right\}$$

$$\cdots\cdots(31)$$

$$\left.\begin{aligned}
&B_2=0, \quad B_2''=-\frac{\rho g\xi}{2(\lambda+2\mu)},\\
&C_2=\frac{\rho g\xi a^2}{(\lambda+2\mu)}\frac{\vartheta_{C_2}}{\vartheta_2}, \quad C_2''=\frac{\rho g\xi a^4}{(\lambda+2\mu)}\frac{\vartheta_{C_2''}}{\vartheta_2},\\
&D_2''=\frac{\rho g\xi}{(\lambda+2\mu)}\frac{\vartheta_{D_2''}}{\vartheta_2}, \quad D_2=\frac{\rho g\xi}{a^2}\frac{\vartheta_{D_2}}{\vartheta_2},
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(32)$$

$$\left.\begin{aligned}
&B_3=0, \quad B_3''=\frac{\rho g}{8(\lambda+2\mu)},\\
&C_3=\frac{\rho g a^4}{2(\lambda+2\mu)}\frac{\vartheta_{C_3}}{\vartheta_3}, \quad C_3=-\frac{\rho g a^6}{2(\lambda+2\mu)}\frac{\vartheta_{C_3''}}{\vartheta_3},\\
&D_3=\frac{\rho g}{2(\lambda+2\mu)a^2}\frac{\vartheta_{D_3}}{\vartheta_3}, \quad D_3''=-\frac{\rho g}{2(\lambda+2\mu)}\frac{\vartheta_{D_3''}}{\vartheta_3},
\end{aligned}\right\}\quad\ldots\ldots(33)$$

where ϑ_2, ϑ_{C_2}, $\vartheta_{C_2''}$, ϑ_{D_2}, $\vartheta_{D_2''}$ and ϑ_3, ϑ_{C_3}, $\vartheta_{C_3''}$, ϑ_{D_3}, $\vartheta_{D_3''}$ are expressed by the following determinants:

$$\left.\begin{aligned}
&\vartheta_2=\begin{vmatrix}
2(\lambda+\mu), & -6\mu, & -2\mu', & 0\\
-(\lambda+\mu), & 6\mu, & -2\mu', & (\lambda'+\mu')\\
-\frac{(\lambda+2\mu)}{2\mu}, & 1, & -1, & \frac{\lambda'}{6\mu'}\\
-\frac{1}{2}, & -1, & -1, & \frac{(2\lambda'+3\mu')}{6\mu'}
\end{vmatrix},\\
&\vartheta_{C_2}=\begin{vmatrix}
\mu, & -6\mu, & -2\mu' & 0\\
\mu, & 6\mu, & -2\mu', & (\lambda'+\mu')\\
\frac{1}{2}, & 1, & -1, & \frac{\lambda'}{6\mu'}\\
\frac{1}{2}, & -1, & -1, & \frac{(2\lambda'+3\mu')}{6\mu'}
\end{vmatrix},\\
&\vartheta_{C_2''}=\begin{vmatrix}
2(\lambda+\mu), & \mu, & -2\mu', & 0\\
-(\lambda+\mu), & \mu, & -2\mu', & (\lambda'+\mu')\\
-\frac{(\lambda+2\mu)}{2\mu}, & \frac{1}{2}, & -1, & \frac{\lambda'}{6\mu'}\\
-\frac{1}{2} & \frac{1}{2}, & -1, & \frac{(2\lambda'+3\mu')}{6\mu'}
\end{vmatrix},
\end{aligned}\right\}\quad\ldots\ldots(34)$$

$$\vartheta_{D_2}=\begin{vmatrix} 2(\lambda+\mu), & -6\mu, & -2\mu', & \mu \\ -(\lambda+\mu), & 6\mu, & -2\mu', & \mu \\ -\dfrac{(\lambda+2\mu)}{2\mu} & 1, & -1, & \dfrac{1}{2} \\ -\dfrac{1}{2}, & -1, & -1, & \dfrac{1}{2} \end{vmatrix},$$

$$\vartheta_{D_2''}=\begin{vmatrix} 2(\lambda+\mu), & -6\mu, & \mu, & 0 \\ -(\lambda+\mu), & 6\mu, & \mu, & (\lambda'+\mu') \\ -\dfrac{(\lambda+2\mu)}{2\mu}, & 1, & \dfrac{1}{2}, & \dfrac{\lambda'}{6\mu'} \\ -\dfrac{1}{2} & -1, & \dfrac{1}{2}, & \dfrac{(2\lambda'+3\mu')}{6\mu'} \end{vmatrix}.$$

$$\vartheta_{3}=\begin{vmatrix} \dfrac{5}{2}(\lambda+\mu), & -8\mu, & \dfrac{(\lambda'+\mu')}{2}, & -4\mu' \\ -\dfrac{3}{2}(\lambda+\mu), & 8\mu, & \dfrac{3}{2}(\lambda'+\mu'), & -4\mu, \\ -\dfrac{(3\lambda+5\mu)}{8\mu} & 1, & \dfrac{(3\lambda'+\mu')}{16\mu'}, & -1 \\ \dfrac{(\lambda-\mu)}{8\mu}, & -1, & \dfrac{(5\lambda'+7\mu')}{16\mu'}, & -1 \end{vmatrix},$$

$$\vartheta_{C_3}=\begin{vmatrix} \mu, & -8\mu, & \dfrac{(\lambda'+\mu')}{2}, & -4\mu' \\ \mu, & 8\mu, & \dfrac{3}{2}(\lambda'+\mu'), & -4'\mu \\ \dfrac{1}{4}, & 1, & \dfrac{(3\lambda'+\mu')}{16\mu'} & -1 \\ \dfrac{1}{4}, & -1, & \dfrac{(3\lambda'+7\mu')}{16\mu'}, & -1 \end{vmatrix},$$

$$\vartheta_{C_3''}=\begin{vmatrix} \dfrac{5}{2}(\lambda+\mu), & \mu, & \dfrac{(\lambda'+\mu')}{2}, & -4\mu' \\ -\dfrac{3}{2}(\lambda+\mu), & \mu, & \dfrac{3}{2}(\lambda'+\mu'), & -4\mu' \\ -\dfrac{(3\lambda+5\mu)}{8\mu}, & \dfrac{1}{4}, & \dfrac{(3\lambda'+\mu')}{16\mu'}, & -1 \\ \dfrac{(\lambda-\mu)}{8\mu}, & \dfrac{1}{4}, & \dfrac{(5\lambda'+7\mu')}{16\mu'}, & -1 \end{vmatrix}, \qquad \ldots(35)$$

$$\vartheta_{D_3}=\begin{vmatrix} \frac{5}{2}(\lambda+\mu), & -8\mu, & \mu, & -4\mu' \\ -\frac{3}{2}(\lambda+\mu), & 8\mu, & \mu, & -4\mu' \\ -\frac{(3\lambda+5\mu)}{8\mu}, & 1, & \frac{1}{4}, & -1 \\ \frac{(\lambda-\mu)}{8\mu}, & -1, & \frac{1}{4}, & -1 \end{vmatrix},$$

$$\left.\vartheta_{D_3''}=\begin{vmatrix} \frac{5}{2}(\lambda+\mu), & -8\mu, & \frac{(\lambda'+\mu')}{2}, & \mu \\ -\frac{3}{2}(\lambda+\mu), & 8\mu, & \frac{3}{2}(\lambda'+\mu'), & \mu \\ -\frac{(3\lambda+5\mu)}{8\mu}, & 1, & \frac{(3\lambda'+\mu')}{16\mu'} & \frac{1}{4} \\ \frac{(\lambda-\mu)}{8\mu} & -1, & \frac{(5\lambda'+7\mu')}{16\mu'}, & \frac{1}{4} \end{vmatrix},\ \right\}$$

Substituting these values (30), (31), (32), (33) in the expressions (25), (26), we obtain the final results of the stresses in the medium and those in the inclusion.

$r\geqq a$:

$$\widehat{rr}=-\rho g\xi\frac{(\lambda+\mu)}{(\lambda+2\mu)}-\frac{2\mu}{r^2}\times\frac{a^2\rho g\xi}{2}\frac{(\lambda'+\mu'-\lambda-\mu)}{(\lambda+2\mu)(\mu+\lambda'+\mu')}+\frac{(2\lambda+3\mu)}{2(\lambda+2\mu)}\rho gr\cos\theta$$

$$+\frac{(2\lambda+3\mu)}{2(\lambda+2\mu)}\frac{(\rho'ga-\rho ga)a}{r}\cos\theta+\frac{\mu'(\lambda'+\mu')\rho ga^4\cos\theta}{2\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}r^3}$$

$$+\frac{\rho'ga^4}{2(\lambda'+2\mu')r^3}\left[\frac{\{(\lambda+2\mu)(2\lambda'+3\mu')-(2\lambda+3\mu)(\lambda'+2\mu')\}}{(\lambda+2\mu)}\right.$$

$$\left.-\frac{\mu'(\lambda'+\mu')(\mu+\lambda'+\mu')}{\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}}\right]\cos\theta$$

$$+\frac{\rho g\xi}{(\lambda+2\mu)}\left[-\mu+\frac{2(\lambda+\mu)\vartheta_{C_2}}{\vartheta_2}\left(\frac{a}{r}\right)^2-\frac{6\mu\vartheta_{C_2''}}{\vartheta_2}\left(\frac{a}{r}\right)^4\right]\cos 2\theta$$

$$+\frac{\rho gr}{(\lambda+2\mu)}\left[\frac{\mu}{2}-\frac{5}{4}\frac{(\lambda+\mu)\vartheta_{C_3}}{\vartheta_3}\left(\frac{a}{r}\right)^4+\frac{4\mu\vartheta_{C_3''}}{\vartheta^3}\left(\frac{a}{r}\right)^6\right]\cos 3\theta,\ \ldots.(36)$$

$$\widehat{\theta\theta}=\frac{\rho g\xi}{(\lambda+2\mu)}\left[-(\lambda+\mu)+\frac{\mu(\lambda'+\mu'-\lambda-\mu)}{(\mu+\lambda'+\mu')}\left(\frac{a}{r}\right)^2\right]+\frac{(2\lambda+\mu)}{2(\lambda+2\mu)}\rho gr\cos\theta$$

$$
-\frac{\mu(\rho'-\rho)ga^2}{2(\lambda+2\mu)r}\cos\theta-\frac{\rho g a\mu'(\lambda'+\mu')}{2\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}}\left(\frac{a}{r}\right)^3\cos\theta
$$

$$
-\frac{\rho' g a^4}{2(\lambda'+2\mu')r^3}\left[\frac{\{(\lambda+2\mu)(2\lambda'+3\mu')-(2\lambda+3\mu)(\lambda'+2\mu')\}}{(\lambda+2\mu)}\right.
$$

$$
\left.-\frac{\mu'(\lambda'+\mu')(\mu+\lambda'+\mu')}{\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}}\right]\cos\theta
$$

$$
+\frac{\rho g\xi}{(\lambda+2\mu)}\left[\mu+\frac{6\mu\vartheta_{C_2''}}{\vartheta_2}\left(\frac{a}{r}\right)^4\right]\cos 2\theta
$$

$$
+\frac{\rho g r}{(\lambda+2\mu)}\left[-\frac{\mu}{2}+\frac{(\lambda+\mu)\vartheta_{C_3}}{4\vartheta_3}\left(\frac{a}{r}\right)^4-\frac{4\mu\vartheta_{C_3''}}{\vartheta}\left(\frac{a}{r}\right)^6\right]\cos 3\theta, \quad \ldots(37)
$$

$$
\widehat{zz}=\frac{\lambda}{2(\lambda+\mu)}\{\widehat{rr}+\widehat{\theta\theta}\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(38)
$$

$$
\widehat{r\theta}=-\frac{\mu\rho g r}{2(\lambda+2\mu)}\sin\theta+\frac{\mu(\rho-\rho')ga^2}{2(\lambda+2\mu)r}\sin\theta+\frac{\rho g a^4\mu'(\lambda'+\mu')}{2\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}r^3}\sin\theta
$$

$$
+\frac{\rho' g a^4}{2(\lambda'+2\mu')r^3}\left[\frac{\{(\lambda+2\mu)(2\lambda'+3\mu')-(2\lambda+3\mu)(\lambda'+2\mu')\}}{(\lambda+2\mu)}\right.
$$

$$
\left.-\frac{\mu'(\lambda'+\mu')(\mu+\lambda'+\mu')}{\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}}\right]\sin\theta
$$

$$
+\frac{\rho g\xi}{(\lambda+9\mu)}\left[\mu+\frac{(\lambda+\mu)\vartheta_{C_2}}{\vartheta_2}\left(\frac{a}{r}\right)^2-\frac{6\mu\vartheta_{C_2''}}{\vartheta}\left(\frac{a}{r}\right)^4\right]\sin 2\theta
$$

$$
+\frac{\rho g r}{(\lambda+2\mu)}\left[-\frac{\mu}{2}-\frac{3(\lambda+\mu)\vartheta_{C_3}}{4\vartheta_3}\left(\frac{a}{r}\right)^4+\frac{4\mu\vartheta_{C_3''}}{\vartheta_3}\left(\frac{a}{r}\right)^6\right]\sin 3\,\theta, \quad \ldots(39)
$$

$r\geq a$:

$$
\widehat{rr'}=-\frac{(\lambda'+\mu')}{(\mu+\lambda'+\mu')}\rho g\xi+\frac{2(\lambda'+3\mu')}{2(\lambda'+9\mu')}\theta' g r \operatorname{coss}\theta
$$

$$
+\frac{(\lambda'+\mu')}{2}\left[\frac{\rho g\mu r}{\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}}\right.
$$

$$
\left.-\frac{\rho' g r\mu'(\mu+\lambda'+\mu')}{(\lambda'+2\mu')\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}}\right]\cos\theta
$$

$$
+\frac{2\mu'\rho g\xi\vartheta_{D_2''}}{(\lambda+2\mu)\vartheta_2}\cos 2\theta
$$

$$
+\frac{\rho g a}{(\lambda+2\mu)}\left[\frac{(\lambda'+'\mu)\vartheta_{D_3}}{4\vartheta_3}\left(\frac{r}{a}\right)^3-\frac{2\mu'\vartheta_{D_3''}}{\vartheta_3}\left(\frac{r}{a}\right)\right]\cos 3\theta, \quad \ldots\ldots\ldots(40)
$$

$$
\widehat{\theta\theta'}=-\frac{(\lambda'+\mu')}{(\mu+\lambda'+\mu')}\rho g\xi+\frac{(2\lambda'+\mu')}{2(\lambda'+2\mu')}\rho' g r\cos\theta
$$

$$+\frac{3}{2}(\lambda'+\mu')\left[\frac{\rho g\mu' r}{\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}}\right.$$

$$\left.-\frac{\rho' g r\mu'(\mu+\lambda'+\mu')}{(\lambda'+2\mu')\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}}\right]\cos\theta$$

$$+\frac{\rho g\xi}{(\lambda+2\mu)}\left[-\frac{2\mu'\vartheta_{D_2''}}{\vartheta_2}+\frac{2(\lambda'+\mu')\vartheta_{D_2}}{\vartheta_2}\left(\frac{r}{a}\right)^2\right]\cos 2\theta$$

$$+\frac{\rho g r}{(\lambda+2\mu)}\left[-\frac{5(\lambda'+\mu')\vartheta_{D_3}}{4\vartheta_3}\left(\frac{r}{a}\right)^3+\frac{2\mu'\vartheta_{D_3''}}{\vartheta_3}\left(\frac{r}{a}\right)\right]\cos\theta \quad \ldots\ldots(41)$$

$$\widehat{zz}'=\frac{\lambda'}{2(\lambda'+\mu')}\{\widehat{rr}'+\widehat{\theta\theta}'\}, \quad \ldots\ldots(42)$$

$$\widehat{r\theta}'=-\frac{\mu'\rho' g r\sin\theta}{2(\lambda'+2\mu')}+\frac{(\lambda'+\mu')}{2}\left[\frac{\rho g\mu' r}{\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}}\right.$$

$$\left.-\frac{\rho' g r\mu'(\lambda'+\mu'+\mu)}{(\lambda'+2\mu')\{\mu(\lambda'+3\mu')+\mu'(\lambda'+\mu')\}}\right]\sin\theta$$

$$+\frac{\rho g\xi}{(\lambda+2\mu)}\left[-\frac{2\mu'\vartheta_{D_2''}}{\vartheta_2}+\frac{(\lambda'+\mu')\vartheta_{D_2}}{\vartheta_2}\left(\frac{r}{a}\right)^2\right]\sin 2\theta$$

$$+\frac{\rho g r}{(\lambda+2\mu)}\left[\frac{2\mu'\vartheta_{D_3''}}{\vartheta_3}-\frac{3(\lambda'+\mu')\vartheta_{D_3}}{\vartheta_3}\left(\frac{r}{a}\right)^3\right]\sin 3\theta. \quad \ldots\ldots (43)$$

When $\lambda=\mu$, $\lambda'=\mu'$ (Poisson's ratio of the semi-infinite solid and that of the heterogeneous inclusion are equally $\frac{1}{4}$), we obtain the expressions of the stress distributions in the solid and that in the inclusion as in the following forms:

$r\geq a$:

$$\widehat{rr}=\rho g\xi\left[\frac{2}{3}\left\{-1+\frac{(\mu-\mu')}{(\mu+2\mu')}\left(\frac{a}{r}\right)^2\right\}\right.$$

$$\left.+\left\{-\frac{1}{3}+\frac{4(\mu-\mu')}{3(\mu+2\mu')}\left(\frac{a}{r}\right)^2-\frac{(\mu-\mu')}{(\mu+2\mu')}\left(\frac{a}{r}\right)^4\right\}\cos 2\theta\right]$$

$$+\rho g a\left[\left\{\frac{5}{6}\left(\frac{r}{a}\right)-\frac{5}{6}\left(\frac{a}{r}\right)+\frac{\mu'}{2(2\mu+\mu')}\left(\frac{a}{r}\right)^3\right\}\cos\theta\right.$$

$$\left.+\left\{\frac{1}{6}\left(\frac{r}{a}\right)-\frac{5(\mu-\mu')}{6(\mu+2\mu')}\left(\frac{a}{r}\right)^3+\frac{2(\mu-\mu')}{3(\mu+2\mu')}\left(\frac{a}{r}\right)^5\right\}\cos 3\theta\right]$$

$$+\rho' g a\left[\frac{5}{6}\left(\frac{a}{r}\right)-\frac{1}{6}\frac{(\mu+2\mu')}{(2\mu+\mu')}\left(\frac{a}{r}\right)^3\right]\cos\theta, \quad \ldots\ldots(44)$$

$$\widehat{\theta\theta}=\rho g\xi\left[\frac{2}{3}\left\{-1-\frac{(\mu-\mu')}{(\mu+2\mu')}\left(\frac{a}{r}\right)^2\right\}+\left\{\frac{1}{3}+\frac{(\mu-\mu')}{(\mu+2\mu')}\left(\frac{a}{r}\right)^4\right\}\cos 2\theta\right]$$
$$+\rho ga\left[\left\{\frac{1}{2}\left(\frac{r}{a}\right)+\frac{1}{6}\left(\frac{a}{r}\right)-\frac{\mu'}{2(2\mu+\mu')}\left(\frac{a}{r}\right)^3\right\}\cos\theta\right.$$
$$\left.+\left\{-\frac{1}{6}\left(\frac{r}{a}\right)+\frac{(\mu-\mu')}{6(\mu+2\mu')}\left(\frac{a}{r}\right)^3-\frac{2(\mu-\mu')}{3(\mu+2\mu')}\left(\frac{a}{r}\right)^5\right\}\cos 3\theta\right]$$
$$+\rho' ga\left[-\frac{1}{6}\left(\frac{a}{r}\right)+\frac{1}{6}\frac{(\mu+2\mu')}{(2\mu+\mu')}\left(\frac{a}{r}\right)^3\right]\cos\theta, \quad \text{..............} \quad (45)$$

$$\widehat{zz}=\rho g\xi\left[-\frac{1}{3}+\frac{(\mu-\mu')}{3(\mu+2\mu')}\left(\frac{a}{r}\right)^2\cos 2\theta\right]$$
$$+\rho ga\left[\left\{\frac{1}{3}\left(\frac{r}{a}\right)-\frac{1}{6}\left(\frac{a}{r}\right)\right\}\cos\theta-\frac{1}{6}\frac{(\mu-'\mu)}{(\mu+2\mu')}\left(\frac{a}{r}\right)^3\cos 3\theta\right]$$
$$+\rho' ga\left[\frac{1}{6}\left(\frac{a}{r}\right)\right]\cos\theta, \quad \text{..............................} \quad (46)$$

$$\widehat{r\theta}=\rho g\xi\left[\frac{1}{3}+\frac{2(\mu-\mu')}{3(\mu+2\mu')}\left(\frac{a}{r}\right)^2-\frac{(\mu-\mu')}{(\mu+2\mu')}\left(\frac{a}{r}\right)^4\right]\sin 2\theta$$
$$+\rho ga\left[\left\{-\frac{1}{6}\left(\frac{r}{a}\right)+\frac{1}{6}\left(\frac{a}{r}\right)+\frac{\mu'}{2(2\mu+\mu')}\left(\frac{a}{r}\right)^3\right\}\sin\theta\right.$$
$$\left.-\left\{\frac{1}{6}\left(\frac{r}{a}\right)+\frac{(\mu-\mu')}{2(\mu+2\mu')}\left(\frac{a}{r}\right)^3-\frac{2(\mu-\mu')}{3(\mu+2\mu')}\left(\frac{a}{r}\right)^5\right\}\sin 3\theta\right]$$
$$+\rho' ga\left[-\frac{1}{6}\left(\frac{a}{r}\right)-\frac{1}{6}\frac{(\mu+9\mu')}{(2\mu+\mu')}\left(\frac{a}{r}\right)^3\right]\sin\theta \quad \text{...............} \quad (47)$$

$r\geq a$:

$$\widehat{rr'}=\rho g\xi\left[-\frac{2\mu'}{(\mu+2\mu')}-\frac{\mu'}{(\mu+2'\mu)}\cos 2\theta\right]$$
$$+\rho ga\left[\frac{\mu'}{2(2\mu+\mu')}\left(\frac{r}{a}\right)\cos\theta+\frac{\mu'}{2(\mu+2\mu')}\left(\frac{r}{a}\right)\cos 3\theta\right]$$
$$+\rho' ga\left[\frac{5}{6}\left(\frac{r}{a}\right)-\frac{1}{6}\frac{(\mu+2\mu')}{(2\mu+\mu')}\left(\frac{r}{a}\right)\right]\cos\theta, \quad \text{.................} \quad (48)$$

$$\widehat{\theta\theta'}=\rho g\xi\left[-\frac{2\mu'}{(\mu+2\mu')}+\frac{\mu'}{(\mu+2\mu')}\cos 2\theta\right]$$
$$+\rho ga\left[\frac{3\mu'}{2(2\mu+\mu')}\left(\frac{r}{a}\right)\cos\theta-\frac{\mu'}{2(\mu+2\mu')}\left(\frac{r}{a}\right)\cos 3\theta\right]$$
$$+\rho' ga\left[\frac{1}{2}\left(\frac{r}{a}\right)-\frac{1}{2}\frac{(\mu+2\mu')}{(2\mu+\mu')}\left(\frac{r}{a}\right)\right]\cos\theta, \quad \text{.................} \quad (49)$$

$$\widehat{zz'}=\rho g\xi\left[-\frac{\mu'}{(\mu+2\mu')}\right]$$

$$+\rho ga\left[\frac{\mu'}{2(2\mu+\mu')}\left(\frac{r}{a}\right)\cos\theta\right]$$

$$+\rho' ga\left[\frac{1}{4}\left(\frac{r}{a}\right)-\frac{1}{6}\frac{(\mu+'2\mu)}{(2\mu+\mu')}\left(\frac{r}{a}\right)\right]\cos\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots (50)$$

$$\widehat{r\theta'}=\rho g\xi\left[\frac{\mu'}{(\mu+2\mu')}\right]\sin 2\theta$$

$$+\rho ga\left[\frac{\mu'}{2(2\mu+\mu')}\left(\frac{r}{a}\right)\sin\theta-\frac{\mu'}{2\mu(+2\mu')}\left(\frac{r}{a}\right)\sin 3\theta\right]$$

$$+\rho' ga\left[-\frac{1}{6}\left(\frac{r}{a}\right)-\frac{(\mu+2\mu')}{2(2\mu+\mu')}\left(\frac{r}{a}\right)\right]\sin\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots (51)$$

We shall investigate closely the stress distributions on the boundary surface $r=a$ by the numerical calculations. Putting $r=a$, we calculate the magnitudes of the separate terms related to $\rho g\xi$, ρga and $\rho' ga$ in all stress components expressed by (44), (45), (46), (47), (48), (49), (50), (51) in the annexed tables when $\frac{\mu'}{\mu}=\frac{1}{5}$, $\frac{\mu'}{\mu}=1$, $\frac{\mu'}{\mu}=5$ and $\frac{\mu'}{\mu}=\infty$. (Table I, II, III, IV, V, VI, VII, VIII, IX, X, XI, XII, XIII, XIV, XV, XVI.) And the following figures (Fig. 2a, 2b, 2c, 3a, 3b, 3c, 4a, 4b, 4c, 5a, 5b, 5c.) shew us these results. In these figures, abscissa is the angle θ, and the parameter of each curve is the magnitude of $\frac{\mu'}{\mu}$.

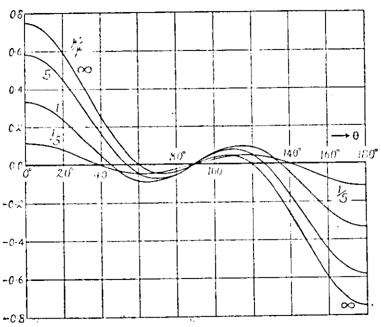

Fig. 2a. Magnitude of term related to ρga of $\widehat{rr}_{r=a}$ $(=\widehat{rr'}_{r=a})$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 1, 5 and ∞.

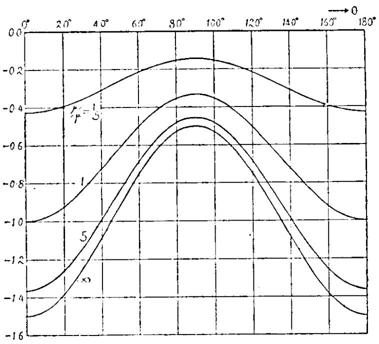

Fig. 2b. Magnitude of term related to $\rho g\xi$ of $\widehat{rr'}_{r=a}$ $(=\widehat{rr'}_{r=a})$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 1, 5 and ∞.

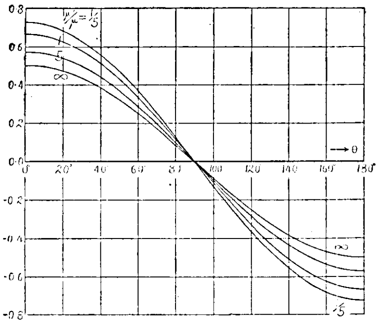

Fig. 2c. Magnitude of term related to $\rho' g a$ of $\widehat{rr}_{r=a}$ $(=\widehat{rr'}_{r=a})$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 1, 5 and ∞.

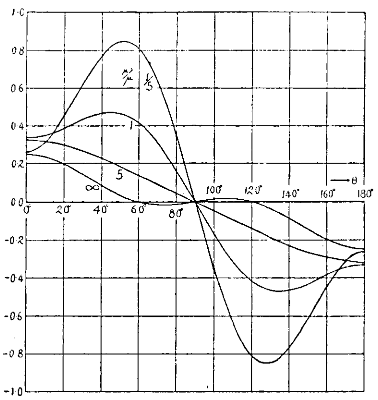

Fig. 3a. Magnitude of term related to $\rho g a$ of $\widehat{\theta\theta}_{r=a}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 1, 5 and ∞.

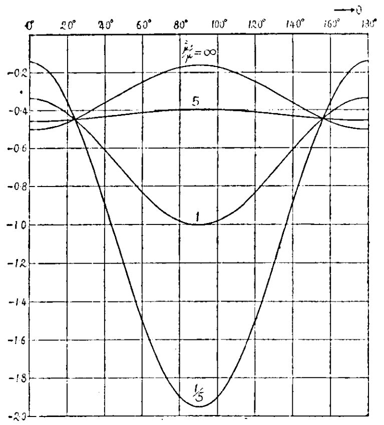

Fig. 3b. Magnitude of term related to $\rho g \xi$ of $\widehat{\theta\theta}_{r=a}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 1, 5 and ∞.

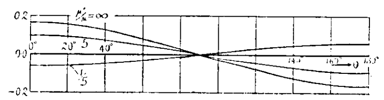

Fig. 3c. Magnitude of term related to $\rho' g a$ of $\widehat{\theta\theta}_{r=a}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 1, 5 and ∞.

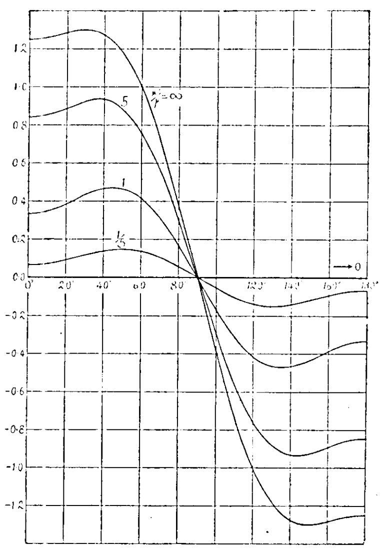

Fig. 4a. Magnitude of term related to $\rho g a$ of $\widehat{\theta\theta'}_{r=a}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 1, 5 and ∞.

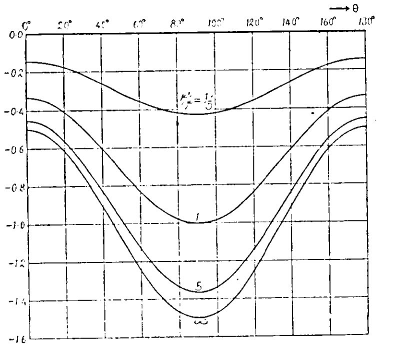

Fig. 4b. Magnitude of term related to $\rho g\xi$ of $\widehat{\theta\theta'}_{r=a}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 1, 5 and ∞.

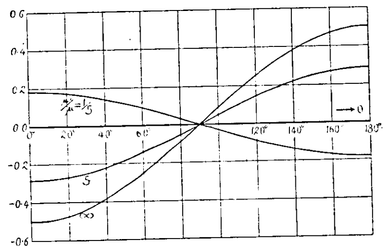

Fig. 4c. Magnitude of term related to $\rho' ga$ of $\widehat{\theta\theta'}_{r=a}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 1, 5 and ∞.

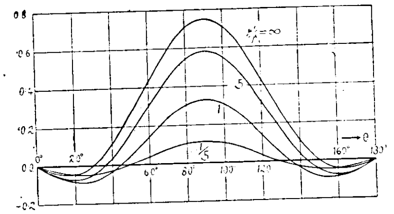

Fig. 5a. Magnitude of term related to ρga of $\widehat{r\theta}_{r=a}$ $(=\widehat{r\theta'}_{r=a})$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 1, 5 and ∞.

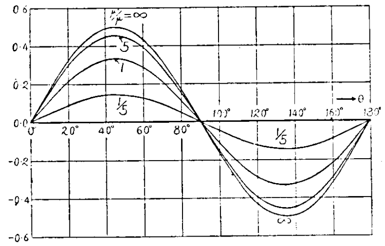

Fig. 5b. Magnitude of term related to $\rho g\xi$ of $\widehat{r\theta}_{r=a}$ $(=\widehat{r\theta'}_{r=a})$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 1, 5 and ∞.

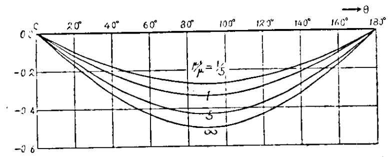

Fig. 5c. Magnitude of term related to $\rho' ga$ of $\widehat{r\theta}_{r=a}$ $(=\widehat{r\theta'}_{r=a})$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 1, 5 and ∞.

From these figures we can see interesting facts and some of them are summarised as follows:

1. The terms related to $\rho g\xi$ of the components of stresses $\widehat{rr'}_{r=a}$, $\widehat{\theta\theta'}_{r=a}$ of the cylindrical inclusion and those of the components of stresses $\widehat{rr}_{r=a}$, $\widehat{\theta\theta}_{r=a}$ of the surrounding medium have symmetrical distributions with respect to the plane $\theta=90°$. The terms related to $\rho g\xi$ of the components of shear stresses $\widehat{r\theta}_{r=a}$ $\widehat{r\theta'}_{r=a}$ have, however, unsymmetrical distributions with respect to this plane.

2. The terms related to ρga of the components of stresses $\widehat{rr}_{r=a}$,

Table I.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{rr}_{r=a}$ $(=\widehat{rr'}_{r=a})$ when $\frac{\mu'}{\mu}=\frac{1}{5}$.

θ	$\rho g\xi$	ρga	$\rho' ga$
1°	−0·429	0·116	0·727
10°	−0·420	0·107	0·717
20°	−0·395	0·079	0·684
30°	−0·358	0·039	0·630
45°	−0·286	−0·019	0·514
60°	−0·214	−0·048	0·364
70°	−0·177	−0·046	0·298
80°	−0·152	−0·028	0·126
90°	−0·143	0	0
100°	−0·152	0·028	−0·126
110°	−0·177	0·046	−0·293
120°	−0·214	0·048	−0·364
135°	−0·286	0·019	−0·514
150°	−0·358	−0·039	−0·630
160°	−0·395	−0·079	−0·684
170°	−0·420	−0·107	−0·717
180°	−0·429	−0·116	−0·727

Table II.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{rr}_{r=a}$ $(=\widehat{rr'}_{r=a})$ when $\frac{\mu'}{\mu}=1$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−1·00	0·334	0·667
10°	−0·981	0·308	0·657
20°	−0·923	0·240	0·627
30°	−0·834	0·144	0·577
45°	−0·667	0	0·472
60°	−0·500	−0·084	0·333
70°	−0·411	−0·087	0·228
80°	−0·353	−0·054	0·116
90°	−0·334	0	0
100°	−0·353	0·054	−0·116
110°	−0·411	0·087	−0·228
120°	−0·500	0·084	−0·333
135°	−0·667	0	−0·472
150°	−0·834	−0·144	−0·577
160°	−0·923	−0·240	−0·627
170°	−0·981	−0·308	−0·657
180°	−1·00	−0·334	−0·667

Table III.

Magnitudes of separate terms related to to ρga, $\rho g\xi$ and $\rho' ga$ of $\widehat{rr}_{r=a}$ $(=\widehat{rr'}_{r=a})$ when $\frac{\mu'}{\mu}=5$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−1·365	0·584	0·571
10°	−1·337	0·549	0·563
20°	−1·258	0·450	0·537
30°	−1·137	0·309	0·494
45°	−1·910	0·091	0.403
60°	−0·683	−0·049	0·286
70°	−0·562	−0·075	0·195
80°	−0·483	−0·052	0·100
90°	−0·455	0	0
100°	−0·483	0·052	−0·100
110°	−0·562	0·075	−0·195
120°	−0·683	0·049	−0·286
135°	−0·910	−0·091	−0·403
150°	−1·137	−0·309	−0·494
160°	−1·258	−0·450	−0·537
170°	−13·37	−0·549	−0·563
180°	−13·65	−0·584	−0·571

Table IV.

Magnitudes of separate terms related to ρga, $\rho g\xi$ and $\rho' ga$ of $\widehat{rr}_{r=a}$ $(=\widehat{rr'}_{r=a})$ when $\frac{\mu'}{\mu}=\infty$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−1·500	0·750	0·500
10°	−1·470	0·708	0·492
20°	−1·383	0·595	0·470
30°	−1·250	0·433	0·433
45°	−1·000	0·176	0·353
60°	−0·750	0	0·250
70°	−0·617	−0·045	0·171
80°	−0·530	−0·038	0·087
90°	−0·500	0	0
100°	−0·530	0·038	−0·087
110°	−0·617	0·045	−0·171
120°	−0·750	0	−0·250
135°	−1·000	−0·176	−0·353
150°	−1·250	−0·433	−0·433
160°	−1·383	−0·595	−0·470
170°	−1·470	−0·708	−0·492
180°	−1·500	−0·750	−0·500

Table V.

Magnitudes of separate terms related to ρga, $\rho' ga$ and $\rho g\xi$ of $\widehat{\theta\theta}_{r=a}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−0·14	0·26	−0·06
10°	−0·20	0·310	−0·059
20°	−0·35	0·444	−0·056
30°	−0·59	0·618	−0·052
45°	−1·05	0·824	−0.042
60°	−1·50	0·809	−0.030
70°	−1·74	0·636	−0.021
80°	−1·90	0·350	−0.010
90°	−1·95	0	0
100°	−1·90	−0·350	0.010
110°	−1·74	−0·636	0.021
120°	−1·50	−0·809	0.030
135°	−1·05	−0·824	0.042
150°	−0·59	−0·618	0.052
160°	−0·35	−0·444	0.056
170°	−0·20	−0·310	0.059
180°	−0·14	−0·260	0.06

Table VII.

Magnitudes of separate terms related to ρga, $\rho' ga$ and $\rho g\xi$ of $\widehat{\theta\theta}_{r=a}$ when $\frac{\mu'}{\mu}=5$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−0·454	0·325	0·095
10°	−0·452	0·318	0·094
20°	−0·447	0·299	0·090
30°	−0·439	0·268	0·082
45°	−0·424	0·208	0·067
60°	−0·409	0·140	0·048
70°	−0·401	0·093	0·033
80°	−0·396	0·046	0·017
90°	−0·394	0	0
100°	−0·396	−0·046	−0·017
110°	−0·401	−0·093	−0·033
120°	−0·409	−0·140	−0·048
135°	−0·424	−0·208	−0·067
150°	−0·439	−0·268	−0·082
160°	−0·447	−0·299	−0·090
170°	−0·452	−0·318	−0·094
180°	−0·454	−0·325	−0·095

Table VI.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{\theta\theta}_{r=a}$ when $\frac{\mu'}{\mu}=1$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−0·334	0·333	0
10°	−0·353	0·348	0
20°	−0·411	0·387	0
30°	−0·550	0·433	0
45°	−0·667	0·471	0
60°	−0·834	0·417	0
70°	−0·923	0·315	0
80°	−0·981	0·170	0
90°	−1·000	0	0
100°	−0·981	−0·170	0
110°	−0·923	−0·315	0
120°	−0·834	−0·417	0
135°	−0·667	−0·471	0
150°	−0·500	−0·433	0
160°	−0·411	−0·387	0
170°	−0·353	−0·348	0
180°	−0·334	−0·333	0

Table VIII.

Magnitudes of separate terms related to ρga, $\rho g\xi$ and $\rho' ga$ of $\widehat{\theta\theta}_{r=a}$ when $\frac{\mu'}{\mu}=\infty$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	−0·50	0·250	0·167
10°	−0·49	0·237	0·165
20°	−0·46	0·198	0·157
30°	−0·42	0·144	0·145
45°	−0·33	0·059	0·118
60°	−0·25	0	0·083
70°	−0·21	−0·015	0·057
80°	−0·18	−0·013	0·029
90°	−0·17	0	0
100°	−0·18	0·013	−0·029
110°	−0·21	0·015	−0·057
120°	−0·25	0	−0·083
135°	−0·33	−0·059	−0·118
150°	−0·42	−0·144	−0·145
160°	−0·46	−0·198	−0·157
170°	−0·49	−0·237	−0·165
180°	−0·50	−0·250	−0·167

Table IX.

Magnitudes of separate terms related to $pg\xi$, pga and $p'ga$ of $\widehat{\theta\theta'}_{r=a}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$.

θ	$pg\xi$	pga	$p'ga$
0°	−0·14	0·065	0·182
10°	−0·15	0·072	0·179
20°	−0·18	0·092	0·171
30°	−0·21	0·118	0·157
45°	−0·29	0·147	0·128
60°	−0·36	0·140	0·091
70°	−0·40	0·109	0·062
80°	−0·42	0·060	0·032
90°	−0·43	0	0
100°	−0·42	−0·060	−0·032
110°	−0·40	−0·109	−0·062
120°	−0·36	−0·140	−0·091
135°	−0·29	−0·147	−0·128
150°	−0·21	−0·118	−0·157
160°	−0·18	−0·092	−0·171
170°	−0·15	−0·072	−0·179
180°	−0·14	−0·065	−0·182

Table XI.

Magnitudes of separate terms related to $pg\xi$, pga and $p'ga$ of $\widehat{\theta\theta'}_{r=a}$ when $\frac{\mu'}{\mu}=5$.

θ	$pg\xi$	pga	$p'ga$
0°	−0·455	0.845	−0·286
10°	−0·483	0·858	−0·281
20°	−0·562	0·892	−0·269
30°	−0·683	0·928	−0·247
45°	−0·910	0·918	−0·202
60°	−1·137	0·763	−0·143
70°	−1·258	0'563	−0·098
80°	−1·337	0·300	−0·050
90°	−1·365	0	0
100°	−1·337	−0·300	0·050
110°	−1·258	−0·563	0·098
120°	−1·137	−0·763	0·143
135°	−0·910	−0·918	0·202
150°	−0·683	−0·928	0·247
160°	−0·562	−0·892	0·269
170°	−0·483	−0·858	0·281
180°	−0·455	−0·845	0·286

Table X.

Magnitudes of separate terms related to $pg\xi$, pga and $p'ga$ of $\widehat{\theta\theta'}_{r=a}$ when $\frac{\mu'}{\mu}=1$.

θ	$pg\xi$	pga	$p'ga$
0°	−0·334	0 333	0
10°	−0·353	0·348	0
20°	−0·411	0·387	0
30°	−0·500	0·433	0
45°	−0·667	0·471	0
60°	−0·834	0·417	0
70°	−0·923	0·315	0
80°	−0·981	0·170	0
90°	−1·000	0	0
100°	−0·981	−0·170	0
110°	−0·923	−0·315	0
120°	−0·834	−0·417	0
135°	−0·667	−0·471	0
150°	−0·500	−0·433	0
160°	−0·411	−0·387	0
170°	−0·353	−0·348	0
180°	−0·334	−0·333	0

Table XII.

Magnitudes of separate terms related to $pg\xi$, pga and $p'ga$ of $\widehat{\theta\theta'}_{r=a}$ when $\frac{\mu'}{\mu}=\infty$.

θ	$pg\xi$	pga	$p'ga$
0°	−0·50	1·250	−0·50
10°	−0·53	1·261	−0·493
20°	−0·62	1·285	−0·470
30°	−0·75	1·30	−0·433
45°	−0·00	1·237	−0·353
60°	−1·25	1·00	−0·25
70°	−1·38	0·730	−0·171
80°	−1·47	0·386	−0·087
90°	−1·50	0	0
100°	−1·47	−0·386	0·087
110°	−1·38	−0·730	0·171
120°	−1·25	−1·00	0·25
135°	−1·00	−1·237	0·353
150°	−0·75	−1·30	0·433
160°	−0·62	−1·285	0·470
170°	−0·53	−1·261	0·493
180°	−0·50	−1·250	0·50

Table XIII.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{r\theta}_{r=a}$ $(=\widehat{r\theta'}_{r=a})$ when $\frac{\mu'}{\mu}=\frac{1}{5}$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	0	0	0
10°	0·049	−0·028	−0·047
20°	0·092	−0·046	−0·093
30°	0·124	−0·048	−0·156
45°	0·143	−0·018	−0·193
60°	0·124	0·039	−0·236
70°	0·092	0·079	−0·256
80°	0·049	0·107	−0·269
90°	0	0·117	−0·273
100°	−0·049	0·107	−0·269
110°	−0·092	0·079	−0·256
120°	−0·124	0·039	−0·236
135°	−0·143	−0·018	−0·193
150°	−0·124	−0·048	−0·156
160°	−0·092	−0·046	−0·093
170°	−0·049	−0·023	−0·047
180°	0	0	0

Table XIV.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{r\theta}_{r=a}$ $(=\widehat{r\theta'}_{r=a})$ when $\frac{\mu'}{\mu}=1$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	0	0	0
10°	0·114	−0·054	−0·058
20°	0·214	−0·087	−0·114
30°	0·289	−0·084	−0·167
45°	0·333	0	−0·236
60°	0·289	0·144	−0·289
70°	0·214	0·240	−0·314
80°	0·114	0·308	−0·328
90°	0	0·334	−0·333
100°	−0·114	0·308	−0·328
110°	−0·214	0·240	−0·314
120°	−0·289	0·144	−0·289
135°	−0·333	0	−0·236
150°	−0·289	−0·084	−0·167
160°	−0·214	−0·087	−0·114
170°	−0·114	−0·054	−0·058
180°	0	0	0

Table XV.

Magnitudes of separate terms related to ρga, $\rho' ga$ and $\rho g\xi$ of $\widehat{r\theta}_{r=a}$ $(=\widehat{r\theta'}_{r=a})$ when $\frac{\mu'}{\mu}=5$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	0	0	0
10°	0·156	−0·051	−0·075
20°	0·292	−0·075	−0·146
30°	0·394	−0·048	−0·214
45°	0·455	0·091	−0·303
60°	0·394	0·309	−0·371
70°	0·292	0·449	−0·403
80°	0·156	0·549	−0·422
90°	0	0·584	−0·428
100°	−0·156	0·549	−0·422
110°	−0·292	0·449	−0·403
120°	−0·394	0·309	−0·371
135°	−0·455	0·091	−0·303
150°	−0·394	−0·048	−0·214
160°	−0·292	−0·075	−0·146
170°	−0·156	−0·051	−0·075
180°	0	0	0

Table XVI.

Magnitudes of separate terms related to $\rho g\xi$, ρga and $\rho' ga$ of $\widehat{r\theta}_{r=a}$ $(=\widehat{r\theta'}_{r=a})$ when $\frac{\mu'}{\mu}=\infty$.

θ	$\rho g\xi$	ρga	$\rho' ga$
0°	0	0	0
10°	0·171	−0·038	−0·087
20°	0·321	−0·046	−0·171
30°	0·433	0	−0·250
45°	0·500	0·176	−0·353
60°	0·433	0·433	−0·433
70°	0·321	0·595	−0·470
80°	0·171	0·709	−0·492
90°	0	0·750	−0·500
100°	−0·171	0·709	−0·492
110°	−0·321	0·595	−0·470
120°	−0·433	0·433	−0·433
135°	−0·500	0·176	−0·353
150°	−0·433	0	−0·250
160°	−0·321	−0·046	−0·171
170°	−0·171	−0·038	−0·087
180°	0	0	0

$\widehat{\theta\theta}_{r=a}$, $\widehat{rr'}_{r=a}$, $\widehat{\theta\theta'}_{r=a}$ have unsymmetrical distributions with respect to the plane of $\theta=90°$, and those related to ρga of $\widehat{r\theta}_{r=a}$, $\widehat{r\theta'}_{r=a}$ have symmetrical distributions with respect to this plane.

3. The terms related to $\rho' ga$ of $\widehat{rr}_{r=a}$, $\widehat{\theta\theta}_{r=a}$, $\widehat{rr'}_{r=a}$, $\widehat{\theta\theta'}_{r=a}$ are also unsymmetrical with respect to the plane $\theta=90°$. The terms concerning $\rho' ga$ of $\widehat{r\theta}_{r=a}$, $\widehat{r\theta'}_{r=a}$ have, however, symmetrical distributions with respect to the plane of $\theta=90°$.

4. When the depth ξ of the center of cylindrical inclusion is deep, the respective distributions of stresses in the inclusion and in the medium are mainly affected by the terms related to $\rho g\xi$ and the terms related to ρga and $\rho' ga$ have no much effects upon the stress distributions. When ρ' is very large, however, the terms related to $\rho' ga$ have some effects upon them.

5. When the depth ξ is moderately shallow, the respective terms related to ρga and $\rho' ga$, which are the unbalancing forces due to body force, have some effects upon the stress distributions.

6. When the cylindrical inclusion is more rigid than the surrounding medium, the magnitudes of the respective terms related to ρga and $\rho g\xi$ of $\widehat{rr}_{r=a}$, $\widehat{r\theta}_{r=a}$, $\widehat{rr'}_{r=a}$, $\widehat{r\theta'}_{r=a}$ are large, and those of the terms related to $\rho' ga$ of $\widehat{r\theta}_{r=a}$ and $\widehat{r\theta'}_{r=a}$ are also large. The magnitudes of the terms related to $\rho' ga$ of $\widehat{rr}_{r=a}$ and $\widehat{rr'}_{r=a}$, however, are small in this case.

7. When the cylindrical inclusion is more rigid than the surrounding medium, the magnitudes of the respective terms related to $\rho g\xi$ and ρga of $\widehat{\theta\theta}_{r=a}$ are generally small, and those of the terms related to $\widehat{\theta\theta'}_{r=a}$, on the contrary, are generally large. The terms related to $\rho' ga$ of the components of stresses $\widehat{\theta\theta}_{r=a}$ and $\widehat{\theta\theta'}_{r=a}$ have special distributions which differ from those of all other components of stresses when the magnitude of $\frac{\mu'}{\mu}$ is large.

To see the properties of distributions of stresses along a radial direction we obtained the annexed figures by numerical calculations of the expressions (44), (45), (46), (47), (48), (49), (50), (51). (Fig. 6a, 6b, 6c, 7a, 7b, 7c, 8a, 8b, 8c). These figures shew us the following facts.

1. Whether or not the elastic constants of the semi-infinite solid are larger than those of the cylindrical inclusion, the magnitudes of the terms related to $\rho g\xi$ of all components of stresses in the cylindrical inclusion are constant along a radial direction but not along azimuthal

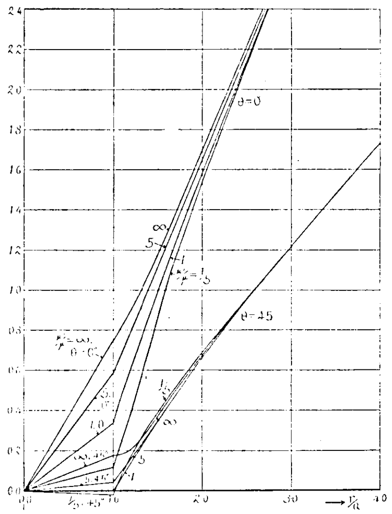

Fig. 6a. Magnitudes of respective terms related to $\rho g a$ of $\widehat{rr}_{\substack{\theta=0^\circ\\ \theta=45^\circ}}$, $\widehat{rr'}_{\substack{\theta=0^\circ\\ \theta=45^\circ}}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 1, 5 and ∞.

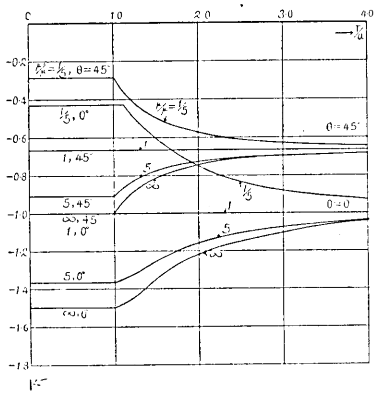

Fig. 6b. Magnitudes of respective terms related to $\rho g \xi$ of $\widehat{rr}_{\substack{\theta=0^\circ\\ \theta=45^\circ}}$, $\widehat{rr'}_{\substack{\theta=0^\circ\\ \theta=45^\circ}}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 1, 5 and ∞.

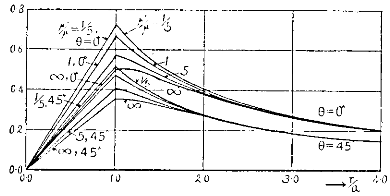

Fig. 6c. Magnitudes of respective terms related to $\rho' g a$ of $\widehat{rr}_{\substack{\theta=0^\circ\\ \theta=45^\circ}}$, $\widehat{rr'}_{\substack{\theta=0^\circ\\ \theta=45^\circ}}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 1, 5 and ∞.

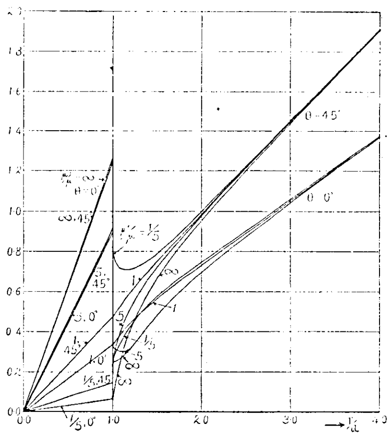

Fig. 7a. Magnitudes of respective terms related to $\rho g a$ of $\widehat{\theta\theta}_{\substack{\theta=0^\circ\\ \theta=45^\circ}}$, $\widehat{\theta\theta'}_{\substack{\theta=0^\circ\\ \theta=45^\circ}}$ when $\frac{\mu'}{\mu}=\frac{1}{5}$, 1, 5 and ∞.

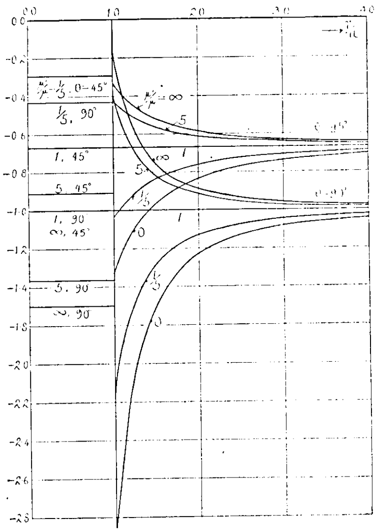

Fig. 7b. Magnitudes of respective terms related to $\rho g\xi$ of $\widehat{\theta\theta}_{\substack{\theta=45^\circ\\ \theta=90^\circ}}$, $\widehat{\theta\theta}'_{\substack{\theta=45^\circ\\ \theta=90^\circ}}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 1, 5 and ∞.

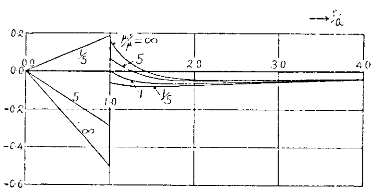

Fig. 7c. Magnitudes of respective terms related to $\rho' ga$ of $\widehat{\theta\theta}_{\theta=0^\circ}$, $\widehat{\theta\theta}'_{\theta=0^\circ}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 1, 5 and ∞.

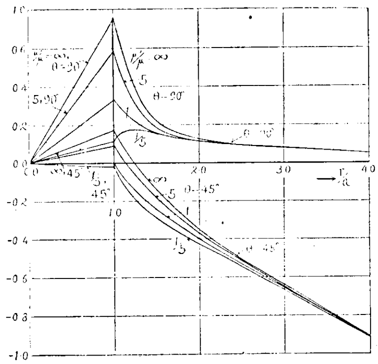

Fig. 8a. Magnitudes of respective terms related to ρga of $\widehat{r\theta}_{\substack{\theta=45^\circ\\ \theta=90^\circ}}$, $\widehat{r\theta}'_{\substack{\theta=45^\circ\\ \theta=90^\circ}}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 1, 5 and ∞.

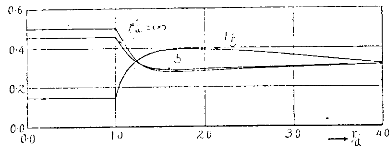

Fig. 8b. Magnitudes of respective terms related to $\rho g\xi$ of $\widehat{r\theta}_{\theta=45^\circ}$ and $\widehat{r\theta}'_{\theta=45^\circ}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 5 and ∞.

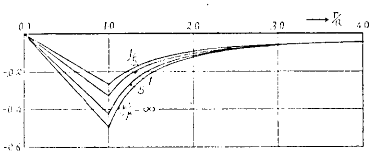

Fig. 8c. Magnitudes of respective terms related to $\rho' ga$ of $\widehat{r\theta}_{\theta=90^\circ}$ and $\widehat{r\theta}'_{\theta=90^\circ}$ when $\dfrac{\mu'}{\mu}=\dfrac{1}{5}$, 1, 5 and ∞.

direction. And the magnitudes of the terms related to ρga and $\rho' ga$ of the respective components of stresses of the inclusion increase linearly along radial direction.

2. The region where the stress distributions in the semi-infinite solid affected by the presence of a cylindrical inclusion is limited in the boundary region of that inclusion and the effective radius upon the distributions of stresses in the boundary medium is approximately four times the radius of the cylindrical inclusion.

The mathematical results obtained in this paper are not applicable to the case where $\frac{\mu'}{\mu}=0$ and the cylindrical inclusion has no rigidity. In this case the problem should be formally solved anew by the following boundary conditions

$$r=a: \quad \left.\begin{aligned} u&=u', \\ \widehat{rr}&=\widehat{rr}', \\ \widehat{r\theta}&=0, \\ \widehat{r\theta'}&=0. \end{aligned}\right\}$$

$$r=\infty: \quad \begin{aligned} \widehat{rr}&=\widehat{rr} \text{ expressed by (27)}, \\ \widehat{\theta\theta}&=\widehat{\theta\theta} \text{ expressed by (27)}, \\ \widehat{r\theta}&=\widehat{r\theta} \text{ expressed by (27)}. \end{aligned}$$

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13. 重力の働ける半無限彈性體內に存在する圓形塡充物附近の應力 (I)

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重力の作用してゐる半無限彈性體內にそれと密度及び彈性常數の異つた圓形な異物質が塡充してゐる場合，この物質の內外に於ける應力釣合を論じた．目的は地殼內部に異物質のある場合，それが地殼の釣合に如何なる影響を與へるかを知る爲めに行つた計算である．直ちに役立つとは思はないが，幾分でも定量的に參考になる所があれば幸である．

計算は，半無限彈性體は plane strain の狀態に置れてあるとしてやり，異物質との硬さの關係を色々にとつて應力の分布狀態を圖示してある．